A New and Unifying Approach to Spin Dynamics and Beam Polarization in Storage Rings

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Abstract

In this paper we summarize and refine our results from [1, 2] where we extended our studies in [3] on polarized beams by introducing a new approach, started in [4], to the understanding of invariant fields. For this we distill tools from the theory of bundles and present five major theorems, one which addresses how invariant fields behave off orbital resonance, one which ties invariant fields with the notion of normal form, one which allows one to compare different invariant fields, and two that relate the existence of invariant fields to the existence of certain invariant sets and relations between them. We then apply the theory to the dynamics of spin-1/2 and spin-1 particles and their density matrix functions, describing statistically the particle-spin content of bunches. Our approach thus unifies the spin-vector dynamics from the T-BMT equation with the spin-tensor dynamics and other dynamics. This unifying aspect of our approach relates the examples elegantly and uncovers relations between the various underlying dynamical systems in a transparent way.

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1. Introduction

The spin polarizations of particles in storage rings are best systematized in a statistical description in terms of invariant spin fields (ISFs) and invariant polarization-tensor fields (IPTFs). The ISF is essential for estimating the maximal attainable proton and deuteron vector polarizations and the IPTF describes the equilibrium tensor polarization of deuterons (spin-1). Moreover, the invariant spin field appears in estimates of the electron and positron equilibrium polarization due to synchrotron radiation.

In this paper we summarize and refine our results from [1, 2], the latter being an extension of our study [3] of spin dynamics in storage rings, by introducing a new approach, started in [4], to the understanding of invariant fields. The new approach employs a method with origins in bundle theory developed in the 1980s by R. Zimmer, R. Feres and others for dynamical-systems theory [5, 6, 7]. In particular, our treatment will be centered on so-called $SO(3)$-spaces denoted by (E, l) and on the associated dynamics given by (j, A), both to be defined in due course. In contrast to [3], we now employ a discrete-time treatment rather than continuous-time but this is a minor point.

We present five major theorems, the Topological Transitivity Theorem (TTT) on the behavior of invariant fields off orbital resonance, the Normal Form Theorem (NFT), tying invariant fields with generalized invariant frame fields, the $SO(3)$-Mapping Theorem (SMT), allowing comparison of different invariant fields, the Invariant Reduction Theorem (IRT), giving new insights into the question of the existence of invariant fields and which is supplemented by the Cross Section Theorem (CST) giving new insights into the question of the existence of invariant frame fields. In addition we also mention a conjecture, the ISF conjecture.

It turns out[1] that the well-established notions of invariant frame field, spin tune, and spin-orbit resonance are augmented by the normal-form concept whereas the well-established notions of invariant spin field and invariant polarization-tensor field are examples of so-called invariant (E, l)-fields. In fact we have a flexibility in the choice of (E, l) which provides a unified way to study the dynamics of spin-1/2 and spin-1 particles and their bunch density matrix functions and which provides a route to the definition of new invariant fields. Accordingly several (E, l)s are discussed in some detail. The origins of our formalism, in bundle theory, are pointed out. We thus open a significant new area of research in our field by bringing in techniques from bundle theory used hitherto in very different research areas. We believe that all five of our theorems are important and new.

We begin in Section 2 by briefly reviewing the familiar treatment of spin-1/2 particles and ISFs and we also mention algorithms for the calculation of ISFs. With this foundation we are then in a position to rewrite our concepts in the language that we need for the rest of the paper, namely in the language of our $SO(3)$-spaces, (E, l). In Section 2 we employ both the continuous-time variable θ as well as the discrete-time variable n where θ is the angular position around the ring and n is the number of turns around the ring. In the remaining sections we focus on the discrete-time treatment.

In Section 3 we discuss particle and field motion in (E, l) and we define (E, l) for the case of spin-1/2 particles. Then in Section 4 we present the five main Theorems. In Section 5 we discuss the five main Theorems for the case of spin-1/2 particles. In Section 6 we discuss the five main Theorems for the case of spin-1 particles and in Section 7 we sketch the bundle-theoretic origins of our approach.

2. Recapitulation of the Spin-1/2 Dynamics

Throughout this text we will assume integrable orbital motion in a model with $2d$-dimensional phase space and with $J \in \mathbb{R}^d$ and $\phi \in \mathbb{R}^d$ being the sets of actions and angles respectively. As is common, our model explicitly does not contain Stern-Gerlach back-reaction of the spin degrees of freedom onto the orbital motion. All external vector fields ("forces") are assumed to be 2π periodic in the free (continuous-time) parameter θ, which may be interpreted as the generalized azimuth related to the longitudinal position s in a storage ring of circumference C by $\theta := \frac{2\pi s}{C}$. The orbital equations of motion read

$$\frac{d\phi}{d\theta} = \omega(J)\,, \quad \frac{dJ}{d\theta} = 0 \tag{1}$$

and are solved for initial conditions $J(\theta_0) = J_0$, $\phi(\theta_0) = \phi_0$ by

$$\phi(\theta) = \phi_0 + \omega(J_0)(\theta - \theta_0)\,, \quad J(\theta) = J_0\,. \tag{2}$$

The integrable orbital motion foliates the $2d$-dimensional phase space into a d parameter family of invariant tori, identified by J on which the angles perform uniform circular motion with the tunes (rotation frequencies) $\omega(J)$. The spin-expectation value, called the spin vector $\vec{S} \in \mathbb{R}^3$, evolves along particle trajectories given by (2) according to the Thomas-BMT equation:

$$\frac{d\vec{S}}{d\theta} = \vec{\Omega}(\theta; J_0, \phi(\theta)) \times \vec{S}\,, \tag{3}$$

where the local precession vector $\vec{\Omega}$ is of course 2π-periodic in θ and in $\phi_1, \ldots, \phi_d$. The general solution of (3) to initial conditions $J(\theta_0) = J_0$, $\phi(\theta_0) = \phi_0$, and $\vec{S}(\theta_0) = \vec{S}_0$ can be written as

$$\vec{S}(\theta) = R(\theta, \theta_0; J_0, \phi_0)\,\vec{S}_0\,, \tag{4}$$

where R is the 3×3 (orthogonal) rotation matrix for the spin transport from θ_0 to θ on a trajectory on the torus specified by J_0 and starting at ϕ_0. $R(\theta, \theta_0; J, \phi)$ is an $SO(3)$-valued function of $(\theta, \theta_0; J, \phi)$ [3, 8], where $SO(3)$ is the group of real orthogonal 3×3-matrices of determinant 1. We assume that R is continuous in ϕ and 2π-periodic in $\phi_1, \ldots, \phi_d$ but we do not assume that R is continuous in θ and θ_0 in order to allow for hard-edged and thin-lens electromagnetic fields.

Here we are interested in the discrete-time dynamics and introduce, in a way that includes the case of storage rings, the one-turn ($\theta - \theta_0 = 2\pi$) maps

$$\begin{aligned}\phi &\mapsto \mathcal{P}_\omega(\phi) = \phi + 2\pi\omega && (5)\\ \vec{S} &\mapsto A(\phi)\,\vec{S} && (6)\\ A(\phi) &:= R(\theta_0 + 2\pi, \theta_0; J_0, \phi)\ , && (7)\end{aligned}$$

where we have suppressed the dependence of A on the starting azimuth θ_0 and on J_0 on the invariant torus. From now on we will suppress almost always any reference to the invariant torus on which the dynamics is studied as well as any reference to the reference azimuth for the one-turn maps. By the properties of R, A is continuous in ϕ and 2π-periodic in $\phi_1, \ldots, \phi_d$. We note that $\mathcal{P}_\omega(\phi)$ in (5) is easily inverted to $\mathcal{P}_\omega^{-1}(\phi) = \phi - 2\pi\omega$.

We call a sequence of fields $\vec{\mathcal{S}}_n : \mathbb{R}^d \to \mathbb{R}^3$, $n \in \mathbb{Z}$ a trajectory of *polarization fields*, iff the sequence evolves via

$$\vec{\mathcal{S}}_{n+1}(\mathcal{P}_\omega(\phi)) = A(\phi)\,\vec{\mathcal{S}}_n(\phi)\ , \tag{8}$$

or equivalently

$$\vec{\mathcal{S}}_{n+1}(\phi) = A(\mathcal{P}_\omega^{-1}(\phi))\,\vec{\mathcal{S}}_n(\mathcal{P}_\omega^{-1}(\phi))\ , \tag{9}$$

i.e. when the $n+1$st generation of the field is constructed by propagating the 3-vectors with the spin/orbit one-turn map. Equation (8) describes evolution in the forward direction of the orbit map, while (9) uses the inverse (backward) orbit map and can serve as an *explicit* evolution equation for the sequence $\vec{\mathcal{S}}_n \mapsto \vec{\mathcal{S}}_{n+1}$. The $\vec{\mathcal{S}}_n$ are continuous in ϕ and 2π-periodic in $\phi_1, \ldots, \phi_d$ if $\vec{\mathcal{S}}_0$ has these properties.

A *spin field* is a polarization field which is normalized to 1, i.e. $|\vec{\mathcal{S}}| = 1$. It is then denoted by $\widehat{\mathcal{S}}$.

A polarization field $\vec{\mathcal{N}}$ is called an *invariant polarization field (IPF)*, iff it is mapped to itself by the spin/orbit one-turn map, i.e. when $\vec{\mathcal{N}}_{n+1} = \vec{\mathcal{N}}_n$ for all n in (8) and (9) or, more explicitly, in the forward direction

$$\vec{\mathcal{N}}(\mathcal{P}_\omega(\phi)) = A(\phi)\,\vec{\mathcal{N}}(\phi)\ , \tag{10}$$

or equivalently in the backward direction

$$\vec{\mathcal{N}}(\phi) = A(\mathcal{P}_\omega^{-1}(\phi))\,\vec{\mathcal{N}}(\mathcal{P}_\omega^{-1}(\phi))\ , \tag{11}$$

Trivially $\vec{\mathcal{N}} = \vec{0}$ is an IPF representing vanishing polarization on the torus. An *invariant spin field (ISF)* is an invariant polarization field which is normalized to 1, i.e. $|\vec{\mathcal{N}}| = 1$. It is then denoted [2] by $\widehat{\mathcal{N}}$.

For thorough accounts of the concept of the *continuous-time* IPFs and ISFs we refer to [3, 9, 11, 8]. In Section 5.1 we will study the dynamics of spin/polarization vector fields in greater detail.

[2] In much of the literature (e.g. in [3, 9, 11, 8]) one finds the symbol $\hat{n}$ instead of $\widehat{\mathcal{N}}$.

The above description of spin-vector motion and the definition of the IPFs/ISFs applies not only to spin-1/2 particles such as electrons and protons but also to the vector polarization of spin-1 particles such as deuterons. However, for spin-1 particles the spin expectation values for some pure states can be zero. Nevertheless the ISF can still be defined as above since it is a concept which is independent of the statistical state of the bunch. Moreover, for the complete description of the polarization state of a beam of spin-1 particles additional degrees of freedom, namely the tensor polarization and *invariant polarization tensor fields (IPTFs)* [12, 13, 14], are required (see also Section 6 below).

For small d (namely $d = 1, 2$) the general spin fields admit a useful pictorial representation. Figure 1 shows a field $\widehat{\mathcal{N}} : \mathbb{T}^1 \to S^2$ (S^2 being the unit sphere in $\mathbb{R}^3$) that assigns a 3-vector of unit Euclidean norm, represented by a point on the closed blue curve on the sphere, to every point of the 1-torus which is identical to the 1-sphere S^1 in $\mathbb{R}^2$. So far the field could be just any field, but here it is in fact the numerical approximation for an ISF of a dynamical system $(\mathcal{P}_\omega, A)$ describing a specific model of the former proton storage ring HERA-p [11]. The invariance of the ISF means that the curve on the sphere is reproduced under the combined spin/orbit one-turn map, while for a non-invariant spin field iteration of the map would create a sequence of not-identical, potentially strongly differing curves.

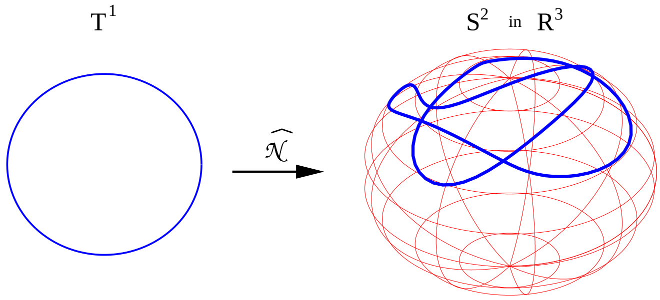

Figure 1: Example of an ISF $\widehat{\mathcal{N}} : \mathbb{T}^1 \to \mathbb{R}^3$: 1996-luminosity-optics of HERA-p with non-resonant vertical tune of about 32.272532721 (rounded) . The beam reference momentum is 805 GeV. Only vertical orbital motion with amplitude of $50.0 \cdot 10^{-6}$ m is excited. The view-point is the East interaction point. Computed with the code `SPRINT` using the `SODOM2` [16] algorithm with 127 particles uniformly distributed over the torus.

The equivalence of spin propagation using the full spin orbit map and evaluation of the ISF at a propagated orbit point is sketched in Figure 2 for a simplified dynamical system in which the image of the 1-torus under the ISF is just a line of latitude on the sphere with the vertical direction as the North pole. The left column of green 1-tori shows, from top to bottom, the invariance of the 1-torus under the iterated (bijective) orbit map $\mathcal{P}_\omega$. In addition it shows the trajectory of a single point under the iteration of $\mathcal{P}_\omega$. In each step (row) the last propagated position of the point is in blue while the sequence of the preceding points is displayed in black. The right column of green ISFs on the sphere shows the invariance of the ISF under the combined spin/orbit map (10). In addition it shows the trajectory of the spin-vector attached to the point on the torus (left) under the iteration of the combined spin/orbit map. In each step (row) the last propagated spin vector is shown as a blue arrow while the sequence of the preceding spins is displayed in black. Thus the figure depicts how the iteration of $\mathcal{P}_\omega$ on a single initial point on the torus will asymptotically produce a dense subset of the torus, i.e. get arbitrarily close to any point, for ω irrational. It also depicts how the iteration of the combined spin/orbit map on a single value of the ISF on some point on the torus asymptotically produces a dense subset of the locus of the ISF.

In general ω is said to be off orbital resonance if $(1, \omega)$ is non resonant, which we now define for future reference:

Definition 2.1 (Non-resonant (NR)). *Let $\omega \in \mathbb{R}^d$. We say $(1, \omega)$ is NR (non-resonant) iff no non-zero integer vector $k \in \mathbb{Z}^{d+1}$ exists such that $k \cdot (1, \omega) = 0$.*

In the following we will not use the phrase off orbital resonance, instead we will write: $(1, \omega)$ is NR. We note that for NR $(1, \omega)$, the map $\mathcal{P}_\omega$ fills the $d-$torus densely. Later we will introduce the term topologically transitive.

Since $r \in SO(3)$ does not change the norm of a spin vector, i.e., $|r\vec{S}| = |\vec{S}|$, Figure 2 also depicts how, if $(1, \omega)$ is NR, the norm of an IPF must be constant on each torus (for each fixed J). Since iteration of $\mathcal{P}_\omega$ yields a dense subset on the torus, evaluation of the continuous IPF on the iterated points yields a dense subset of the locus of the ISF. But since evaluation of the IPF at transported orbit points is equivalent to transporting the field value by the norm preserving map $\vec{S} \mapsto A(\phi)\vec{S}$, the norms of the IPF values must be constant on a dense subset. It can then only be continuous if it is constant everywhere (on the torus).

Figure 3 depicts a field $\widehat{\mathcal{N}} : \mathbb{T}^2 \to S^2 \in \mathbb{R}^3$ that assigns a 3-vector of unit norm, represented by a point on the blue shaded area on the sphere, to every point of the 2-torus. At this stage the field could be just any field, but is is fact the numerical approximation for an ISF of a dynamical system $(\mathcal{P}_\omega, A)$ describing a model of the storage ring HERA-p, similar to Figure 1 but now with both transverse orbital modes excited. The invariance of the ISF entails that the blue shaded area on the sphere is reproduced under the combined spin/orbit one-turn map.

The invariance condition (10) or equivalently (11) and the continuity and periodicity constraints in ϕ make the question of existence of an ISF or nowhere vanishing IPF a nontrivial issue. While there is so far no rigorous existence theorem, there are several numerical algorithms which compute numerical approximations to the ISF that are very stable under long-term tracking, given that the spin/orbit dynamical system is reasonably well behaved.

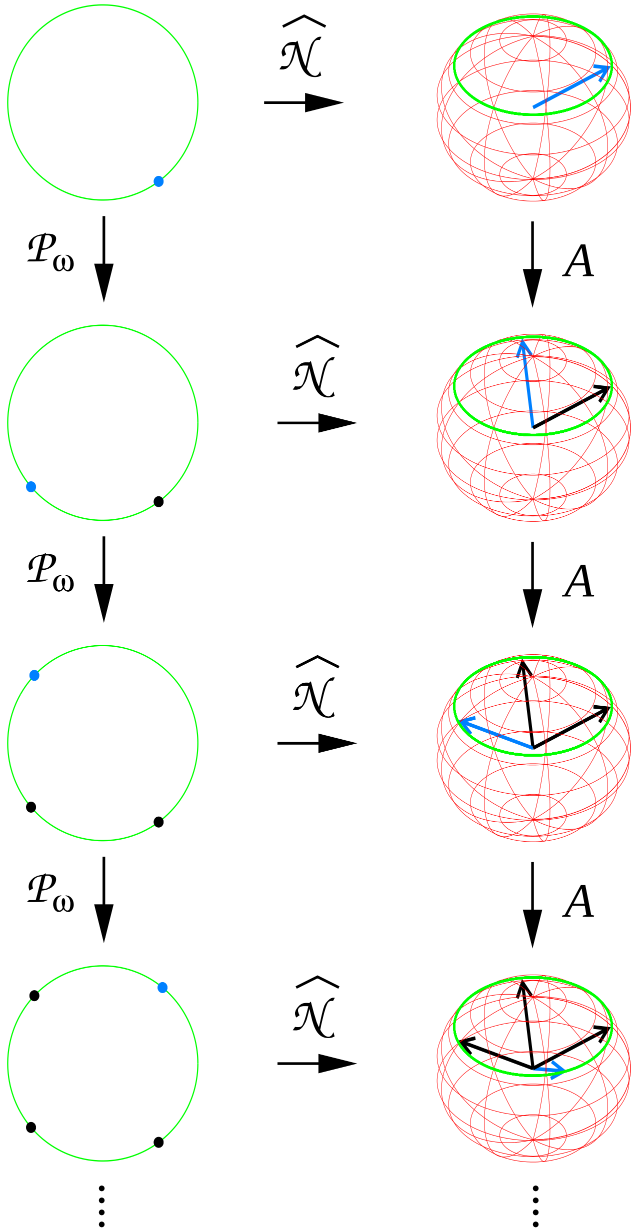

Figure 2: The motion of a point on the torus under the action of $\mathcal{P}_\omega$ samples the ISF at a sequence of points on the torus. This is equivalent to the motion of a spin vector under the action of the combined spin/orbit map, given that the spin vector was initialized with the value of the ISF at the initial point on the torus. The ISF is that of the *single resonance model* [9, 15].

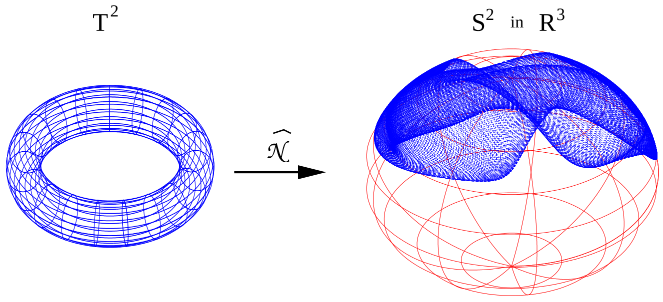

Figure 3: Example of a slightly more complex ISF $\widehat{\mathcal{N}} : \mathbb{T}^2 \to \mathbb{R}^3$: same optics and energy as in Figure 1 but this time with vertical *and* horizontal orbital motion with amplitudes of $5.0 \cdot 10^{-6}$ m and $10.0 \cdot 10^{-6}$ m respectively. The rounded horizontal and vertical tunes are 31.278984723 and 32.272532721 respectively. Computed with the code `SPRINT` using stroboscopic averaging to compute the ISF for 1 particle at an arbitrary point on the torus and then tracking it for $1 \cdot 10^5$ turns.

There are cases however, when for whatever reason, spin motion becomes too "wild" and the algorithms tend to fail to compute reasonable (stable) approximations to ISFs at least with achievable computational effort. There is a perturbative method [15] from the mid 1980s (implemented in the code `SMILE`). Then there are Fourier-methods [16] and [17] (implemented in the codes `SODOM`, `SODOM2`, `SPRINT`, and `MILES`). The ISF can also be computed via a normal-form algorithm based on Truncated Power Series Algebra (TPSA) from the 1990s (implemented in the code `COSY-Infinity` [18] and in the `PTC/FPP` code [19]). Perhaps the most flexible and general algorithm uses so-called stroboscopic averaging [9]. This was invented in the mid-1990s and was first implemented in the code `SPRINT` [11]. It has now been implemented in many other codes. For mathematical details on stroboscopic averaging, see [8]. Stroboscopic averaging can also be used to obtain the IPTF [12, 13]. For more details on these four algorithms, see [20]. The numerical evidence produced by the above mentioned codes indicates that invariant fields can be rather complex entities. Moreover, as already mentioned, the question of existence, although resolvable in some simple cases, is up to this day, unresolved in general and, as evidenced by the simulation codes, situations can occur where ISFs might not exist or might be extremely ill-conditioned. It is in the light of these facts that we present a new approach here, started in [4], to the understanding of invariant fields. In any case we conjecture that an ISF exists if $(1, \omega)$ is NR. This is a special case of the "ISF Conjecture" mentioned in the Introduction and defined in Section 5.1.1.

The importance of the ISF can be best explained by the key role which polarization fields play in the statistical description of a bunch of spin-1/2 particles where the bunch can be described in terms of a semiclassical spin-1/2 density matrix function ρ which is a function whose values are hermitian matrices. The function ρ arises from the semiclassical treatment of Dirac's equation where ρ is the semiclassical approximation of a spin-1/2 Wigner function. Thus the particle variables J and ϕ are purely classical whereas the spin-degrees-of-freedom are treated fully quantum mechanically (see [21] and the references therein). For simplicity we suppress the term "semiclassical" when we talk about ρ.

The spin-1/2 density matrix function is a complex hermitian 2×2 matrix and it is given by

$$\rho(\phi) := \rho_{\rm orb}(\phi)\,\frac{1}{2}\Big(I_{2\times 2} + \vec{\sigma}\cdot\vec{\mathcal{S}}(\phi)\Big)\ , \tag{12}$$

where again we have suppressed the implicit dependence on J. Here $\rho_{\rm orb}$ is the orbital phase space density of the bunch, $I_{2\times 2}$ is the 2×2 unit matrix, $\vec{\sigma}$ is the vector of the three spin-1/2 Pauli matrices, and $\vec{\mathcal{S}}$, $|\vec{\mathcal{S}}| \le 1$, is the polarization field describing the spin state of the bunch.

The orbital density $\rho_{\rm orb}$ is a real nonnegative field defined on the (J, ϕ) phase space and it is preserved along orbital trajectories for every measure preserving map (or flow, in the *continuous-time* case), in particular for the symplectic flow of (2). Therefore the one-turn map $\mathcal{P}_\omega$ in (5) maps $\rho_{\rm orb}$ into

$$\rho'_{\rm orb}(\phi) = \rho_{\rm orb}(\mathcal{P}_\omega^{-1}(\phi))\ . \tag{13}$$

Because (13) is based on the Liouville flow associated with (1) we refer to this as the Liouville one-turn map associated with (1). We will use this terminology in the following and in particular in Section 3.2. We assume that $\rho_{\rm orb}$ and $\vec{\mathcal{S}}$ are continuous in ϕ and, being defined on the torus, are periodic in the $\phi_1, \ldots, \phi_d$. Thus ρ in (12) is continuous in ϕ and 2π-periodic in $\phi_1, \ldots, \phi_d$. Also the phase space density $\rho_{\rm orb}$ is normalized to one, i.e. $\int d\phi\, dJ\, \rho_{\rm orb} = 1$. We call $\rho_{\rm orb}$ in (12) "invariant", if $\rho'_{\rm orb} = \rho_{\rm orb}$. One can show that if $(1, \omega)$ is NR, then every invariant $\rho_{\rm orb}$ is independent of ϕ.

The polarization field evolves according to (9), and thus the spin-1/2 density matrix function is transported by the combined spin/orbit one-turn map as

$$\begin{aligned}\rho'(\phi) &= \rho'_{\rm orb}(\phi)\,\frac{1}{2}\Big(I_{2\times 2} + \vec{\sigma}\cdot\vec{\mathcal{S}}'(\phi)\Big)\\ &= \rho_{\rm orb}(\mathcal{P}_\omega^{-1}(\phi))\,\frac{1}{2}\\ &\times\ \Big(I_{2\times 2} + \vec{\sigma}\cdot A(\mathcal{P}_\omega^{-1}(\phi))\,\vec{\mathcal{S}}(\mathcal{P}_\omega^{-1}(\phi))\Big)\ .\end{aligned} \tag{14}$$

If $\rho_{\rm orb}$ in (12) is invariant, i.e., $\rho'_{\rm orb} = \rho_{\rm orb}$, then one speaks of orbital equilibrium of the bunch and if in addition ρ is invariant, i.e., $\rho' = \rho$ then we say that the beam is in spin equilibrium. If ρ is invariant and if $(1, \omega(J))$ is NR for all J for which $\rho_{\rm orb}(J, \cdot)$ is not the zero function then ρ can be rewritten as

$$\rho_{\rm equi}(J,\phi) = \rho_0(J)\,\frac{1}{2}\Big(I_{2\times 2} + P(J)\Big(\vec{\sigma}\cdot\widehat{\mathcal{N}}(J,\phi)\Big)\Big)\ , \tag{15}$$

where $\widehat{\mathcal{N}}(J, \cdot)$ is an ISF and $0 \le P(J) \le 1$ is the degree of polarization on the torus with J and where we have briefly reinstated the explicit dependence on J with $\rho_0(J) = \rho_{\rm orb}(J,\cdot)$. Every physical observable $\mathcal{O}$ of the bunch can be written as $\mathcal{O}(J,\phi) = o_0(J,\phi) I_{2\times 2} + \sum_{i=1}^3 o_i(J,\phi)\sigma_i$ where the functions o_0 and o_i are real valued. Then its expectation value $\langle \mathcal{O}\rangle(n)$ at turn n is given by

$$\langle \mathcal{O}\rangle(n) = \int d\phi\, dJ \;{\rm Tr}[\rho_n \mathcal{O}] \;, \tag{16}$$

in obvious notation and where Tr is the trace operation. If the beam is in spin equilibrium, i.e., ρ is invariant, then ρ does not vary from turn to turn and neither does $\langle \mathcal{O}\rangle$. Experimenters at colliders and fixed-target storage rings then have the all-important stable conditions.

In the statistical description the polarization vector $\vec{P}$ of the particle bunch is defined as the expectation value of $\vec{\sigma}$ whence, by (16),

$$\begin{aligned} \vec{P} &= \int d\phi\, dJ \;{\rm Tr}[\vec{\sigma}\rho(J,\phi)] \\ &= \int d\phi\, dJ \;\rho_{\rm orb}(J,\phi)\; \vec{\mathcal{S}}(J,\phi) \;, \end{aligned} \tag{17}$$

whence the polarization $|\vec{P}|$ of the bunch satisfies

$$|\vec{P}| \le \int d\phi\, dJ \;\rho_{\rm orb}(J,\phi)\; |\vec{\mathcal{S}}(J,\phi)| \;. \tag{18}$$

In spin equilibrium, when $\vec{P}$ does not vary from turn to turn, and using (15),(18) (and setting $P(J) = 1$ for all J) we get $|\vec{P}| \le P_{\rm max}$ where the maximum attainable equilibrium polarization $P_{\rm max}$ is given by

$$\begin{aligned} P_{\rm max} &= \int dJ\; \rho_0(J)\; |\int d\phi\; \widehat{\mathcal{N}}(J,\phi)| \\ &= (2\pi)^d \int dJ\; \rho_0(J)\; P_{\rm lim}(J) \;, \end{aligned} \tag{19}$$

where the *static polarization limit* on each torus is given by

$$P_{\rm lim}(J) = \frac{1}{(2\pi)^d} |\int d\phi\; \widehat{\mathcal{N}}(J,\phi)| \;. \tag{20}$$

The five major theorems of this work are rigorously stated and applied in Sections 4-6, but we now briefly discuss them in the more common wording of this introductory section. We begin with the Topological Transitivity Theorem (TTT) to be introduced in Section 4.2 which states that, if $(1,\omega)$ is NR, then invariant fields are functions whose range is severely restricted. For example if ρ_{orb} is an invariant orbital density and if $(1,\omega)$ is NR then $\rho_{\rm orb}$ is independent of ϕ (we used this property above). Moreover if $\vec{\mathcal{N}}$ is an IPF and if $(1,\omega)$ is NR then $|\vec{\mathcal{N}}|$ is independent of ϕ (we used also this property above). Moreover if $\mathcal{M}$ is an IPTF and if $(1,\omega)$ is NR then ${\rm Tr}[\mathcal{M}^2]$ and $\det(\mathcal{M})$ are independent of ϕ.

The Normal Form Theorem (NFT) will be introduced in Section 4.3. An important example for spin-1/2 particles is the following. Let $\widehat{\mathcal{S}}$ be a spin field, let $T = T(\phi)$ be an $SO(3)$-valued function which is continuous in ϕ and 2π-periodic in $\phi_1,\ldots,\phi_d$ and such that $\widehat{\mathcal{S}}(\phi)$ is the third column of $T(\phi)$. Let

$$A'(\phi) := T^{\rm t}(\mathcal{P}_\omega(\phi))\; A(\phi)\; T(\phi) \tag{21}$$

where the superscript $^{\rm t}$ denotes the transpose. Then an application of the NFT gives A' is $SO(2)$-valued iff $\widehat{\mathcal{S}}$ is an ISF, i.e. iff $\widehat{\mathcal{S}}$ fulfills (10) or equivalently (11). Here $SO(2)$ is the subgroup of $SO(3)$, defined by

$$SO(2) := \left\{ \begin{pmatrix} \cos(y) & -\sin(y) & 0 \\ \sin(y) & \cos(y) & 0 \\ 0 & 0 & 1 \end{pmatrix} : y \in \mathbb{R} \right\} \;. \tag{22}$$

In Section 5.3 we will see that T is a so-called invariant frame field. In fact the NFT (see Section 4.3) allows one to transform invariant fields to invariant frame fields (or to generalized invariant frame fields) and vice versa.

We now discuss the $SO(3)$-Mapping Theorem (SMT) to be introduced in Section 4.4 which allows one to transform invariant fields to invariant fields. It is in the spirit of the SMT to transform known invariant fields into previously unrecognized invariant fields or vice versa. However the SMT also transforms known invariant fields into known invariant fields as the following simple example shows. If $\vec{\mathcal{N}}$ is an IPF then $|\vec{\mathcal{N}}|$ satisfies

$$|\vec{\mathcal{N}}|(\mathcal{P}_\omega(\phi)) = |\vec{\mathcal{N}}|(\phi) \;, \tag{23}$$

i.e., $|\vec{\mathcal{N}}|$ behaves, dynamically, as an invariant Liouville density.

The remaining two theorems, the Invariant Reduction Theorem (IRT) and the Cross Section Theorem (CST) are presented in Section 4.5. They give, for spin-1/2 particles, a topological criterion for the existence of an ISF and a topological criterion for the existence of an invariant frame field.

This completes our brief look at the standard knowledge of the dynamics of spin-1/2 particles.

3. $SO(3)$-Spaces (E, l) and the Associated (j, A) Dynamics

We now introduce the $SO(3)$-spaces (E,l) and the associated particle and field dynamics with parameters (j,A) where j is defined below. Starting from this chapter we will use a different, slightly more formal definition of the angle variables and their domain, the d-torus, $\mathbb{T}^d$. We do this because this definition conveniently simplifies the statement and proofs of the theorems. In the main course of the text we will replace functions like $A(\phi)$, $\mathcal{P}_\omega(\phi)$, $\vec{\mathcal{S}}(\phi)$, $\rho(\phi)$, $\widehat{\mathcal{N}}(\phi)$, etc. defined for $\phi \in \mathbb{R}^d$ and continuous and periodic as in Section 2, with "new" continuous functions $A(z)$, $\mathcal{P}_\omega(z)$, $\vec{\mathcal{S}}(z)$, $\rho(z)$, $\widehat{\mathcal{N}}(z)$, etc. defined for $z \in \mathbb{T}^d$. An immediate gain of this notation is that from now on we will never again have to state or require explicit periodicity in the angles. Since $\mathcal{P}_\omega(z)$ takes values on $\mathbb{T}^d$ its computation needs a little explanation. To a given z there correspond countably many $\phi \in \mathbb{R}^d$. So we pick one and let $\phi' = \phi + 2\pi\omega$. There is then a unique z' corresponding to ϕ' for all choices of ϕ and we define $\mathcal{P}_\omega(z) = z'$.

3.1. $SO(3)$-spaces (E, l)

The 2-tuple (E, l) denotes an $SO(3)$-space defined as follows:

Definition 3.1 ($SO(3)$-space). *We denote the space of continuous functions from $X \to Y$ by $\mathcal{C}(X, Y)$, where X and Y are topological spaces [22]. Then (E, l) is said to be an $SO(3)$-space if E is a topological space, $l \in \mathcal{C}(SO(3) \times E, E)$ and*

$$l(I_{3\times3}; x) \;=\; x\,. \tag{24}$$
$$l(r_1 r_2; x) \;=\; l(r_1; l(r_2; x))\,, \tag{25}$$

where $r_1, r_2 \in SO(3)$, $x \in E$, and $I_{3\times3} \in SO(3)$ is the unit matrix. We call l an $SO(3)$-action on E or just an $SO(3)$-action if E is clear.

We note that the identity function l_{id} defined by $l_{\text{id}}(r; x) := x$ is a (trivial) $SO(3)$-action for every E. Even though it's trivial it will be useful in Sections 5 and 6. In the following we need the notion of a homeomorphism. A homeomorphism is a continuous bijective map with continuous inverse [23]. Let X, Y be topological spaces then here and in the following $\text{Homeo}(X, Y)$ will denote the set of homeomorphisms $X \to Y$, and $\text{Homeo}(X)$ will denote the set of homeomorphisms $X \to X$.

In general our topological spaces do not need to be *Hausdorff* so we will mention it when needed. For a discussion of Hausdorff see, e.g., [24].

3.2. (j, A) dynamics on (E, l)

For every (E, l) and (j, A) we define two dynamical systems, a particle dynamics and a field dynamics. We think of a particle moving on the d-torus $\mathbb{T}^d$ via

$$z \to z' = j(z)\,,\; j \in \text{Homeo}(\mathbb{T}^d)\,, \tag{26}$$

with an associated E-valued quantity x (an "E-spin") which evolves via

$$x \to x' = l(A(z); x)\,,\; A \in \mathcal{C}(\mathbb{T}^d, SO(3))\,. \tag{27}$$

and we have a "one-turn particle-E-spin map" $\mathcal{P}[E, l, j, A] \in \mathcal{C}(\mathbb{T}^d \times E, \mathbb{T}^d \times E)$ defined by $\mathcal{P}[E, l, j, A](z, x) := (z', x')$, i.e.,

$$\mathcal{P}[E, l, j, A](z, x) = (j(z),\, l(A(z); x))\,. \tag{28}$$

The inverse of $\mathcal{P}[E, l, j, A]$ is

$$\mathcal{P}[E, l, j, A]^{-1} = \mathcal{P}[E, l, j^{-1}, A^{\text{t}} \circ j^{-1}]\,, \tag{29}$$

where $\circ$ denotes composition. Since $j^{-1} \in \text{Homeo}(\mathbb{T}^d)$ and $(A^t \circ j^{-1}) \in \mathcal{C}(\mathbb{T}^d, SO(3))$ it follows from (29) that $\mathcal{P}[j, A]^{-1}$ is continuous whence the bijection $\mathcal{P}[E, l, j, A]$ is a homeomorphism.

In all physical applications we have in mind, $j = \mathcal{P}_\omega$ and so in this case j is just a shorthand. However, our theorems work for general j. Note that $\mathcal{P}_\omega$ is continuous and since $\mathcal{P}_{-\omega}$ is its inverse, $\mathcal{P}_\omega \in \text{Homeo}(\mathbb{T}^d)$.

We are primarily interested in the field dynamics induced by the particle-spin dynamics. So let $f : \mathbb{T}^d \to E$ be an E-valued field on $\mathbb{T}^d$ and set $x = f(z)$ in (28), i.e. an E-spin is assigned to each point on the torus. Then, after one turn, z becomes $j(z)$ and the field value at $j(z)$ becomes $l(A(z); f(z))$ whence, (27) reads in this case as,

$$(z, f(z)) \mapsto (j(z), l(A(z); f(z)))\,. \tag{30}$$

Thus after one turn the field f becomes the field $f' : \mathbb{T}^d \to E$ where $f'(z) := l(A(j^{-1}(z)); f(j^{-1}(z)))$ whence we have the one-turn field map

$$f \mapsto f' := l(A \circ j^{-1}; f \circ j^{-1})\,. \tag{31}$$

Note, by (31), that

$$f' \circ j = l(A; f)\,. \tag{32}$$

We work in the framework of topological dynamics. So A, j, l, f, f' are continuous functions whence we formalize (31) by the one-turn field map $\tilde{\mathcal{P}}[E, l, j, A] : \mathcal{C}(\mathbb{T}^d, E) \to \mathcal{C}(\mathbb{T}^d, E)$ defined by $\tilde{\mathcal{P}}[E, l, j, A](f) := f'$, i.e. by (31),

$$\tilde{\mathcal{P}}[E, l, j, A](f) := l(A \circ j^{-1}; f \circ j^{-1})\,, \tag{33}$$

where $f \in \mathcal{C}(\mathbb{T}^d, E)$. The inverse is

$$\tilde{\mathcal{P}}[E, l, j, A]^{-1} = \tilde{\mathcal{P}}[E, l, j^{-1}, A^{\text{t}} \circ j^{-1}]\,, \tag{34}$$

as is easily checked. While we work in the framework of topological dynamics, weaker or stronger conditions than continuity would be possible.

Two simple cases increase the insight into $\tilde{\mathcal{P}}$. If $A = I_{3\times3}$, (33) becomes

$$\tilde{\mathcal{P}}[E, l, j, I](f) := l(I; f \circ j^{-1}) = f \circ j^{-1}\,, \tag{35}$$

which is a generalized Liouville one-turn field map. If $l = l_{\text{id}}$ then

$$\tilde{\mathcal{P}}[E, l_{\text{id}}, j, A](f) := l_{\text{id}}(A \circ j^{-1}; f \circ j^{-1}) = f \circ j^{-1}\,, \tag{36}$$

for all A, which is the same generalized Liouville one-turn field map.

We call an $f \in \mathcal{C}(\mathbb{T}^d, E)$ an *invariant (E, l)-field of (j, A)*, or *invariant (E, l)-field* or just *invariant field*, if it is mapped by (33) into itself, i.e. $\tilde{\mathcal{P}}[E, l, j, A](f) = f$ which, by (33), is equivalent to the condition

$$f \circ j = l(A; f)\,. \tag{37}$$

Of course this means that in the iteration $f \to f\,' \to f\,'' \cdots$ the fs are the same, i.e., $f = f\,' = f\,'' = \cdots$.

Our main focus is on exploring invariant fields as these describe the spin equilibrium of a bunch [3] and as we will see in Section 5.1 below, the well-established notions of IPF and ISF are invariant (E, l)-fields for a special choice of (E, l). Analogously we will see in Section 6 below that the notion of invariant polarization-tensor field are (E, l)-field and invariant (E, l)-fields for another special choice of (E, l).

3.3. Topological transitivity

Let us define

Definition 3.2.

$$E_x := l(SO(3); x) := \{l(r; x) : r \in SO(3)\} , \qquad (38)$$

where $x \in E$.

If $(z, y) \in \mathbb{T}^d \times E_x$ then an $r \in SO(3)$ exists such that $y = l(r; x)$ whence, by (25) and (28),

$$\begin{aligned}\mathcal{P}[E, l, j, A](z, y) &= (j(z),\, l(A(z); y)) \\ &= (j(z),\, l(A(z); l(r; x))) \\ &= (j(z),\, l(A(z)r; x)) \qquad (39)\end{aligned}$$

so that each set $\mathbb{T}^d \times E_x$ is invariant under the particle-E-spin motion. Thus E is "decomposed" naturally into the E_x. Note also that the E_x partition E, i.e. for all $x, y \in E$, either $E_x = E_y$ or $E_x \cap E_y = \varnothing$ and $\cup_x E_x = E$ (here trivial since $x \in E_x$).

Of central interest are invariant (E, l)-fields in the situation when j is topologically transitive:

Definition 3.3 (Topological Transitivity). *A map $j \in \mathcal{C}(\mathbb{T}^d, \mathbb{T}^d)$ is said to be topologically transitive when a $z_0 \in \mathbb{T}^d$ exists such that the $j^n(z_0)$ are dense in $\mathbb{T}^d$, i.e., when* $\mathrm{Cl}(\{j^n(z_0) : n = 0, \pm 1, \pm 2, ...\}) = \mathbb{T}^d$ *where* Cl *indicates topological closure [25].*

If $j = \mathcal{P}_\omega$ then j is topologically transitive iff $(1, \omega)$ is NR. Topological transitivity is an important notion for two reasons: firstly $\mathcal{P}_\omega$ is generically topologically transitive since a NR $(1, \omega)$ is "generic" and secondly, by the TTT in Section 4.2, the topological transitivity has a strong impact on the possible form of any invariant (E, l)-field f since it restricts the possible range of f. Here a NR $(1, \omega)$ is generic in two senses, topological and measure-theoretic. Firstly, the set of ω such that $(1, \omega)$ is NR is a dense subset of $\mathbb{R}^d$. Secondly and with respect to Lebesgue measure almost every ω has the property that $(1, \omega)$ is NR. See [26] on "Generic Property" also.

4. The Main Theorems

4.1. Introduction

We now present the five theorems. Some of our results will require the Hausdorff condition and when needed that will be made clear. The theorems may appear abstract at a first reading but their application to the familiar spin-1/2 and spin-1 dynamics in Sections 5 and 6 respectively will render them more concrete.

We first define two notions that are crucial in the course of this section.

Definition 4.1 (Isotropy Group).
Let (E, l) be an $SO(3)$-space and $x \in E$. We denote by H_x the set of those $r \in SO(3)$ for which x is a fixed point of $l(r; \cdot)$ i.e.,

$$H_x \equiv \mathrm{Iso}(E, l; x) := \{r \in SO(3) : l(r; x) = x\} . \qquad (40)$$

One can show that H_x is a subgroup of $SO(3)$, and it is called the *isotropy group* of (E, l) at x. Note, by (38) and (40), that for any $SO(3)$-space (E, l) and any $x \in E$ we have $\mathrm{Iso}(E, l; x) = SO(3)$ iff $E_x = \{x\}$.

Definition 4.2 ($SO(3)$-map).
Let (E, l) and $(\check{E}, \check{l})$ be $SO(3)$-spaces and let $\gamma \in \mathcal{C}(E, \check{E})$ be a continuous function: $E \to \check{E}$. γ is called an $SO(3)$-map from (E, l) to $(\check{E}, \check{l})$ iff

$$\check{l}(r; \gamma(x)) = \gamma(l(r, x)) , \ \forall r \in SO(3), x \in E . \qquad (41)$$

If $\gamma \in \mathrm{Homeo}(E, \check{E})$, i.e. if γ^{-1} exists and is continuous, then one can show directly by applying γ^{-1} to both sides of (41) and identifying $\gamma(x) = \check{x}$, that $\gamma^{-1}(\check{l}(r; \check{x})) = l(r; \gamma^{-1}(\check{x}))$, i.e. that γ^{-1} is an $SO(3)$-map from $(\check{E}, \check{l})$ to (E, l).

Let $x \in E$ be mapped to $x' = l(r; x)$ for some $r \in SO(3)$ and let $\check{x} := \gamma(x) \in \check{E}$, γ being an $SO(3)$-map, be mapped to $\check{x}' = \check{l}(r; \check{x})$. Then by (41) $\check{x}' = \check{l}(r; \gamma(x)) = \gamma(l(r, x)) = \gamma(x')$. In other words, an $SO(3)$-map preserves the time evolution.

4.2. The Topological Transitivity Theorem (TTT)

Theorem 1 (Topological Transitivity Theorem).
[1] Let (E, l) be an $SO(3)$-space and let E be Hausdorff. Moreover, let $j \in$ $\mathrm{Homeo}(\mathbb{T}^d)$ be topologically transitive and $A \in \mathcal{C}(\mathbb{T}^d, SO(3))$. If f is an invariant (E, l)-field of (j, A) then an $x \in E$ exists such that f is E_x-valued.

In the following we will often refer to the Topological Transitivity Theorem as the TTT. Recall from Section 3.3 that $E_x = l(SO(3); x))$. The proof [1] of the TTT uses the compactness [27] of $SO(3)$.

The Hausdorff condition on E is very weak and it is satisfied by almost all examples which are important for us and satisfied for all examples in this work. The TTT implies that the topological transitivity of j has a strong impact on invariant (E, l)-fields of any (j, A). Since $\mathcal{P}_\omega$ is generically topologically transitive and because of the TTT, most of our remaining theorems are formulated for fields taking values in only one E_x (though all theorems could be restated without this restriction). One key aspect of the TTT is that it guarantees, for topologically transitive motion on the torus, that our fairly abstract invariant (E, l)–fields cannot violate certain fundamental facts of spin dynamics. In particular we will see in subsections 5.2 and 6.2 that the norms of the values of the invariant vector/tensor polarization fields must be constant over a torus with topologically transitive motion. If $(1, \omega)$ is resonant for $\omega \in \mathbb{R}^d$, then either the motion is periodic or there are so-called sub-tori on which the motion is topologically transitive.

Since the TTT emphasizes the E_x we now mention one property of E_x which, due to lack of space, cannot be covered in this work but which is too important to be left unmentioned. Let (E, l) be an $SO(3)$-space and let

E be Hausdorff and $x \in E$. As before let us abbreviate $H_x \equiv \mathrm{Iso}(E,l;x)$. Then we define

$$
\begin{aligned}
rH_x &:= \{rh : h \in H_x\}\ ,\ (r \in SO(3)) \\
SO(3)/H_x &:= \{rH_x : r \in SO(3)\}\ .
\end{aligned}
$$

The sets rH_x are known as the "left cosets" of H_x [28]. We define the function $\lambda : SO(3)/H_x \to E_x$ by $\lambda(rH_x) := l(r,x)$. Note that λ is a function, i.e., single-valued due to (40). The point to be made is that one can show that $\lambda \in \mathrm{Homeo}(SO(3)/H_x, E_x)$. Thus E_x is determined, up to homeomorphism, by $\mathrm{Iso}(E,l;x)$ alone! Note that the natural topology on $SO(3)/H_x$ is the final topology with respect to the function $p : SO(3) \to SO(3)/H_x$ defined by $p(r) := rH_x$. For the notion of final topology, see [29]. A simple example, occurring for example in Section 5.3, is when $\mathrm{Iso}(E,l;x) = SO(2)$ whence $SO(3)/SO(2)$ and E_x are homeomorphic. On the other hand one can show that $SO(3)/SO(2)$ is homeomorphic to the unit sphere S^2 in $\mathbb{R}^3$ whence E_x is homeomorphic to S^2. Note that some of the topological spaces E_x arising in Sections 5 and 6 are more complex than S^2 and thus worth a study but we must leave the details to the reader.

4.3. The Normal Form Theorem (NFT)

The NFT relates invariant (E,l)-fields to invariant (E,l)-frame fields (the latter will be defined below).

Theorem 2 (Normal Form Theorem (NFT)). *[1] Let $T \in \mathcal{C}(\mathbb{T}^d, SO(3))$, $j \in \mathrm{Homeo}(\mathbb{T}^d)$, $A \in \mathcal{C}(\mathbb{T}^d, SO(3))$, $x \in E$ and define both the field $f \in \mathcal{C}(\mathbb{T}^d, E)$ and $A' \in \mathcal{C}(\mathbb{T}^d, SO(3))$ by*

$$
\begin{aligned}
f(z) &:= l(T(z);x)\ , &(42)\\
A'(z) &:= T^{\mathrm{t}}(j(z))A(z)T(z)\ . &(43)
\end{aligned}
$$

Then f is an invariant (E,l)-field, i.e. $f \circ j = l(A;f)$ as in (37), iff

$$
A'(z) \in H_x \equiv \mathrm{Iso}(E,l;x)\ ,\ \forall z \in \mathbb{T}^d\ . \tag{44}
$$

The proof of "left implies right" is simple: Let $f \circ j = l(A;f)$, then $l(T(j(z));x) = l(A(z);l(T(z);x))$. Now applying $l(T^{\mathrm{t}}(j(z));\cdot)$ to both sides yields $l(T^{\mathrm{t}}(j(z));l(T(j(z));x)) = l(T^{\mathrm{t}}(j(z));l(A(z);l(T(z);x)))$ $\Leftrightarrow x = l(T^{\mathrm{t}}(j(z))A(z)T(z);x) = l(A'(z);x)$ whence $A'(z) \in H_x \forall z$. The proof of the NFT is completed in [1].

In Sections 5.3 and 6.3 we will identify the isotropy groups for various important $SO(3)$-spaces in order to apply the NFT. Note also that the scope of the NFT is very large since one can show that for every subgroup H of $SO(3)$ an $SO(3)$-space (E,l) and an $x \in E$ exist such that $H = H_x \equiv \mathrm{Iso}(E,l;x)$.

The NFT leads to the concept of the invariant (E,l)-frame field. In fact we call a $T \in \mathcal{C}(\mathbb{T}^d, SO(3))$ an *invariant (E,l)-frame field* of (j,A) at x if the A' in (43) satisfies (44). Thus the NFT can also be reformulated as: The function f is an invariant (E,l)-field iff T is an invariant (E,l)-frame field at x. Thus, by the NFT, if an invariant (E,l)-frame field exists at x, then an invariant (E,l)-field exists which takes values only in E_x. We will find a partial converse of this statement in Section 4.5. The notion of the invariant (E,l)-frame field generalizes the notion of the invariant frame field, the latter being introduced in Section 5.3.

The NFT also leads to an important concept in our work, the normal form. If $A'(z)$ in (43) belongs to a subgroup H of $SO(3)$ for all z then we call (j,A') an H-normal form of (j,A). Thus the NFT can be reformulated as: The function f is an invariant (E,l)-field of (j,A) iff (j,A') is an H_x-normal form of (j,A). The normal form concept also captures the notions of spin tune and spin-orbit resonance [30] which, due to lack of space, cannot be covered in this work. We finally note that the transformation $A \mapsto A'$ in (43) which underlies the notion of normal form also has the important property [30] that

$$
\begin{aligned}
\tilde{\mathcal{P}}[E,l,j,A'] &= \tilde{\mathcal{P}}[E,l,\mathrm{id}_{\mathbb{T}^d},T]^{-1} \\
&\circ\ \tilde{\mathcal{P}}[E,l,j,A] \circ \tilde{\mathcal{P}}[E,l,\mathrm{id}_{\mathbb{T}^d},T]\ ,
\end{aligned} \tag{45}
$$

where $\mathrm{id}_{\mathbb{T}^d} \in \mathrm{Homeo}(\mathbb{T}^d)$ is the identity function on $\mathbb{T}^d$ and where $T \in \mathcal{C}(\mathbb{T}^d, SO(3))$ is arbitrary. For example (45) implies that a function $f \in \mathcal{C}(\mathbb{T}^d, E)$ is an invariant (E,l)-field of (j,A) iff $\tilde{\mathcal{P}}[E,l,\mathrm{id}_{\mathbb{T}^d},T]^{-1}(f)$ is an invariant (E,l)-field of (j,A').

4.4. The $SO(3)$-Mapping Theorem (SMT)

The SMT and its corollary enable comparison of the dynamics of different $SO(3)$-spaces for the same (j,A).

Theorem 3 ($SO(3)$-Mapping Theorem (SMT)). *Let γ be an $SO(3)$-map from the $SO(3)$-space (E,l) to the $SO(3)$-space $(\check{E},\check{l})$. Let f be arbitrary in $\mathcal{C}(\mathbb{T}^d, E)$ and transport it by the one-turn map of (E,l) and (j,A) to $f' := \tilde{\mathcal{P}}[E,l,j,A](f)$. Furthermore let $\check{f} \in \mathcal{C}(\mathbb{T}^d, \check{E})$ be defined by $\check{f}(z) := \gamma(f(z))$ and transport it by the one-turn map of $(\check{E},\check{l})$ and (j,A) to $\check{f}' := \tilde{\mathcal{P}}[\check{E},\check{l},j,A](\check{f})$. Then $\check{f}'(z) = \gamma(f'(z))$ for all (j,A), or equivalently*

$$
\tilde{\mathcal{P}}[\check{E},\check{l},j,A](\gamma \circ f) = \gamma \circ (\tilde{\mathcal{P}}[E,l,j,A](f)) \quad \forall (j,A)\ . \tag{46}
$$

The proof is too short to be omitted: Let $f \mapsto f' = l(A \circ j^{-1}; f \circ j^{-1})$ and $\check{f} \mapsto \check{f}' = \check{l}(A \circ j^{-1}; \check{f} \circ j^{-1})$. Now let $\check{f} = \gamma \circ f$. Then $\check{f}' = \check{l}(A \circ j^{-1}; \gamma \circ f \circ j^{-1}) = \gamma(l(A \circ j^{-1}; f \circ j^{-1})) = \gamma \circ f'$, q.e.d.

The SMT has several essential implications.

Firstly, if f is an invariant (E,l)-field of (j,A), then $\check{f}$ is an invariant $(\check{E},\check{l})$-field of (j,A).

Secondly, if the $SO(3)$-map γ is a homeomorphism then, by recalling the discussion after Definition 4.2, γ^{-1} is an $SO(3)$-map from $(\check{E},\check{l})$ to (E,l) whence f is an invariant (E,l)-field of (j,A) iff $\check{f}$ is an invariant $(\check{E},\check{l})$-field of (j,A).

Thirdly, the SMT has a converse. In fact if $\gamma \in \mathcal{C}(E,\check{E})$ and if, for all $f \in \mathcal{C}(\mathbb{T}^d, E)$ and for all (j,A), $\check{f}'(z) = \gamma(f'(z))$ then γ is an $SO(3)$-map from (E,l) to $(\check{E},\check{l})$.

Fourthly, we now state a corollary to the SMT, the Decomposition Theorem (DT). For that purpose let $x \in E$ and $\check{x} \in \check{E}$. One can show that (E_x, l_x) is an $SO(3)$-space where the function $l_x : SO(3) \times E_x \to E_x$ is the restriction of l from $SO(3) \times E$ to $SO(3) \times E_x$, i.e. $l_x(r; y) := l(r; y)$ and where the natural topology of E_x is its subspace topology [25] as a subset of E. Analogously, $(\check{E}_{\check{x}}, \check{l}_{\check{x}})$ is an $SO(3)$-space. Now let $\beta \in \mathcal{C}(E_x, \check{E}_{\check{x}})$ and let the field $f \in \mathcal{C}(\mathbb{T}^d, E)$ take values only in E_x whence, by (38) and for every $z \in \mathbb{T}^d$, there exists an $r_z \in SO(3)$ such that $f(z) = l(r_z; x)$. Moreover f is mapped after one turn to $f' := \tilde{\mathcal{P}}[E, l, j, A](f)$ which takes values only in E_x too because, by (25) and (32), $f'(j(z)) = l(A(z); f(z)) = l(A(z); l(r_z; x)) = l(A(z)r_z; x)$. On the other hand $\check{f} \in \mathcal{C}(\mathbb{T}^d, \check{E})$, defined by $\check{f}(z) := \beta(f(z))$, takes values only in $\check{E}_{\check{x}}$. And, $\check{f}$ is analogously mapped after one turn to $\check{f}' := \tilde{\mathcal{P}}[\check{E}, \check{l}, j, A](\check{f})$ which, again by using (25) and (32), takes values only in $\check{E}_{\check{x}}$.

Corollary 1 (Decomposition Theorem (DT)). *[1] If β is an $SO(3)$-map from (E_x, l_x) to $(\check{E}_{\check{x}}, \check{l}_{\check{x}})$, then $\check{f}'(z) = \beta(f'(z))$ for all (j, A).*

One can prove the DT by setting $\gamma = \beta$ in the SMT.

The terminology DT refers to E being "decomposed", i.e., partitioned, into the E_x (similarly for $\check{E}$). The DT is an important corollary to the SMT, because if E_x and $\check{E}_{\check{x}}$ are Hausdorff (which they are, whenever E and $\check{E}$ are Hausdorff) then, for any (E_x, l_x) and $(\check{E}_{\check{x}}, \check{l}_{\check{x}})$, there is a way to construct any $SO(3)$-map β, given that it exists. In fact one can show [1] that an $SO(3)$-map β from (E_x, l_x) to $(\check{E}_{\check{x}}, \check{l}_{\check{x}})$ exists, iff the group $H_x \equiv \mathrm{Iso}(E, l; x)$ is *conjugate* to a subgroup of $\check{H}_{\check{x}} \equiv \mathrm{Iso}(\check{E}, \check{l}; \check{x})$, i.e. iff an $r_0 \in SO(3)$ exists so that $r_0 H_x r_0^{\mathrm{t}} \subset \check{H}_{\check{x}}$. Then β can be defined by

$$\beta(l(r; x)) = \check{l}(r\, r_0^{\mathrm{t}}; \check{x}) \; . \tag{47}$$

There is an alternative construction of $\beta \in \mathcal{C}(E_x, \check{E}_{\check{x}})$ for the case that an $SO(3)$-map $\gamma \in \mathcal{C}(E, \check{E})$ is already given. The restriction of γ from E to any E_x is an $SO(3)$-map β from (E_x, l_x) to $(\check{E}_{\check{x}}, \check{l}_{\check{x}})$ if $\check{x} \in \check{l}(SO(3); \gamma(x))$. If E_x and $\check{E}_{\check{x}}$ are Hausdorff both methods yield the same β.

Note that every $SO(3)$-map β from (E_x, l_x) to $(\check{E}_{\check{x}}, \check{l}_{\check{x}})$ is a homeomorphism if H_x and $\check{H}_{\check{x}}$ are conjugate. In contrast, if H_x and $\check{H}_{\check{x}}$ are not conjugate then no $SO(3)$-map β can be a homeomorphism [1].

We will apply the SMT in Sections 5 and 6 to $SO(3)$-spaces defined in those sections. It is also in the spirit of the SMT to design $SO(3)$-spaces in order to learn more about invariant fields but this aspect is beyond the scope of this work.

4.5. The Invariant Reduction Theorem (IRT) and the Cross Section Theorem (CST)

Throughout this section we consider a fixed $SO(3)$-space (E, l) and suppress reference to it whenever it helps to keep the notation concise. We also assume a field $f \in \mathcal{C}(\mathbb{T}^d, E)$ to be E_x-valued for some $x \in E$. For $y \in E_x$ define

$$\mathcal{R}_x(y) := \{r \in SO(3) : l(r; x) = y\} \; , \tag{48}$$

and let

$$\begin{aligned} \Sigma_x[f] &\equiv \Sigma_x[E, l, f] \\ &:= \bigcup_{z \in \mathbb{T}^d} \{z\} \times \mathcal{R}_x(f(z)) && (49) \\ &= \{(z, r) \in (\mathbb{T}^d \times SO(3)) : \\ &\qquad l(r; x) = f(z)\} \; . && (50) \end{aligned}$$

Here we mostly use the shorthand $\Sigma_x[f]$, in Sections 5,6 and 7 we will use $\Sigma_x[E, l, f]$. As usual let $j \in \mathrm{Homeo}(\mathbb{T}^d)$ and $A \in \mathcal{C}(\mathbb{T}^d, SO(3))$. We introduce an evolution of $\Sigma_x[f]$ under (j, A) by defining a one-turn map on $\mathbb{T}^d \times SO(3)$ as a special case of (28) in Section 3.2. In fact we define the function $l_{SO(3)} \in \mathcal{C}(SO(3) \times SO(3), SO(3))$ by

$$l_{SO(3)}(r'; r) := r' r \; . \tag{51}$$

One can show that $(SO(3), l_{SO(3)})$ is an $SO(3)$-space whence $\mathcal{P}[SO(3), l_{SO(3)}, j, A] \in \mathrm{Homeo}(\mathbb{T}^d \times SO(3))$ reads as

$$\mathcal{P}[SO(3), l_{SO(3)}, j, A](z, r) := (j(z), A(z)r) \; . \tag{52}$$

Theorem 4 (Invariant Reduction Theorem (IRT)). *[4, 1] f is an invariant (E, l)-field iff the set $\Sigma_x[f]$ is invariant under $\mathcal{P}[SO(3), l_{SO(3)}, j, A]$, i.e. iff $\mathcal{P}[SO(3), l_{SO(3)}, j, A](\Sigma_x[f]) = \Sigma_x[f]$.*

To sketch the IRT proof, one first shows, by (25) and (50), that

$$\mathcal{P}[SO(3), l_{SO(3)}, j, A](\Sigma_x[f]) = \Sigma_x[f'] \; , \tag{53}$$

where f' is f after one turn, i.e., $f' = \tilde{\mathcal{P}}[E, l, j, A](f)$. Thus if f is invariant, i.e., $f' = f$ then, by (53), $\Sigma_x[f]$ is invariant under $\mathcal{P}[SO(3), l_{SO(3)}, j, A]$.
Conversely if $\mathcal{P}[SO(3), l_{SO(3)}, j, A](\Sigma_x[f]) = \Sigma_x[f]$ then, by (53), $\Sigma_x[f] = \Sigma_x[f']$ which leads to $f' = f$.

Let $(z_0, y) \in \mathbb{T}^d \times E_x$ and define

$$\overline{\Sigma}_x[z_0, y] := \bigcup_{n \in \mathbb{Z}} \mathcal{P}[SO(3), l_{SO(3)}, j, A]^n \big(\{z_0\} \times \mathcal{R}_x(y)\big) \; . \tag{54}$$

Then a corollary [1] to the IRT states:

Corollary 2. *If $\Sigma_x[f] = \mathrm{Cl}(\overline{\Sigma}_x[z_0, y])$ then j is topologically transitive and f is an invariant (E, l)-field.*

Furthermore if $(z, r) \in \mathrm{Cl}(\overline{\Sigma}_x[z_0, y])$, then $f(z) = l(r; x)$ so that $y \in E_x$ explicitly determines the invariant (E, l)-field f via the set $\mathrm{Cl}(\overline{\Sigma}_x[z_0, y])$. Note that (54) does not depend on f (but see the partial converse below).

Let $p_x[E, l, f] : \Sigma_x[E, l, f] \to \mathbb{T}^d$ be defined by $p_x[f](z, r) \equiv p_x[E, l, f](z, r) := z$. Hence $p_x[f]$ is continuous with respect to the subspace topology on $\Sigma_x[f]$. One calls $\sigma \in \mathcal{C}(\mathbb{T}^d, \Sigma_x[f])$ a *cross section* of $p_x[f]$ if $p_x[f](\sigma(z)) = z$, i.e. if σ is a right inverse of $p_x[f]$.

Remark: let $\sigma \in \mathcal{C}(\mathbb{T}^d, \Sigma_x[f])$, then $\sigma(z) = (\xi(z), r(z))$, with $\xi \in \mathcal{C}(\mathbb{T}^d, \mathbb{T}^d)$. Therefore $l(r(z); x) = f(\xi(z))$ and $p_x[f](\sigma(z)) = \xi(z)$. Thus a necessary condition for σ to be continuous right inverse of $p_x[f]$ is, $\xi(z) = z$ and $r \in \mathcal{C}(\mathbb{T}^d, SO(3))$ so that $l(r(z); x) = f(z)$. The converse of this is true and gives Theorem 5 which we now state.

Theorem 5 (Cross Section Theorem (CST)). *[1] $p_x[f]$ has a cross section iff $T \in \mathcal{C}(\mathbb{T}^d, SO(3))$ exists such that $f(z) = l(T(z); x)$.*

The proof [1] uses the fact that $\sigma(z) := (z, T(z))$ is a cross section if T is continuous. Note that, under the assumptions of the NFT, $p_x[f]$ has a cross section. For more relations between the NFT and CST see the discussion after (55). The definition of $p_x[f]$ and the name CST are suggested by bundle theory, as discussed in Section 7.

We now state a partial converse to Corollary 2. Let j be topologically transitive, let $p_x[f]$ have a cross section and let f be an invariant (E, l)-field. Then

$$\Sigma_x[f] = \mathrm{Cl}(\,\overline{\Sigma}_x[z_0, f(z_0)]\,)\ . \tag{55}$$

Thus, as mentioned after Corollary 2, $f(z_0)$ explicitly determines the invariant field f via the set $\mathrm{Cl}(\,\overline{\Sigma}_x[j, A, z_0, f(z_0)]\,)$. The fact that f can be determined by the single value $f(z_0)$ is not surprising since the $j^n(z_0)$ are dense in $\mathbb{T}^d$ and since the iteration $f(j^{n+1}(z_0)) = l(A(j^n(z_0))r; f(j^n(z_0)))$ gives f on a dense subset of $\mathbb{T}^d$ and, by continuity, everywhere. Recall the discussion of Figure 2 in Section 2. In contrast, (55) is an alternative method for obtaining an explicit form of f from $f(z_0)$ and it does so for arbitrary (E, l). If the $j^n(z_0)$ are not dense in $\mathbb{T}^d$ then $f(z_0)$ does not necessarily determine f. For example let $j(z) = z$ and $A(z) := I_{3\times 3}$ then every $f \in \mathcal{C}(\mathbb{T}^d, E)$ is an invariant (E, l)-field and hence it is not determined by $f(z_0)$.

Recall from Section 4.3 that, by the NFT, an invariant (E, l)-frame field at x provides an invariant (E, l)-field which takes values only in E_x. The NFT and the CST give us a partial converse namely the claim that an invariant (E, l)-frame field at x exists iff an invariant (E, l)-field f exists which takes values only in E_x and such that $p_x[E, l, f]$ has a cross section. To prove the claim, let first an invariant (E, l)-frame field T at x exist whence, by the NFT, the function $f \in \mathcal{C}(\mathbb{T}^d, SO(3))$ defined by $f(z) = l(T(z); x)$ is an invariant (E, l)-field which takes values only in E_x and, by the CST, $p_x[E, l, f]$ has a cross section. Conversely if an invariant (E, l)-field f exists which takes values only in E_x and such that $p_x[E, l, f]$ has a cross section then, by the CST, a $T \in \mathcal{C}(\mathbb{T}^d, SO(3))$ exists such that $f(z) = l(T(z); x)$. Thus, by the NFT, T is an invariant (E, l)-frame field at x which proves the claim. Note also that through this claim the CST gives us new insights into the question of the existence of invariant (E, l)-frame fields.

The IRT gives a topological interpretation to the invariant (E, l)-fields. Moreover it is of interest for the study of the problems of existence of the invariant (E, l)-fields, in particular via Corollary 2. The CST on the other hand gives a topological interpretation to the invariant (E, l)-frame fields. The topological interpretations of the IRT and the CST have their origins in bundle theory since the topological spaces $\Sigma_x[E, l, f]$ are the bundle spaces of principal bundles if E is Hausdorff. For more details see Section 7.

5. Applying the Main Theorems to Spin-1/2 Dynamics

5.1. *Reconsidering the spin-1/2 dynamics in terms of (E, l)*

In this section we introduce and study the $SO(3)$-spaces $(\mathbb{R}^3, l_{\mathrm{v}})$ and $(E_{1/2}, l_{1/2})$ which play the leading role for the spin-1/2 dynamics. We also introduce $(\mathbb{R}, l_{\mathrm{id}})$ and $(\mathbb{R}^4, \overline{l_{1/2}})$, which are useful for understanding $(E_{1/2}, l_{1/2})$. Although the results from some of the examples are obvious in the context of standard spin dynamics, the examples still serve to illustrate our methods.

5.1.1. $(\mathbb{R}^3, l_{\mathrm{v}})$ *for particle and field dynamics*

We begin by defining the $SO(3)$-space $(E, l) = (\mathbb{R}^3, l_{\mathrm{v}})$ for which the one-turn particle-E-spin map $\mathcal{P}[E, l, j, A]$ in (28) reproduces the discrete-time spin vector motion (5) - (7), and for which the one-turn field-E-spin map $\tilde{\mathcal{P}}[E, l, j, A]$ reproduces the discrete-time polarization field motion (8) - (9).

We define $l_{\mathrm{v}} \in \mathcal{C}(SO(3) \times \mathbb{R}^3, \mathbb{R}^3)$ by

$$l_{\mathrm{v}}(r; \vec{S}) := r\vec{S}\ , \tag{56}$$

where $r \in SO(3)$. The spin vector $\vec{S}$ is an $\mathbb{R}^3$-spin variable.

It follows from (28) and (56) that for every $j \in \mathrm{Homeo}(\mathbb{T}^d)$ and every $A \in \mathcal{C}(\mathbb{T}^d, SO(3))$

$$\mathcal{P}[\mathbb{R}^3, l_{\mathrm{v}}, j, A](z, \vec{S}) = (j(z),\ A(z)\vec{S})\ . \tag{57}$$

With $j = \mathcal{P}_\omega$ this reproduces (5) - (7).

The field motion $\vec{f} \to \vec{f}'$ in (32) is given by $\vec{f}'(j(z)) = l_v(A(z); \vec{f}(z)) = A(z)\vec{f}(z)$, so that the field map of (33) gives

$$\tilde{\mathcal{P}}[E, l_v, j, A](\vec{f}) := l_{\mathrm{v}}(A \circ j^{-1}; \vec{f} \circ j^{-1})\ , \tag{58}$$

which reproduces (8) - (9). By iteration we have $\vec{f} \to \vec{f}' \to \vec{f}'' = (\vec{f}')' = l_{\mathrm{v}}(A \circ j^{-1}; \vec{f}' \circ j^{-1}) \cdots$.

From (37), $\vec{f} \in \mathcal{C}(\mathbb{T}^d, \mathbb{R}^3)$ is an invariant $(\mathbb{R}^3, l_{\mathrm{v}})$-field of (j, A), i.e. $\tilde{\mathcal{P}}[\mathbb{R}^3, l_{\mathrm{v}}, j, A](\vec{f}) = \vec{f}$, iff

$$\vec{f} \circ j = A\vec{f}\ . \tag{59}$$

In consistency with Section 2 we call every invariant $(\mathbb{R}^3, l_{\mathrm{v}})$-field an invariant polarization field (IPF) and a unit IPF is called an invariant spin field (ISF).

The required continuity of the ISF and the fact that it is nowhere zero make the existence of the ISF a nontrivial issue, as we mentioned before. But as we mentioned in the Introduction and in Section 2, on the basis of much computational and analytical experience we propose the "ISF-conjecture", namely: If j is topologically transitive then (j, A) has an ISF for every $A \in \mathcal{C}(\mathbb{T}^d, SO(3))$.

5.1.2. $(E_{1/2}, l_{1/2})$ for density matrix dynamics

We now define the $SO(3)$-space $(E, l) = (E_{1/2}, l_{1/2})$ for which the one-turn field map $\tilde{\mathcal{P}}[E, l, j, A]$ in (33) reproduces the discrete-time motion of the spin-1/2 density matrix function in (14).

We also introduce the $SO(3)$-spaces $(\mathbb{R}, l_{\rm id})$ and $(\mathbb{R}^4, \overline{l_{\rm v}})$ which facilitate computations and help in the understanding of $(E_{1/2}, l_{1/2})$.

From Section 2 we recall that the orbital dynamics is given by the one-turn maps

$$\rho_{\rm orb} \to \rho'_{\rm orb} \ \text{ via } \ \rho'_{\rm orb}(z) = \rho_{\rm orb}(j^{-1}(z)) \ , \qquad (60)$$

where $j = \mathcal{P}_\omega$, and that the spin field dynamics is given by

$$\vec{\mathcal{S}} \to \vec{\mathcal{S}}' \ \text{ via } \ \vec{\mathcal{S}}'(z) = A(j^{-1}(z))\vec{\mathcal{S}}(j^{-1}(z)) \ , \qquad (61)$$

where $|\vec{\mathcal{S}}| \le 1$ whence $|\vec{\mathcal{S}}'| \le 1$. Note that, in our Stern-Gerlach-free model, the spin motion does not impact the orbital densities.

The density matrix function that starts at

$$\rho(z) := \rho_{\rm orb}(z)\,\frac{1}{2}\Big(I_{2\times 2} + \vec{\sigma}\cdot\vec{\mathcal{S}}(z)\Big) \qquad (62)$$

is mapped to

$$\rho'(z) \ = \ \rho'_{\rm orb}(z)\,\frac{1}{2}\Big(I_{2\times 2} + \vec{\sigma}\cdot\vec{\mathcal{S}}'(z)\Big) \ . \qquad (63)$$

We now define $E_{1/2}$ as the set of hermitian 2×2-matrices, i.e., $E_{1/2} := \{R \in \mathbb{C}^{2\times 2} : R^\dagger = R\}$, with † denoting the hermitian conjugate. Thus ρ is $E_{1/2}$ valued. The natural topology of $E_{1/2}$ is its subspace topology as a subset of $\mathbb{C}^{2\times 2}$. Since $\rho_{\rm orb}, \rho'_{\rm orb}$ and $\vec{\mathcal{S}}, \vec{\mathcal{S}}'$ are continuous, $\rho, \rho' \in \mathcal{C}(\mathbb{T}^d, E_{1/2})$.

The dynamics of ρ, given by (63), is described by a field dynamics given by the $SO(3)$-space (E, l) where $E = E_{1/2}$ and $l = l_{1/2}$ with $l_{1/2} \in \mathbb{C}(SO(3)\times E_{1/2}, E_{1/2})$ being defined by

$$l_{1/2}(r; \gamma_{1/2}(S_0, \vec{S})) = \gamma_{1/2}(S_0, r\vec{S}) \ , \qquad (64)$$

where $S_0 \in \mathbb{R}$ and $\vec{S} \in \mathbb{R}^3$, and $\gamma_{1/2} \in \text{Homeo}(\mathbb{R}^4, E_{1/2})$ is defined by

$$\gamma_{1/2}(S_0, \vec{S}) = \frac{1}{2}\Big(S_0 I_{2\times 2} + \vec{\sigma}\cdot\vec{S}\Big) \ . \qquad (65)$$

It follows [12] from (65) that if $h \in E_{1/2}$ then $\gamma_{1/2}^{-1}(h) = (S_0, \vec{S})$ where $S_0 \in \mathbb{R}$ and $\vec{S} \in \mathbb{R}^3$ are defined by

$$S_0 := \text{Tr}[h] \ , \ S_i := \text{Tr}[\sigma_i h] \ , \ (i = 1, 2, 3) \ , \qquad (66)$$

which can be used to show that $\gamma_{1/2} \in \text{Homeo}(\mathbb{R}^4, E_{1/2})$. It follows from (62),(63) and (65) that

$$\begin{aligned} \rho(z) &= \gamma_{1/2}\big(\rho_{\rm orb}(z), \rho_{\rm orb}(z)\vec{\mathcal{S}}(z)\big) \ , \\ \rho'(z) &= \gamma_{1/2}\big(\rho'_{\rm orb}(z), \rho'_{\rm orb}(z)\vec{\mathcal{S}}'(z)\big) \ , \end{aligned}$$

whence, by (60),(61) and (64),

$$\begin{aligned} l_{1/2}(A(z); \rho(z)) &= \gamma_{1/2}\big(\rho_{\rm orb}(z), \rho_{\rm orb}(z)A(z)\vec{\mathcal{S}}(z)\big) \\ &= \gamma_{1/2}\big(\rho'_{\rm orb}(j(z)), \rho'_{\rm orb}(j(z)) \\ &= \qquad \vec{\mathcal{S}}'(j(z))\big) \\ &= \rho'(j(z)) \ , \end{aligned} \qquad (67)$$

so that, by comparing with (32), the field dynamics of $(E_{1/2}, l_{1/2})$ indeed reproduces (63), i.e., $\tilde{\mathcal{P}}[E_{1/2}, l_{1/2}, j, A]$ $(\rho) = \rho'$. The density matrix function ρ is an invariant $(E_{1/2}, l_{1/2})$-field if $\vec{\mathcal{S}}$ is an invariant $(\mathbb{R}^3, l_v)$-field and if ρ_0 is an invariant $(\mathbb{R}, l_{\rm id})$-field, i.e., if $\rho_0(j(z)) = \rho_0(z)$ (recall the definition of $l_{\rm id}$ in Section 3.1).

To gain further insights we define the $SO(3)$-space $(\mathbb{R}^4, \overline{l_{1/2}})$, with

$$\overline{l_{1/2}}(r; S_0, \vec{S}) := (l_{\rm id}(r; S_0), l_{\rm v}(r; \vec{S})) = (S_0, r\vec{S}) \ . \qquad (68)$$

With these $SO(3)$-spaces $\gamma_{1/2}$ is an $SO(3)$-map from $(\mathbb{R}^4, \overline{l_{1/2}})$ to $(E_{1/2}, l_{1/2})$ and $\gamma_{1/2}^{-1}$ is an $SO(3)$-map from $(E_{1/2}, l_{1/2})$ to $(\mathbb{R}^4, \overline{l_{1/2}})$ and $l_{1/2}$ is given by

$$l_{1/2}(r; h) = \gamma_{1/2}(\overline{l_{1/2}}(r; \gamma_{1/2}^{-1}(h))) \ . \qquad (69)$$

One can use (69) to show that $l_{1/2}$ is an $SO(3)$-action. Note also that the functions in $\mathcal{C}(\mathbb{T}^d, E_{1/2})$ are in one-one correspondence with the functions in $\mathcal{C}(\mathbb{T}^d, \mathbb{R}^4)$ because $\gamma_{1/2}$ is a homeomorphism whence $f \in \mathcal{C}(\mathbb{T}^d, \mathbb{R}^4)$ can be mapped bijectively to $g \in \mathcal{C}(\mathbb{T}^d, E_{1/2})$, defined by $g := \gamma_{1/2} \circ f$. It is shown in Section 5.4 below by using the SMT and for any $f \in \mathcal{C}(\mathbb{T}^d, \mathbb{R}^4)$ that $\gamma_{1/2} \circ f$ is an invariant $(E_{1/2}, l_{1/2})$-field iff f is an invariant $(\mathbb{R}^4, \overline{l_{1/2}})$-field. Note also that if $f_0 \in \mathcal{C}(\mathbb{T}^d, \mathbb{R})$ and $\vec{f} \in \mathcal{C}(\mathbb{T}^d, \mathbb{R}^3)$ then $(f_0, \vec{f})$ is an invariant $(\mathbb{R}^4, \overline{l_{1/2}})$-field iff f_0 is an invariant $(\mathbb{R}, l_{\rm id})$-field and $\vec{f}$ is an invariant $(\mathbb{R}^3, l_{\rm v})$-field.

5.2. The Topological Transitivity Theorem (TTT)

Let j be topologically transitive and $A \in \mathcal{C}(\mathbb{T}^d, SO(3))$.

We first consider the $SO(3)$-space $(\mathbb{R}^3, l_{\rm v})$. By (38) and (56) and for $x = \vec{s} \in \mathbb{R}^3$,

$$\begin{aligned} E_{\vec{s}} &= l_{\rm v}(SO(3); \vec{s}) = \{r\vec{s} : r \in SO(3)\} \\ &= \{\vec{S} \in \mathbb{R}^3 : |\vec{S}| = |\vec{s}|\} \ , \end{aligned} \qquad (70)$$

which is a sphere of radius $|\vec{s}|$ centered at $(0, 0, 0)$. By the TTT (Theorem 1), if $\vec{\mathcal{N}}$ is an invariant $(\mathbb{R}^3, l_{\rm v})$-field of (j, A), i.e. an IPF, then $\vec{\mathcal{N}}$ is $E_{\vec{s}}$-valued for some $\vec{s} \in \mathbb{R}^3$ and thus $|\vec{\mathcal{N}}(z)|$ is indeed independent of z as it should be.

Next we briefly consider the $SO(3)$-space $(\mathbb{R}, l_{\rm id})$ where $l_{\rm id}$ is defined in Section 3.1. Since $l_{\rm id}$ represents the identity, every invariant $(\mathbb{R}, l_{\rm id})$-field f_0 is constant, i.e., $f_0(z) \equiv s_0$ is an invariant $(\mathbb{R}, l_{\rm id})$-field and is E_{s_0}-valued with $E_{s_0} = l_{\rm id}(SO(3); s_0) = \{s_0\}$ for some real constant s_0. Conversely if f is constant then f_0 is an invariant $(\mathbb{R}, l_{\rm id})$-field.

Now we consider the $SO(3)$-space $(\mathbb{R}^4, \overline{l_{1/2}})$. Recall that it is a construct invented to elegantly glue together the orbital and spin components necessary to build density matrix functions. Let $x = (s_0, \vec{s}) \in \mathbb{R}^4$, then

$$\begin{aligned} E_{(s_0,\vec{s})} &= \overline{l_{1/2}}(SO(3); s_0, \vec{s}) \\ &= \{(S_0, \vec{S}) \in \mathbb{R}^4 : S_0 = s_0, |\vec{S}| = |\vec{s}|\} \; . \end{aligned} \tag{71}$$

By the TTT, if $(\rho_{\rm orb}, \vec{\mathcal{N}})$ is an invariant $(\mathbb{R}^4, \overline{l_{1/2}})$-field of (j, A) then, it is $E_{(s_0,\vec{s})}$-valued for some real s_0 and some $\vec{s} \in \mathbb{R}^3$ and thus by (71), $\rho_{\rm orb}(z)$ and $|\vec{\mathcal{N}}(z)|$ are independent of z.

We finally consider the $SO(3)$-space $(E_{1/2}, l_{1/2})$. By (38) and for $x = \gamma_{1/2}(s_0, \vec{s}) \in E_{1/2}$,

$$\begin{aligned} E_x &= l_{1/2}(SO(3); \gamma_{1/2}(s_0, \vec{s})) \\ &= \gamma_{1/2}(\overline{l_{1/2}}(SO(3); s_0, \vec{s})) \\ &= \{\gamma_{1/2}(S_0, \vec{S}) : \\ &\qquad S_0 \in \mathbb{R}, \vec{S} \in \mathbb{R}^3, S_0 = s_0, |\vec{S}| = |\vec{s}|\} \; . \end{aligned} \tag{72}$$

By the TTT, if g is an invariant $(E_{1/2}, l_{1/2})$-field of (j, A) then an $x \in E_{1/2}$ exists such that g is E_x valued for some $x \in E_{1/2}$. Thus if we write) $g = \gamma_{1/2} \circ (f_0, \vec{f})$ then, by (72), the functions $f_0(z)$ and $|\vec{f}(z)|$ are independent of z.

5.3. The Normal Form Theorem (NFT)

Following Section 4.3 we now identify the isotropy groups for various important $SO(3)$-spaces in order to apply the NFT.

We first consider the $SO(3)$-space $(\mathbb{R}^3, l_{\rm v})$. Let $T \in \mathcal{C}(\mathbb{T}^d, SO(3))$. Then we pick $x = (0,0,1)^{\rm t}$ and define $\hat{f} \in \mathcal{C}(\mathbb{T}^d, \mathbb{R}^3)$ by

$$\hat{f}(z) := l_{\rm v}(T(z); (0,0,1)^{\rm t}) = T(z)(0,0,1)^{\rm t} \; . \tag{73}$$

With this, $\hat{f}(z)$ is the third column of $T(z)$. One can show, by (22),(40), and (73) and a small amount of linear algebra, that

$$\mathrm{Iso}(\mathbb{R}^3, l_{\rm v}; (0,0,1)^{\rm t}) = SO(2) \; , \tag{74}$$

where $SO(2)$ is defined by (22). The NFT (Theorem 2) states that, for all $z \in \mathbb{T}^d$,

$$T^{\rm t}(j(z))A(z)T(z) \in SO(2) \; , \tag{75}$$

iff $\hat{f}$ is an invariant $(\mathbb{R}^3, l_{\rm v})$-field, i.e. iff $\hat{f}$ is an ISF (since $|\hat{f}| = 1$).

Recalling the discussion in Section 4.3, we conclude that the NFT states that a $T \in \mathcal{C}(\mathbb{T}^d, SO(3))$ is an invariant $(\mathbb{R}^3, l_{\rm v})$-frame field at $(0,0,1)^{\rm t}$ iff the third column of T is an ISF. Note that an invariant $(\mathbb{R}^3, l_{\rm v})$-frame field at $(0,0,1)^{\rm t}$ is also just called an invariant frame field (IFF) whence invariant (E, l)-frame fields are generalized IFFs. Thus the NFT states in this case that a $T \in \mathcal{C}(\mathbb{T}^d, SO(3))$ is an IFF iff its third column is an ISF. For more details on IFFs see also [30].

We now generalize $x = (0,0,1)^{\rm t}$ to $x = \vec{s} \in \mathbb{R}^3$ where $\vec{s} \neq \vec{0}$ and we define $\vec{h} \in \mathcal{C}(\mathbb{T}^d, \mathbb{R}^3)$ by $\vec{h}(z) := l_{\rm v}(T(z); \vec{s}) = T(z)\vec{s}$ for some $T \in \mathcal{C}(\mathbb{T}^d, SO(3))$. One can show that $\mathrm{Iso}(\mathbb{R}^3, l_{\rm v}; \vec{s})$ is *conjugate* to $\mathrm{Iso}(\mathbb{R}^3, l_{\rm v}; (0,0,1)^{\rm t})$, in other words that $r_0 \in SO(3)$ exists such that

$$\mathrm{Iso}(\mathbb{R}^3, l_{\rm v}; \vec{s}) = r_0 SO(2) r_0^{\rm t} \; . \tag{76}$$

In complete analogy with the case of $x = (0,0,1)^{\rm t}$ it can be shown that $\vec{h}(z)$ is an IPF iff $T^{\rm t}(j(z))A(z)T(z) \in (r_0 SO(2) r_0^{\rm t})$.

In passing we note that because of $\mathrm{Iso}(\mathbb{R}^3, l_{\rm v}; \vec{0}) = SO(3)$ and since the trivial IPF $\vec{\mathcal{N}}_0(z) := \vec{0}$ is an invariant $(\mathbb{R}^3, l_{\rm v})$-field, the NFT is trivial in the case $x = \vec{0}$.

We now consider the $SO(3)$-space $(E, l) = (\mathbb{R}, l_{\rm id})$. Let $x = s_0 \in \mathbb{R}$. Then for any $T \in \mathcal{C}(\mathbb{T}^d, SO(3))$ we may define $f_0 \in \mathcal{C}(\mathbb{T}^d, \mathbb{R})$ by

$$f_0(z) := l_{\rm id}(T(z); s_0) = s_0 \; . \tag{77}$$

Moreover $\mathrm{Iso}(\mathbb{R}, l_{\rm id}; s_0) = SO(3)$ so that for any (j, A) the condition $(T^{\rm t} \circ j)\, A\, T \in \mathrm{Iso}(\mathbb{R}, l_{\rm id}; s_0)$ holds. Also any constant $f_0(z)$ is an invariant $(\mathbb{R}, l_{\rm id})$-field whence the NFT is trivial here too.

Next we consider the $SO(3)$-space $(\mathbb{R}^4, \overline{l_{1/2}})$ that glues together our (j, A)-dynamics on $(\mathbb{R}, l_{\rm id})$ and on $(\mathbb{R}^3, l_{\rm v})$. Let $T \in \mathcal{C}(\mathbb{T}^d, SO(3))$. We only consider the case where $x = (s_0, \vec{s})$ with $s_0 \in \mathbb{R}$, $\vec{0} \neq \vec{s} \in \mathbb{R}^3$ and define $f \in \mathcal{C}(\mathbb{T}^d, \mathbb{R}^4)$ by

$$f(z) := \overline{l_{1/2}}(T(z); (s_0, \vec{s})) = (s_0, T(z)\vec{s}) = (s_0, \vec{h}(z)) \; . \tag{78}$$

With $\mathrm{Iso}(\mathbb{R}, l_{\rm id}; s_0) = SO(3)$ and $\mathrm{Iso}(\mathbb{R}^3, l_{\rm v}; \vec{s}) = r_0 SO(2) r_0^{\rm t}$ for some $r_0 \in SO(3)$ one quickly finds that

$$\mathrm{Iso}(\mathbb{R}^4, \overline{l_{1/2}}; (s_0, \vec{s})) = \mathrm{Iso}(\mathbb{R}^3, l_{\rm v}; \vec{s}) = r_0 SO(2) r_0^{\rm t} \; . \tag{79}$$

By the discussion after (76) it is no surprise that the NFT states that f is an invariant $(\mathbb{R}^4, \overline{l_{1/2}})$-field, i.e., that $\vec{h}$ is an IPF, iff $(T^{\rm t} \circ j)\, A\, T \in (r_0 SO(2) r_0^{\rm t})$.

Finally we consider the $SO(3)$-space $(E_{1/2}, l_{1/2})$. Let $T \in \mathcal{C}(\mathbb{T}^d, SO(3))$. We only consider the case where $x = \gamma_{1/2}(s_0, \vec{s})$ with $s_0 \in \mathbb{R}$, $\vec{0} \neq \vec{s} \in \mathbb{R}^3$. We define $g \in \mathcal{C}(\mathbb{T}^d, E_{1/2})$ by

$$\begin{aligned} g(z) &:= l_{1/2}(T(z); \gamma_{1/2}(s_0, \vec{s})) \\ &= \gamma_{1/2}(\overline{l_{1/2}}(T(z); (s_0, \vec{s})) \\ &= \gamma_{1/2}(s_0, \vec{h}(z)) \; . \end{aligned} \tag{80}$$

Also one finds that

$$\begin{aligned} \mathrm{Iso}(E_{1/2}, l_{1/2}; \gamma_{1/2}(s_0, \vec{s})) &= \mathrm{Iso}(\mathbb{R}^4, \overline{l_{1/2}}; (s_0, \vec{s})) \\ &= r_0 SO(2) r_0^{\rm t} \; . \end{aligned} \tag{81}$$

Then in analogy with the case of the $SO(3)$-space $(\mathbb{R}^4, \overline{l_{1/2}})$ the NFT states that g is an invariant $(E_{1/2}, l_{1/2})$-field, i.e., $\vec{h}$ is an IPF, iff $(T^{\rm t} \circ j)\, A\, T \in (r_0 SO(2) r_0^{\rm t})$.

5.4. The $SO(3)$-Mapping Theorem (SMT)

Let $j \in \mathrm{Homeo}(\mathbb{T}^d)$ and $A \in \mathcal{C}(\mathbb{T}^d, SO(3))$. We consider three cases, in the first the $SO(3)$ map is a homeomorphism and in the latter two not. Furthermore, although the latter two are elementary applications of the SMT, they are important illustrations of the theory.

First we consider the case where $(E, l) = (\mathbb{R}^4, \overline{l_{1/2}})$ and $(\check{E}, \check{l}) = (E_{1/2}, l_{1/2})$. We recall from Section 5.1.2 that $\gamma_{1/2}$ is an $SO(3)$-map from $(\mathbb{R}^4, \overline{l_{1/2}})$ to $(E_{1/2}, l_{1/2})$. Let $f \in \mathcal{C}(\mathbb{T}^d, \mathbb{R}^4)$ and let us define $\check{f} \in \mathcal{C}(\mathbb{T}^d, E_{1/2})$ by $\check{f} := \gamma_{1/2} \circ f$. Since the $SO(3)$-map $\gamma_{1/2}$ is a homeomorphism and recalling the discussion after (46), the SMT (Theorem 3) implies that f is an invariant $(\mathbb{R}^4, \overline{l_{1/2}})$-field iff $\check{f}$ is an invariant $(E_{1/2}, l_{1/2})$-field of (j, A). Thus we can study the dynamics of the more important $(E_{1/2}, l_{1/2})$ space by studying the simpler $(\mathbb{R}^4, \overline{l_{1/2}})$ space.

We now consider the case where $(E, l) = (E_{1/2}, l_{1/2})$ and $(\check{E}, \check{l}) = (\mathbb{R}, l_{\rm id})$. We define the function $\gamma \in \mathcal{C}(E_{1/2}, \mathbb{R})$ by $\gamma(h) := \mathrm{Tr}[h]$ and we compute, by (65),

$$\gamma(\gamma_{1/2}(s_0, \vec{s})) = \mathrm{Tr}\left[\frac{1}{2}(s_0 I_{2\times 2} + \vec{\sigma} \cdot \vec{s})\right] = s_0 \,, \tag{82}$$

where $h = \gamma_{1/2}(s_0, \vec{s})$ and where we also used the fact that $0 = \mathrm{Tr}[\sigma_i]$. One can show by direct computation that γ is an $SO(3)$-map from $(E_{1/2}, l_{1/2})$ to $(\mathbb{R}, l_{\rm id})$. Let $g \in \mathcal{C}(\mathbb{T}^d, E_{1/2})$ and define $\check{g} \in \mathcal{C}(\mathbb{T}^d, \mathbb{R})$ by $\check{g}(z) := \gamma(g(z)) = \mathrm{Tr}[g(z)]$. The SMT states that if g is an invariant $(E_{1/2}, l_{1/2})$-field of (j, A) then $\mathrm{Tr}[g]$ is an invariant $(\mathbb{R}, l_{\rm id})$-field of (j, A), i.e. $\mathrm{Tr}[g \circ j] = \mathrm{Tr}[g]$. Since g and $\mathrm{Tr}[g]$ are continuous this implies that $\mathrm{Tr}[g]$ is constant for $(E_{1/2}, l_{1/2})$-invariant g if j is topologically transitive.

Lastly we consider the case where $(E, l) = (\mathbb{R}^3, l_{\rm v})$ and $(\check{E}, \check{l}) = (\mathbb{R}, l_{\rm id})$. We define the function $\gamma \in \mathcal{C}(E, \check{E}) = \mathcal{C}(\mathbb{R}^3, \mathbb{R})$ by $\gamma(\vec{s}) := |\vec{s}|$. One can show that γ is an $SO(3)$-map from $(\mathbb{R}^3, l_{\rm v})$ to $(\mathbb{R}, l_{\rm id})$. Let $\vec{f} \in \mathcal{C}(\mathbb{T}^d, \mathbb{R}^3)$ and let us define $\check{\vec{f}} \in \mathcal{C}(\mathbb{T}^d, \mathbb{R})$ by $\check{\vec{f}}(z) := \gamma(\vec{f}(z)) = |\vec{f}|(z)$. Recalling Section 4.4, the SMT states that if $\vec{f}$ is an invariant $(\mathbb{R}^3, l_{\rm v})$-field of (j, A), i.e., is an IPF then $|\vec{f}|$ is an invariant $(\mathbb{R}, l_{\rm id})$-field of (j, A).

5.5. The Invariant Reduction Theorem (IRT) and the Cross Section Theorem (CST)

Let $j \in \mathrm{Homeo}(\mathbb{T}^d)$ and $A \in \mathcal{C}(\mathbb{T}^d, SO(3))$.

We first consider the $SO(3)$-space $(E, l) = (\mathbb{R}, l_{\rm id})$. So let $x \in E = \mathbb{R}$. Let $f_0 \in \mathcal{C}(\mathbb{T}^d, \mathbb{R})$ take values only in E_x whence for every $z \in \mathbb{T}^d$, $f_0(z) = x$ so that, by (50) and (68),

$$\begin{aligned}
\Sigma_x[\mathbb{R}, l_{\rm id}, f_0] &= \{(z, r) \in (\mathbb{T}^d \times SO(3)) : \\
&\qquad l_{\rm id}(r; x) = f_0(z)\} \\
&= \{(z, r) \in (\mathbb{T}^d \times SO(3)) : x = x\} \\
&= \mathbb{T}^d \times SO(3) \,.
\end{aligned} \tag{83}$$

Recalling Section 4.5, the IRT states that f_0 is an invariant $(\mathbb{R}, l_{\rm id})$-field iff $\mathcal{P}[SO(3), l_{SO(3)}, j, A](\mathbb{T}^d \times SO(3)) = \mathbb{T}^d \times SO(3)$. This claim of the IRT is no surprise because $\mathbb{T}^d \times SO(3)$ is the domain of the bijection $\mathcal{P}[SO(3), l_{SO(3)}, j, A]$ whence $\mathcal{P}[SO(3), l_{SO(3)}, j, A](\mathbb{T}^d \times SO(3)) = \mathbb{T}^d \times SO(3)$ and because, by the discussion after (77), f_0 is an invariant $(\mathbb{R}, l_{\rm id})$-field. To discuss the CST we first recall from Section 4.5 and (83) that the function $p_x[\mathbb{R}, l_{\rm id}, f_0] \in \mathcal{C}(\mathbb{T}^d \times SO(3), \mathbb{T}^d)$ is defined by $p_x[\mathbb{R}, l_{\rm id}, f_0](z, r) := z$. Thus recalling Section 4.5 the CST states that $p_x[\mathbb{R}, l_{\rm id}, f_0]$ has a cross section iff a $T \in \mathcal{C}(\mathbb{T}^d, SO(3))$ exists such that $x = f_0(z) = l_{\rm id}(T(z); x)$, i.e. that $x = x$. This claim of the CST is no surprise because, for every $T \in \mathcal{C}(\mathbb{T}^d, SO(3))$, $x = x$ and because the function $\sigma \in \mathcal{C}(\mathbb{T}^d, \mathbb{T}^d \times SO(3))$ defined by $\sigma(z) := (z, I_{3\times 3})$ is a cross section of $p_x[\mathbb{R}, l_{\rm id}, f_0]$ since $p_x[\mathbb{R}, l_{\rm id}, f_0](\sigma(z)) = p_x[\mathbb{R}, l_{\rm id}, f_0](z, I_{3\times 3}) = z$.

We now consider the $SO(3)$-space $(E, l) = (\mathbb{R}^3, l_{\rm v})$. We first pick $x = (0, 0, 1)^t \in \mathbb{R}^3$. Let $\vec{f} \in \mathcal{C}(\mathbb{T}^d, \mathbb{R}^3)$ take values only in E_x whence for every $z \in \mathbb{T}^d$, by (50) and (56),

$$\begin{aligned}
\Sigma_x[\mathbb{R}^3, l_{\rm v}, \vec{f}] &= \{(z, r) \in (\mathbb{T}^d \times SO(3)) : \\
&\qquad r(0, 0, 1)^t = \vec{f}(z)\} \,.
\end{aligned} \tag{84}$$

Recalling Section 4.5 and using (84) and the fact that $|\vec{f}| = 1$, the IRT states that $\vec{f}$ is an ISF of (j, A) iff $\mathcal{P}[SO(3), l_{SO(3)}, j, A](\Sigma_x[\mathbb{R}^3, l_{\rm v}, \vec{f}]) = \Sigma_x[\mathbb{R}^3, l_{\rm v}, \vec{f}]$. To discuss the CST we first recall from Section 4.5 and (83) that the function $p_x[\mathbb{R}^3, l_{\rm v}, \vec{f}] \in \mathcal{C}(\Sigma_x[\mathbb{R}^3, l_{\rm v}, \vec{f}], \mathbb{T}^d)$ is defined by $p_x[\mathbb{R}^3, l_{\rm v}, \vec{f}](z, r) := z$. Thus recalling Section 4.5 and using (56) the CST states that $p_x[\mathbb{R}^3, l_{\rm v}, \vec{f}]$ has a cross section iff a $T \in \mathcal{C}(\mathbb{T}^d, SO(3))$ exists such that $\vec{f}(z) = l_{\rm v}(T(z); x)$. Since $x = (0, 0, 1)^t$ and by (56) we note that $\vec{f}(z) = l_{\rm v}(T(z); x)$ iff $\vec{f}(z) = T(z)(0, 0, 1)^t$, i.e., iff $\vec{f}$ is the third column of T. Thus recalling Section 4.5 the CST states that $p_x[\mathbb{R}^3, l_{\rm v}, \vec{f}]$ has a cross section iff a $T \in \mathcal{C}(\mathbb{T}^d, SO(3))$ exists such that $\vec{f}$ is the third column of T. One can show by some Topology [4] that if $d = 1$ then $p_x[\mathbb{R}^3, l_{\rm v}, \vec{f}]$ always has a cross section while, if $d \geq 2$, there exist $\vec{f}$ such that $p_x[\mathbb{R}^3, l_{\rm v}, \vec{f}]$ has a cross section and there exist $\vec{f}$ such that $p_x[\mathbb{R}^3, l_{\rm v}, \vec{f}]$ has no cross section.

The CST allows us to characterize IFFs in terms of cross sections by claiming that a (j, A) has an IFF iff it has an ISF $\vec{f}$ for which $p_x[\mathbb{R}^3, l_{\rm v}, \vec{f}]$ has a cross section. The claim is a special case of a claim made in Section 4.5 since an IFF is an invariant $(\mathbb{R}^3, l_{\rm v})$-frame field at $(0, 0, 1)^{\rm t}$. Thus the CST gives us new insights into the question of the existence of IFFs.

We can summarize the case $x = (0, 0, 1)^t$ by saying that the CST gives a topological criterion for the existence of an IFF and that the IRT gives a topological criterion for the existence of an ISF. The above discussion of the IRT and CST, which was made for the choice $(E, l) = (\mathbb{R}^3, l_{\rm v})$ and $x = (0, 0, 1)^t \in \mathbb{R}^3$ can be generalized to the case when $0 \neq x \in \mathbb{R}^3$. Recalling from Section 5.3 that in the general case $\mathrm{Iso}(\mathbb{R}^3, l_{\rm v}; x)$ is conjugate to $\mathrm{Iso}(\mathbb{R}^3, l_{\rm v}; (0, 0, 1)^t)$ the general case is just a minor modification of its subcase

$x = (0,0,1)^t$ and so we leave the remaining details to the reader as well as the trivial case when $x = (0,0,0)^t \in \mathbb{R}^3$. Moreover we must leave the $SO(3)$-spaces $(\mathbb{R}^4, \overline{l_{1/2}})$ and $(E_{1/2}, l_{1/2})$ to the reader too.

6. Applying the Main Theorems to Spin-1 Dynamics

6.1. The spin-1 dynamics in terms of (E, l)

In this section we introduce and study the $SO(3)$-spaces $(E_{\rm t}, l_{\rm t})$, and (E_1, l_1), which play the leading role for the spin-1 dynamics. We also introduce $(\mathbb{R}^4 \times E_{\rm t}, \overline{l_1})$ which along with $(\mathbb{R}, l_{\rm id})$ is useful for understanding (E_1, l_1).

6.1.1. $(E_{\rm t}, l_{\rm t})$ for particle and field dynamics

The spin-1 density matrix function on a single torus is a complex 3×3 hermitian matrix function whose values can be expressed in terms of the polarization vector and the polarization tensor. As explained in [12, 13, 14] the polarization tensor is a real 3×3 traceless matrix. Then guided by these properties we define the $SO(3)$-space $(E_{\rm t}, l_{\rm t})$ where

$$E_{\rm t} := \{M \in \mathbb{R}^{3\times 3} : M^t = M, \mathrm{Tr}[M] = 0\} \tag{85}$$

and the $SO(3)$-action $l_{\rm t} \in \mathcal{C}(SO(3) \times E_{\rm t}, E_{\rm t})$ by

$$l_{\rm t}(r; M) := rMr^t \ , \tag{86}$$

with $r \in SO(3), M \in E_{\rm t}$. The natural topology of $E_{\rm t}$ is its subspace topology as a subset of $\mathbb{R}^{3\times 3}$. Note that $l_{\rm t}(r; M)$ is a matrix similar to M for all r. The "spin tensor" M in (85) is the $E_{\rm t}$-spin variable that we need. It follows from (28) and (86) that, for every $j \in \mathrm{Homeo}(\mathbb{T}^d)$ and every $A \in \mathcal{C}(\mathbb{T}^d, SO(3))$,

$$\mathcal{P}[E_{\rm t}, l_{\rm t}, j, A](z, M) = (j(z),\ A(z)MA^t(z)) \ . \tag{87}$$

The field motion $\mathcal{M} \to \mathcal{M}'$ is given via (32) and (86) by

$$\mathcal{M}'(j(z)) = l_{\rm t}(A(z); \mathcal{M}(z)) = A(z)\mathcal{M}(z)A^t(z) \ , \tag{88}$$

thus the field map of (33) reads as

$$\begin{aligned}
\tilde{\mathcal{P}}[E_{\rm t}, l_{\rm t}, j, A](\mathcal{M}) \ &:= \ l_{\rm t}(A \circ j^{-1}; \mathcal{M} \circ j^{-1}) \\
&= \ (A \circ j^{-1})(\mathcal{M} \circ j^{-1}) \\
&\times \ (A^t \circ j^{-1}) \ .
\end{aligned} \tag{89}$$

The one-turn field-$E_{\rm t}$-spin map $\tilde{\mathcal{P}}[E_{\rm t}, l_{\rm t}, j, A]$ in (89) reproduces the discrete-time polarization-tensor field motion of [12, 13, 14].

Using (87), $\mathcal{M} \in \mathcal{C}(\mathbb{T}^d, E_{\rm t})$ is an invariant $(E_{\rm t}, l_{\rm t})$-field of (j, A), i.e. $\tilde{\mathcal{P}}[E_{\rm t}, l_{\rm t}, j, A](\mathcal{M}) = \mathcal{M}$, iff

$$\mathcal{M}(j(z)) = A(z)\mathcal{M}(z)A^t(z) \ . \tag{90}$$

We call every invariant $(E_{\rm t}, l_{\rm t})$-field an invariant polarization-tensor field (IPTF).

6.1.2. (E_1, l_1) for density matrix dynamics

We now define the $SO(3)$-space $(E, l) = (E_1, l_1)$ for which the one-turn field map $\tilde{\mathcal{P}}[E, l, j, A]$ in (33) reproduces the discrete-time motion of the spin-1 density matrix function. We also introduce the $SO(3)$-space $(\mathbb{R}^4 \times E_{\rm t}, \overline{l_1})$ which facilitates computations and the understanding of (E_1, l_1). In storage rings we need the semiclassical spin-1 density matrix function ρ. This arises from the semiclassical treatment of the Proca equation where ρ is a semiclassical approximation of the spin-1 Wigner function. Thus the particle-variables are purely classical whereas the spin-degrees-of-freedom are treated fully quantum mechanically. For simplicity we suppress the term "semiclassical" when we discuss ρ.

As in the fully quantum-mechanical case the values of the spin-1 density matrix function ρ are complex hermitian 3×3 matrices and following [12, 13, 14] ρ reads as

$$\begin{aligned}
\rho(z) \ &= \ \rho_{\rm orb}(z)\frac{1}{3}\Big(I_{3\times 3} + \frac{3}{2}\vec{\Sigma}\cdot\vec{\mathcal{S}}(z) \\
&+ \ \sqrt{\frac{3}{2}}\sum_{i,k=1}^{3} \mathcal{M}_{ik}(z)(\Sigma_i\Sigma_k + \Sigma_k\Sigma_i)\Big) \ ,
\end{aligned} \tag{91}$$

where again we have suppressed the implicit dependence on J and θ. Here $I_{3\times 3}$ is the 3×3 unit matrix and $\Sigma_1, \Sigma_2, \Sigma_3$ are the matrices:

$$\Sigma_1 := \sqrt{\frac{1}{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix} \ , \ \Sigma_2 := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \ ,$$
$$\Sigma_3 := \sqrt{\frac{1}{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \ ,$$

with the same axis asignments as in [12, 13, 14]. As before $\rho_{\rm orb}$ is the coninuous orbital phase space density of the bunch and $\vec{\mathcal{S}}$ is the continuous polarization field describing the spin vector state of the bunch where $|\vec{\mathcal{S}}| \le 1$. The $\mathcal{M}_{ik}$ denote the (ik)-th matrix elements of the continuous tensor-polarization field $\mathcal{M}$ describing the spin tensor state of the bunch where $\mathrm{Tr}[\mathcal{M}^2] \le 1$. The square of the Euclidean norm (also called the Frobenius norm [31]) of any $M \in E_{\rm t}$ reads as $\sum_{i,k=1}^3 \ M_{ik}M_{ik} = \mathrm{Tr}[M^2]$ where we used the fact that M is symmetric. Thus the Euclidean norm of $\vec{\mathcal{S}}(z)$ and the Frobenius norm of $\mathcal{M}(z)$ are ≤ 1. We abbreviate the set of hermitian 3×3 matrices by E_1, i.e.,

$$E_1 := \{R \in \mathbb{C}^{3\times 3} : R^\dagger = R\} \ , \tag{92}$$

and we equip E_1 with the subspace topology from $\mathbb{C}^{3\times 3}$. Since $\rho_{\rm orb}, \vec{\mathcal{S}}$ and $\mathcal{M}$ are continuous, $\rho \in \mathcal{C}(\mathbb{T}^d, E_1)$. The observables and their expectation values with respect to ρ are defined in analogy with the spin-1/2 case. For the latter see the discussion after (15).

From Section 5.1.1 we recall that the orbital dynamics is given by the one-turn maps

$$\rho_{\rm orb} \to \rho'_{\rm orb} \ \text{ via } \ \rho'_{\rm orb}(z) = \rho_{\rm orb}(j^{-1}(z)) \ . \tag{93}$$

and the polarization field dynamics is given by

$$\vec{\mathcal{S}} \to \vec{\mathcal{S}}' \;\text{ via }\; \vec{\mathcal{S}}'(z) = A(j^{-1}(z))\vec{\mathcal{S}}(j^{-1}(z)) \; . \qquad (94)$$

From Section 6.1.1 we recall that the polarization-tensor field dynamics is given by the one-turn map $\mathcal{M} \to \mathcal{M}'$ via

$$\mathcal{M}'(z) = A(j^{-1}(z))\mathcal{M}(j^{-1}(z))A^t(j^{-1}(z)) \; . \qquad (95)$$

The spin-1 density matrix function ρ in (91) is mapped in one turn to

$$\begin{aligned} \rho'(z) &= \rho'_{\rm orb}(z)\,\frac{1}{3}\Big(I_{3\times 3} + \frac{3}{2}\vec{\Sigma}\cdot\vec{\mathcal{S}}'(z) \\ &+ \sqrt{\frac{3}{2}}\sum_{i,k=1}^{3}\mathcal{M}'_{ik}(z)(\Sigma_i\Sigma_k + \Sigma_k\Sigma_i)\Big) \; . \end{aligned} \qquad (96)$$

The condition $|\vec{\mathcal{S}}| \le 1$ is consistent with the dynamics since it implies, by (94), that $|\vec{\mathcal{S}}'(z)| = |A(j^{-1}(z))\vec{\mathcal{S}}(j^{-1}(z))|$ $= |\vec{\mathcal{S}}(j^{-1}(z))| \le 1$. Moreover the condition $\mathrm{Tr}[\mathcal{M}^2] \le 1$ is consistent with the dynamics since it implies, by (95), that $\mathrm{Tr}[(\mathcal{M}'(z))^2] = \mathrm{Tr}[A(j^{-1}(z))\mathcal{M}^2(j^{-1}(z))A^t(j^{-1}(z))]$ $= \mathrm{Tr}[\mathcal{M}^2(j^{-1}(z))] \le 1$. The dynamics of ρ given by (96) is the same as that of the field dynamics given by the $SO(3)$-space (E, l) where $E = E_1$ and $l = l_1$ and where $l_1 \in \mathbb{C}(SO(3)\times E_1, E_1)$ is defined by

$$l_1(r; \gamma_1(S_0, \vec{S}, M)) := \gamma_1(S_0, r\vec{S}, rMr^t) \; , \qquad (97)$$

where $S_0 \in \mathbb{R}, \vec{S} \in \mathbb{R}^3, M \in E_{\rm t}$ and where $\gamma_1 \in \mathrm{Homeo}(\mathbb{R}^4 \times E_{\rm t}, E_1)$ is defined by

$$\begin{aligned} \gamma_1(S_0, \vec{S}, M) &= \frac{1}{3}\Big(S_0 I_{3\times 3} + \frac{3}{2}\vec{\Sigma}\cdot\vec{S} + \sqrt{\frac{3}{2}} \\ &\times \sum_{i,k=1}^{3} M_{ik}(\Sigma_i\Sigma_k + \Sigma_k\Sigma_i)\Big) \; . \end{aligned} \qquad (98)$$

It follows from (98) that if $h \in E_1$ then $\gamma_1^{-1}(h) = (S_0, \vec{S}, M)$ where $S_0 \in \mathbb{R}, \vec{S} \in \mathbb{R}^3$ and $M \in E_{\rm t}$ are defined by

$$S_0 := \mathrm{Tr}[h] \; , \; S_i := \mathrm{Tr}[\Sigma_i h] \; , \qquad (99)$$

$$\begin{aligned} M_{ik} &:= -\mathrm{Tr}[h]\sqrt{\frac{2}{3}}\delta_{ik} \\ &+ \sqrt{\frac{3}{8}}\mathrm{Tr}[(\Sigma_i\Sigma_k + \Sigma_k\Sigma_i)h] \; , \\ &(i, k = 1, 2, 3) \end{aligned} \qquad (100)$$

which can be used to show that $\gamma_1 \in \mathrm{Homeo}(\mathbb{R}^4 \times E_{\rm t}, E_1)$. It follows from (91),(96) and (98) that

$$\begin{aligned} \rho(z) &= \gamma_1\big(\rho_{\rm orb}(z), \rho_{\rm orb}(z)\vec{\mathcal{S}}(z), \rho_{\rm orb}(z)\mathcal{M}(z)\big) \; , \\ \rho'(z) &= \gamma_1\big(\rho'_{\rm orb}(z), \rho'_{\rm orb}(z)\vec{\mathcal{S}}'(z), \rho'_{\rm orb}(z)\mathcal{M}'(z)\big) \; , \end{aligned}$$

whence, by (93),(94),(95) and (97),

$$\begin{aligned} l_1(A(z); \rho(z)) &= \gamma_1\Big(\rho_{\rm orb}(z), \rho_{\rm orb}(z)A(z)\vec{\mathcal{S}}(z), \\ &\qquad \rho_{\rm orb}(z)A(z)\mathcal{M}(z)A^t(z)\Big) \\ &= \gamma_1\Big(\rho'_{\rm orb}(j(z)), \rho'_{\rm orb}(j(z))\vec{\mathcal{S}}'(j(z)), \\ &\qquad \rho'_{\rm orb}(j(z))\mathcal{M}'(j(z))\Big) \\ &= \rho'(j(z)) \; , \end{aligned} \qquad (101)$$

so that, by comparing with (32), the field dynamics of (E_1, l_1) indeed reproduces (96), i.e., $\tilde{\mathcal{P}}[E_1, l_1, j, A](\rho) = \rho'$. Clearly ρ is an invariant (E_1, l_1)-field if the following hold: $\rho_{\rm orb}$ is an invariant $(\mathbb{R}, l_{\rm id})$-field, $\vec{\mathcal{S}}$ is an invariant $(\mathbb{R}^3, l_v)$-field, and $\mathcal{M} \in \mathcal{C}(\mathbb{T}^d, E_{\rm t})$ is an invariant $(E_{\rm t}, l_{\rm t})$-field. Note that in our Stern-Gerlach-free model, the $SO(3)$-actions $l_{\rm v}$ and $l_{\rm t}$ do not impact the orbital densities.

To gain further insights we define the $SO(3)$-space $(\mathbb{R}^4 \times E_{\rm t}, \overline{l_1})$, with

$$\begin{aligned} \overline{l_1}(r; S_0, \vec{S}, M) &:= \big(l_{\rm id}(r; S_0), l_{\rm v}(r; \vec{S}), l_{\rm t}(r; M)\big) \\ &= \big(S_0, r\vec{S}, rMr^t\big) \; . \end{aligned} \qquad (102)$$

With this $SO(3)$-space, γ_1 is an $SO(3)$-map from $(\mathbb{R}^4 \times E_{\rm t}, \overline{l_1})$ to (E_1, l_1) and γ_1^{-1} is an $SO(3)$-map from (E_1, l_1) to $(\mathbb{R}^4 \times E_{\rm t}, \overline{l_1})$ and l_1 is given by

$$l_1(r; h) = \gamma_1(\overline{l_1}(r; \gamma_1^{-1}(h))) \; . \qquad (103)$$

One can use (103) to show that l_1 is an $SO(3)$-action. Note that the functions in $\mathcal{C}(\mathbb{T}^d, E_1)$ are in one-one correspondence with the functions in $\mathcal{C}(\mathbb{T}^d, \mathbb{R}^4 \times E_{\rm t})$ because $f \in \mathcal{C}(\mathbb{T}^d, \mathbb{R}^4 \times E_{\rm t})$ can be mapped bijectively to $g \in \mathcal{C}(\mathbb{T}^d, E_1)$, defined by $g := \gamma_1 \circ f$ since γ_1 is a homeomorphism. It is shown in Section 6.4 below by using the SMT and for any $f \in \mathcal{C}(\mathbb{T}^d, \mathbb{R}^4 \times E_{\rm t})$ that $\gamma_1 \circ f$ is an invariant (E_1, l_1)-field iff f is an invariant $(\mathbb{R}^4 \times E_{\rm t}, \overline{l_1})$-field. Note also that if $f_0 \in \mathcal{C}(\mathbb{T}^d, \mathbb{R}), \vec{f} \in \mathcal{C}(\mathbb{T}^d, \mathbb{R}^3)$ and $\mathcal{M} \in \mathcal{C}(\mathbb{T}^d, E_{\rm t})$ then $(f_0, \vec{f}, \mathcal{M})$ is an invariant $(\mathbb{R}^4 \times E_{\rm t}, \overline{l_1})$-field iff the following hold: f_0 is an invariant $(\mathbb{R}, l_{\rm id})$-field, $\vec{f}$ is an invariant $(\mathbb{R}^3, l_{\rm v})$-field, and $\mathcal{M}$ is an invariant $(E_{\rm t}, l_{\rm t})$-field.

6.2. The Topological Transitivity Theorem (TTT)

Let j be topologically transitive and $A \in \mathcal{C}(\mathbb{T}^d, SO(3))$. Although the results from some of the examples are obvious in the context of standard spin dynamics, the examples still serve to illustrate our methods.

We first consider the $SO(3)$-space $(\mathbb{R}^3, l_{\rm t})$. With (38), (86) and for $x = m \in E_{\rm t}$,

$$\begin{aligned} E_m &= l_{\rm t}(SO(3); m) = \{rmr^t : r \in SO(3)\} \\ &= \{M \in E_{\rm t} : \det(M) = \det(m), \\ &\qquad \mathrm{Tr}[M^2] = \mathrm{Tr}[m^2]\} \; . \end{aligned} \qquad (104)$$

If $M \in E_m$ then, by (86), the matrices m and M are similar whence they have identical characteristic polynomials which can be used to prove (104). Clearly all matrices in E_m have the same eigenvalues and thus have the same number of distinct eigenvalues as m. In fact the characteristic polynomial of any $m \in E_{\rm t}$ is the function $t \mapsto t^3 - (t/2){\rm Tr}[m^2] - \det(m)$. Moreover, by (86) and since all matrices in $E_{\rm t}$ are real and symmetric, the eigenvalues are real and E_m contains at least one diagonal matrix [32].

By the TTT (Theorem 1), if $\mathcal{M}$ is an invariant $(E_{\rm t}, l_{\rm t})$-field of (j, A), i.e. an IPTF, then $\mathcal{M}$ is E_m-valued for some $m \in E_{\rm t}$ and thus $\det(\mathcal{M})$ and ${\rm Tr}[\mathcal{M}^2]$ are independent of z. Recalling from Section 6.1.2 that ${\rm Tr}[m^2]$ is the square of the Frobenius norm of $m \in E_{\rm t}$ we note, by (104), that all $M \in E_m$ have the same Frobenius norm whence, by the TTT, the Frobenius norm of an invariant $(E_{\rm t}, l_{\rm t})$-field is independent of z.

We next consider the $SO(3)$-space $(\mathbb{R}^4 \times E_{\rm t}, \overline{l_1})$. Recall that it is a construct invented to elegantly glue together the orbital and spin components necessary to build density matrix functions. Let $x = (s_0, \vec{s}, m) \in \mathbb{R}^4 \times E_{\rm t}$, then

$$\begin{aligned} E_{(s_0,\vec{s},m)} &= \overline{l_1}(SO(3); s_0, \vec{s}, m) \\ &= \big\{(S_0, \vec{S}, M) \in \mathbb{R}^4 \times E_{\rm t} : \\ &\quad S_0 = s_0,\ |\vec{S}| = |\vec{s}|, \\ &\quad \det(M) = \det(m), \\ &\quad {\rm Tr}[M^2] = {\rm Tr}[m^2]\big\}\ . \end{aligned} \tag{105}$$

By the TTT, if $(f_0, \vec{\mathcal{N}}, \mathcal{M})$ is an invariant $(\mathbb{R}^4 \times E_{\rm t}, \overline{l_1})$-field of (j, A) then it is $E_{(s_0,\vec{s},m)}$-valued for some $s_0 \in \mathbb{R}, \vec{s} \in \mathbb{R}^3$ and $m \in E_{\rm t}$ and thus, by (104), $f_0(z)$ and $|\vec{\mathcal{N}}(z)|$ as well as $\det(\mathcal{M}(z))$ and ${\rm Tr}[\mathcal{M}^2(z)]$ are independent of z, in particular the norms of $\vec{\mathcal{N}}(z)$ and of $\mathcal{M}(z)$ are independent of z.

We finally consider the $SO(3)$-space (E_1, l_1). By (38) and (105) and for $x = \gamma_1(s_0, \vec{s}, m) \in E_1$,

$$\begin{aligned} E_x &= l_1(SO(3); \gamma_1(s_0, \vec{s}, m)) \\ &= \gamma_1(\overline{l_1}(SO(3); s_0, \vec{s}, m)) \\ &= \big\{\gamma_1(S_0, \vec{S}, M) : \\ &\quad S_0 \in \mathbb{R},\ \vec{S} \in \mathbb{R}^3,\ M \in E_{\rm t}, \\ &\quad S_0 = s_0, |\vec{S}| = |\vec{s}|, \det(M) = \det(m), \\ &\quad {\rm Tr}[M^2] = {\rm Tr}[m^2]\big\}\ . \end{aligned} \tag{106}$$

By the TTT, if g is an invariant (E_1, l_1)-field of (j, A) then an $x \in E_1$ exists such that g is E_x valued for some $x \in E_1$. Thus if we write $g = \gamma_1 \circ (f_0, \vec{f}, \mathcal{M})$ then, by (106), the functions $f_0(z)$ and $|\vec{\mathcal{N}}(z)|$ as well as $\det(\mathcal{M}(z))$ and ${\rm Tr}[\mathcal{M}^2(z)]$ are independent of z.

6.3. *The Normal Form Theorem (NFT)*

Let $T \in \mathcal{C}(\mathbb{T}^d, SO(3))$. Following Section 4.3 we now identify the isotropy groups for various important $SO(3)$-spaces in order to apply the NFT.

6.3.1. $(E_{\rm t}, l_{\rm t})$

Case 1 ($x \in E_{\rm t}$ has two distinct eigenvalues)

We first consider the subcase of Case 1 where $x = x_0$ is the diagonal matrix

$$x_0 = \begin{pmatrix} y & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & -2y \end{pmatrix}, \ (0 \neq y \in \mathbb{R}) \tag{107}$$

and define $\mathcal{M}_0 \in \mathcal{C}(\mathbb{T}^d, E_{\rm t})$ by

$$\mathcal{M}_0(z) := l_{\rm t}(T(z); x_0) = y(I_{3\times3} - 3\vec{f}_0(z)\vec{f}_0^{\,\rm t}(z))\ , \tag{108}$$

where in the second equality we used (86) and where the function $\vec{f}_0 \in \mathcal{C}(\mathbb{T}^d, \mathbb{R}^3)$ is defined by

$$\vec{f}_0(z) := T(z)(0, 0, 1)^t\ . \tag{109}$$

Note that x_0 has two distinct eigenvalues: y and $-2y$. One can show, by (40) and (86) and a small amount of linear algebra, that

$${\rm Iso}(E_{\rm t}, l_{\rm t}; x_0) = SO(2) \rtimes \mathbb{Z}_2\ , \tag{110}$$

where

$$SO(2) \rtimes \mathbb{Z}_2 := \{rr' : r \in \mathbb{Z}_2, r' \in SO(2)\}\ , \tag{111}$$

and where $\mathbb{Z}_2$ consists of the two elements

$$I_{3\times3}, \text{ and } \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}\ .$$

The NFT (Theorem 2) states that, for all $z \in \mathbb{T}^d$,

$$T^{\rm t}(j(z))A(z)T(z) \in (SO(2) \rtimes \mathbb{Z}_2)\ , \tag{112}$$

iff $\mathcal{M}_0$ is an invariant $(E_{\rm t}, l_{\rm t})$-field, i.e. iff $\mathcal{M}_0$ is an IPTF.

Note every IFF is an invariant $(E_{\rm t}, l_{\rm t})$-frame field at x_0. In fact if T is an IFF then, by the discussion after (75), T satisfies (75), i.e., $T^{\rm t}(j(z))A(z)T(z) \in SO(2)$ whence, and since $SO(2) \subset (SO(2) \rtimes \mathbb{Z}_2)$, we get (112). One can show that the converse does not hold. We see that if T is an IFF then, by the NFT, the function $\mathcal{M}_0$ in (108) is an invariant $(E_{\rm t}, l_{\rm t})$-field. We will reconsider (108) in Section 6.5. We now consider Case 1, i.e., when x has two distinct eigenvalues, in full generality. Recalling from the discussion after (104) that an $r \in SO(3)$ exists such that the matrix $r^t x r$ is diagonal, one can show that a unique $0 \neq y \in \mathbb{R}$ and an $r_0 \in SO(3)$ exist such that $x = r_0 x_0 r_0^{\rm t}$. We thus define $\mathcal{M} \in \mathcal{C}(\mathbb{T}^d, E_{\rm t})$ by

$$\mathcal{M}(z) := l_{\rm t}(T(z); x) = y(I_{3\times3} - 3\vec{f}(z)\vec{f}^{\,\rm t}(z))\ , \tag{113}$$

where in the second equality we used (86) and where the function $\vec{f} \in \mathcal{C}(\mathbb{T}^d, \mathbb{R}^3)$ is defined by

$$\vec{f}(z) := T(z)r_0(0, 0, 1)^t\ . \tag{114}$$

One can show, by (24),(25),(40) and (110), that

$$\begin{aligned} {\rm Iso}(E_{\rm t}, l_{\rm t}; x) &= r_0 {\rm Iso}(E_{\rm t}, l_{\rm t}; x_0) r_0^t \\ &= r_0 (SO(2) \rtimes \mathbb{Z}_2) r_0^{\rm t}\ , \end{aligned} \tag{115}$$

i.e., $\mathrm{Iso}(E_{\rm t}, l_{\rm t}; x)$ is conjugate to $\mathrm{Iso}(E_{\rm t}, l_{\rm t}; x_0)$. Thus the general Case 1 is just a minor modification of its subcase $x = x_0$ which was discussed above and so we leave the remaining details to the reader.

Case 2 ($x \in E_{\rm t}$ has three distinct eigenvalues)

We first consider the subcase of Case 2 where $x = x_0$ with

$$x_0 = \begin{pmatrix} y_1 & 0 & 0 \\ 0 & y_2 & 0 \\ 0 & 0 & -y_1 - y_2 \end{pmatrix} , \tag{116}$$

and where $y_1 \in (0, \infty)$, $y_2 \in (-y_1/2, y_1)$ and where we define $\mathcal{M}_0 \in \mathcal{C}(\mathbb{T}^d, E_{\rm t})$ by

$$\begin{aligned} \mathcal{M}_0(z) &:= l_{\rm t}(T(z); x_0) \\ &= y_1 I_{3\times 3} - (2y_1 + y_2)\ \vec{f}_0(z)\vec{f}_0^t(z) \\ &\quad + (\ y_2 - y_1\ )\ \vec{g}_0(z)\vec{g}_0^t(z) , \end{aligned} \tag{117}$$

where in the second equality we used (86) and where the functions $\vec{f}_0, \vec{g}_0 \in \mathcal{C}(\mathbb{T}^d, \mathbb{R}^3)$ are defined by (109) and by

$$\vec{g}_0(z) := T(z)(0, 1, 0)^t . \tag{118}$$

Note that x_0 has three eigenvalues: $y_1, y_2, -y_1 - y_2$. One can show, by (40) and (86) and some linear algebra, that

$$\mathrm{Iso}(E_{\rm t}, l_{\rm t}; x_0) = SO_{diag}(3) , \tag{119}$$

where $SO_{diag}(3)$ is the set of the four diagonal matrices in $SO(3)$. The NFT (Theorem 2) then states that, for all $z \in \mathbb{T}^d$,

$$T^{\rm t}(j(z))A(z)T(z) \in SO_{diag}(3) , \tag{120}$$

iff $\mathcal{M}_0$ is an invariant $(E_{\rm t}, l_{\rm t})$-field, i.e. iff $\mathcal{M}_0$ is an IPTF. We now consider Case 2, i.e., when x has three distinct eigenvalues, in full generality. Recalling from the discussion after (104) that an $r \in SO(3)$ exists such that the matrix $r^t x r$ is diagonal, one can show that a unique $y_1 \in (0, \infty)$, a unique $y_2 \in (-y_1/2, y_1)$ and an $r_0 \in SO(3)$ exist such that $x = r_0 x_0 r_0^t$. We thus define $\mathcal{M} \in \mathcal{C}(\mathbb{T}^d, E_{\rm t})$ by

$$\begin{aligned} \mathcal{M}(z) &:= l_{\rm t}(T(z); x) \\ &= y_1 I_{3\times 3} - (2y_1 + y_2)\ \vec{f}(z)\vec{f}^t(z) \\ &\quad + (\ y_2 - y_1\ )\ \vec{g}(z)\vec{g}^t(z) , \end{aligned} \tag{121}$$

where in the second equality we used (86) and where the functions $\vec{f}, \vec{g} \in \mathcal{C}(\mathbb{T}^d, \mathbb{R}^3)$ are defined by (114) and by

$$\vec{g}(z) := T(z)r_0(0, 1, 0)^t . \tag{122}$$

One can show, by (24),(25),(40) and (119), that

$$\begin{aligned} \mathrm{Iso}(E_{\rm t}, l_{\rm t}; x) &= r_0 \mathrm{Iso}(E_{\rm t}, l_{\rm t}; x_0) r_0^t \\ &= r_0 SO_{diag}(3) r_0^{\rm t} , \end{aligned} \tag{123}$$

i.e., $\mathrm{Iso}(E_{\rm t}, l_{\rm t}; x)$ is conjugate to $\mathrm{Iso}(E_{\rm t}, l_{\rm t}; x_0)$. Thus the general Case 2 is just a minor modification of its subcase $x = x_0$ which was discussed above and so we again leave the remaining details to the reader.

Case 3 ($x \in E_{\rm t}$ has only one eigenvalue)

Recalling from the discussion after (104) that an $r \in SO(3)$ exists such that the matrix $r^t x r$ is diagonal, one can show that $x = 0$ whence $\mathrm{Iso}(E_{\rm t}, l_{\rm t}; x) = SO(3)$ so that the trivial IPTF $\mathcal{M}(z) := 0$ is indeed an invariant $(E_{\rm t}, l_{\rm t})$-field for arbitrary (j, A).

6.3.2. $(\mathbb{R}^4 \times E_{\rm t}, \overline{l_1}), (E_1, l_1)$

We leave the $SO(3)$-spaces $(\mathbb{R}^4 \times E_{\rm t}, \overline{l_1})$ and (E_1, l_1) to the reader. In fact they can be handled in terms of the $SO(3)$-spaces $(\mathbb{R}, l_{\rm id}), (\mathbb{R}^3, l_{\rm v})$ and $(E_{\rm t}, l_{\rm t})$ in the same way as we handled the $SO(3)$-spaces $(\mathbb{R}^4, \overline{l_{1/2}})$ and $(E_{1/2}, l_{1/2})$ in terms of the $SO(3)$-spaces $(\mathbb{R}, l_{\rm id}), (\mathbb{R}^3, l_{\rm v})$ in Section 5.3. Note, by (40),(97) and (102) and for arbitrary $s_0 \in \mathbb{R}, \vec{s} \in \mathbb{R}^3$ and $m \in E_{\rm t}$,

$$\begin{aligned} &\mathrm{Iso}(E_1, l_1; \gamma_1(s_0, \vec{s}, m)) = \\ &\mathrm{Iso}(\mathbb{R}^4 \times E_{\rm t}, \overline{l_1}; s_0, \vec{s}, m) = \\ &\mathrm{Iso}(\mathbb{R}^3, l_{\rm v}; \vec{s}) \cap \mathrm{Iso}(E_{\rm t}, l_{\rm t}; m) , \end{aligned} \tag{124}$$

where $\mathrm{Iso}(\mathbb{R}^3, l_{\rm v}; \vec{s})$ and $\mathrm{Iso}(E_{\rm t}, l_{\rm t}; m)$ were identified in Sections 5.3 and 6.3. Thus $\mathrm{Iso}(\mathbb{R}^4 \times E_{\rm t}, \overline{l_1}; s_0, \vec{s}, m)$ and $\mathrm{Iso}(E_1, l_1; h)$ can be computed by using (124) and some linear algebra.

6.4. The $SO(3)$-Mapping Theorem (SMT)

Let $j \in \mathrm{Homeo}(\mathbb{T}^d)$ and $A \in \mathcal{C}(\mathbb{T}^d, SO(3))$. We will illustrate the SMT with four examples of $SO(3)$-maps between specific (E, l) and $(\check{E}, \check{l})$ spaces. In the first the $SO(3)$ map is a homeomorphism while in the second, third and fourth examples it is not. Although the second and third examples are elementary applications of the SMT, they are important illustrations of the theory. The first three examples are basically the same as those of Section 5.4 with spin-1/2 replaced by spin-1. The fourth example in an interesting case involving both the SMT via the DT.

In the first case we consider $(E, l) = (\mathbb{R}^4 \times E_{\rm t}, \overline{l_1})$ and $(\check{E}, \check{l}) = (E_1, l_1)$. We recall from Section 6.1.2 that γ_1 is an $SO(3)$-map from $(\mathbb{R}^4 \times E_{\rm t}, \overline{l_1})$ to (E_1, l_1). Let $f \in \mathcal{C}(\mathbb{T}^d, \mathbb{R}^4 \times E_{\rm t})$ and define $\check{f} \in \mathcal{C}(\mathbb{T}^d, E_1)$ by $\check{f} := \gamma_1 \circ f$. Since the $SO(3)$-map γ_1 is a homeomorphism the (j, A) dynamics of the two $SO(3)$-spaces are equivalent. Thus we can study the dynamics of the more important (E_1, l_1) space by studying the simpler $(\mathbb{R}^4 \times E_{\rm t}, \overline{l_1})$ space. Furthermore, recalling Section 4.4, the SMT (Theorem 3) implies that f is an invariant $(\mathbb{R}^4 \times E_{\rm t}, \overline{l_1})$-field of (j, A) iff $\check{f}$ is an invariant (E_1, l_1)-field of (j, A).

We now consider the case where $(E, l) = (E_1, l_1)$ and $(\check{E}, \check{l}) = (\mathbb{R}, l_{\rm id})$. We define the function $\gamma \in \mathcal{C}(E_1, \mathbb{R})$ by $\gamma(h) := \mathrm{Tr}[h]$ and we compute, by (99)

$$\gamma(\gamma_1(s_0, \vec{s}, m)) = s_0 , \tag{125}$$

where $h = \gamma_1(s_0, \vec{s}, m)$. One can show by direct computation that γ is an $SO(3)$-map from (E_1, l_1) to $(\mathbb{R}, l_{\rm id})$. Let $g \in \mathcal{C}(\mathbb{T}^d, E_1)$ and let us define $\check{g} \in \mathcal{C}(\mathbb{T}^d, \mathbb{R})$ by $\check{g}(z) := \gamma(g(z)) = \mathrm{Tr}[g(z)]$. Recalling Section 4.4 the SMT states that if g is an invariant (E_1, l_1)-field of (j, A) then $\mathrm{Tr}[g]$

is an invariant $(\mathbb{R}, l_{\rm id})$-field of (j, A), i.e. $\mathrm{Tr}[g \circ j] = \mathrm{Tr}[g]$. Since g and $\mathrm{Tr}[g]$ are continuous this implies that $\mathrm{Tr}[g]$ is constant for (E_1, l_1)-invariant g if j is topologically transitive.

Next we consider the case where $(E, l) = (E_{\rm t}, l_{\rm t})$ and $(\check{E}, \check{l}) = (\mathbb{R}, l_{\rm id})$. We define the function $\gamma \in \mathcal{C}(E, \check{E}) = \mathcal{C}(E_{\rm t}, \mathbb{R})$ by $\gamma(m) := \sqrt{\mathrm{Tr}[m^2]}$. One can show that γ is an $SO(3)$-map from $(E_{\rm t}, l_{\rm t})$ to $(\mathbb{R}, l_{\rm id})$. Let $\mathcal{M} \in \mathcal{C}(\mathbb{T}^d, E_{\rm t})$ and define $\check{\mathcal{M}} \in \mathcal{C}(\mathbb{T}^d, \mathbb{R})$ by

$$\check{\mathcal{M}}(z) := \gamma(\mathcal{M}(z)) = \sqrt{\mathrm{Tr}[\mathcal{M}^2(z)]} \,. \tag{126}$$

The SMT states that if $\mathcal{M}$ is an invariant $(E_{\rm t}, l_{\rm t})$-field of (j, A), i.e., is an IPTF then $\sqrt{\mathrm{Tr}[\mathcal{M}^2]}$ is an invariant $(\mathbb{R}, l_{\rm id})$-field of (j, A). Since $\mathcal{M}$ and $\sqrt{\mathrm{Tr}[\mathcal{M}^2]}$ are continuous this implies that $\sqrt{\mathrm{Tr}[\mathcal{M}^2]}$ is constant for (E_1, l_1)-invariant $\mathcal{M}$ if j is topologically transitive.

Finally, we consider the case where $(E, l) = (\mathbb{R}^3, l_{\rm v})$ and $(\check{E}, \check{l}) = (E_{\rm t}, l_{\rm t})$ and in order to apply the DT we choose $x = (0, 0, 1)^t \in \mathbb{R}^3$ and $\check{x} = x_0$ with $x_0 \in E_{\rm t}$ given by (107). This choice of x and $\check{x}$ is motivated by the aim to obtain an IPTF of the form (130). It follows from (70),(104) and (107) that

$$\begin{aligned} E_x &= \{\vec{S} \in \mathbb{R}^3 : |\vec{S}| = 1\} \,, \\ \check{E}_{\check{x}} &= \{M \in E_{\rm t} : \det(M) = -2y^3, \\ &\quad \mathrm{Tr}[M^2] = 6y^2\} \,. \end{aligned} \tag{127}$$

Recalling Section 4.4, and since $E = \mathbb{R}^3$ and $\check{E} = E_{\rm t}$ are Hausdorff, we can compute *all* $SO(3)$-maps from (E_x, l_x) to $(\check{E}_{\check{x}}, \check{l}_{\check{x}})$. In fact, by (74),

$$\mathrm{Iso}(E, l; x) = \mathrm{Iso}(\mathbb{R}^3, l_{\rm v}; (0, 0, 1)^t) = SO(2) \tag{128}$$

and, by (110), $\mathrm{Iso}(\check{E}, \check{l}; \check{x}) = \mathrm{Iso}(E_{\rm t}, l_{\rm t}; x_0) = SO(2) \bowtie \mathbb{Z}_2$ whence $\mathrm{Iso}(E, l; x) \subset \mathrm{Iso}(\check{E}, \check{l}; \check{x})$ so that $\mathrm{Iso}(E, l; x)$ is, trivially, conjugate to a subgroup of $\mathrm{Iso}(\check{E}, \check{l}; \check{x})$ which implies, by Section 4.4, that $SO(3)$-maps exist. In fact using (47) one can show that the function $\beta \in \mathcal{C}(E_x, \check{E}_{\check{x}})$, defined by

$$\beta(\vec{S}) := y(I_{3\times 3} - 3\vec{S}\vec{S}^t) \,, \tag{129}$$

is the *only* $SO(3)$-map from (E_x, l_x) to $(\check{E}_{\check{x}}, \check{l}_{\check{x}})$. To state the SMT let $\vec{f}_0 \in \mathcal{C}(\mathbb{T}^d, \mathbb{R}^3)$ take values only in E_x, i.e., $|\vec{f}_0(z)| = 1$, and define $\check{\vec{f}}_0 \in \mathcal{C}(\mathbb{T}^d, E_{\rm t})$ by

$$\check{\vec{f}}_0(z) \;:=\; \beta(\vec{f}_0(z)) = y(I_{3\times 3} - 3\vec{f}_0(z)\vec{f}_0^t(z)) \,, \tag{130}$$

where in the second equality we used (129). Note that $\check{\vec{f}}_0$ takes values only in $\check{E}_{\check{x}}$. Recalling Section 4.4, the SMT states that if $\vec{f}_0$ is an invariant $(\mathbb{R}^3, l_{\rm v})$-field, i.e., is an ISF, then $\check{\vec{f}}_0$ is an $(E_{\rm t}, l_{\rm t})$-field, i.e., an IPTF. Thus for any ISF the SMT gives us an IPTF whose values are matrices with two distinct eigenvalues. In the special case $y = 1/\sqrt{6}$, (130) is the expression for the IPTF in eq. 1.1 of [14][3] where it is proved that that expression is unique. We will reconsider (130) in Section 6.5.

[3] There the IPTF is called the ITF

6.5. *The Invariant Reduction Theorem (IRT) and the Cross Section Theorem (CST)*

Let $j \in \mathrm{Homeo}(\mathbb{T}^d)$ and $A \in \mathcal{C}(\mathbb{T}^d, SO(3))$. We consider the $SO(3)$-space $(E, l) = (E_{\rm t}, l_{\rm t})$ and we pick $x = x_0$ with $x_0 \in E_{\rm t}$ given by (107). To apply the IRT let $\mathcal{M} \in \mathcal{C}(\mathbb{T}^d, E_{\rm t})$ take values only in E_x. Note, by (50),(86) and (107), that

$$\begin{aligned} \Sigma_x[E_{\rm t}, l_{\rm t}, \mathcal{M}] &= \{(z, r) \in (\mathbb{T}^d \times SO(3)) : \\ &\quad r x_0 r^t = \mathcal{M}(z)\} \\ &= \{(z, r) \in (\mathbb{T}^d \times SO(3)) : \\ &\quad y I_{3\times 3} - 3yr(0, 0, 1)^t(0, 0, 1)r^t \\ &\quad = \mathcal{M}(z)\} \,. \end{aligned} \tag{131}$$

Recalling Section 4.5 the IRT states that $\mathcal{M}$ is an IPTF of (j, A) iff
$\mathcal{P}[SO(3), l_{SO(3)}, j, A](\Sigma_x[E_{\rm t}, l_{\rm t}, \mathcal{M}]) = \Sigma_x[E_{\rm t}, l_{\rm t}, \mathcal{M}]$.

To discuss the CST we first recall from Section 4.5 and (131) that the function $p_x[E_{\rm t}, l_{\rm t}, \mathcal{M}] \in \mathcal{C}(\Sigma_x[E_{\rm t}, l_{\rm t}, \mathcal{M}], \mathbb{T}^d)$ is defined by $p_x[E_{\rm t}, l_{\rm t}, \mathcal{M}](z, r) := z$. Thus recalling Section 4.5 the CST states that $p_x[E_{\rm t}, l_{\rm t}, \mathcal{M}]$ has a cross section iff a $T \in \mathcal{C}(\mathbb{T}^d, SO(3))$ exists such that $\mathcal{M}(z) = l_{\rm t}(T(z); x)$.

This allows us to characterize invariant $(E_{\rm t}, l_{\rm t})$-frame fields at x in terms of cross sections by claiming that a (j, A) has an invariant $(E_{\rm t}, l_{\rm t})$-frame field at x iff it has an IPTF $\mathcal{M}$ such that $\mathcal{M}$ takes values only in E_x and such that $p_x[E_{\rm t}, l_{\rm t}, \mathcal{M}]$ has a cross section. The claim is a special case of a claim made in Section 4.5.

We can summarize the present case where $(E, l) = (E_{\rm t}, l_{\rm t})$ and $x = x_0$ with $x_0 \in E_{\rm t}$ given by (107) by saying that the CST gives a topological criterion for the existence of an invariant $(E_{\rm t}, l_{\rm t})$-frame field at x and that the IRT gives a topological criterion for the existence of an IPTF which takes values only in E_x.

With the above discussion of the CST for this case we can now reconsider (108) and (130) by making three claims about the situation when $\vec{f} \in \mathcal{C}(\mathbb{T}^d, \mathbb{R}^3)$ and $|\vec{f}| = 1$ and when $\mathcal{M} \in \mathcal{C}(\mathbb{T}^d, E_{\rm t})$ is defined by

$$\mathcal{M}(z) = y(I_{3\times 3} - 3\vec{f}(z)\vec{f}^t(z)) \,. \tag{132}$$

Note that (108) and (130) are of the form (132) and that $\mathcal{M}$ in (132) takes values only in E_x.

The first claim, which follows from the SMT (see also the discussion after (130)), states that if $\vec{f}$ is an ISF then $\mathcal{M}$ is an IPTF.

The second claim states that if $\vec{f}$ is an ISF and if $p_x[E_{\rm t}, l_{\rm t}, \mathcal{M}]$ has a cross section then $\mathcal{M}$ is an IPTF and an invariant $(E_{\rm t}, l_{\rm t})$-frame field T at x exists such that $\mathcal{M}(z) = l_{\rm t}(T(z); x)$. To prove the second claim we first note, by the first claim, that $\mathcal{M}$ is an IPTF. Since $p_x[E_{\rm t}, l_{\rm t}, \mathcal{M}]$ has a cross section we recall from the CST that a $T \in \mathcal{C}(\mathbb{T}^d, SO(3))$ exists such that $\mathcal{M}(z) = l_{\rm t}(T(z); x)$. Since $\mathcal{M}$ is an IPTF we conclude from the NFT that T is an invariant $(E_{\rm t}, l_{\rm t})$-frame field T at x.

The third claim states that if $\vec{f}$ is an ISF such that $p_x[\mathbb{R}^3, l_{\rm v}, \vec{f}]$ has a cross section, i.e., such that $\vec{f}$ is the third column of an IFF T then $\mathcal{M}$ is an IPTF which satisfies $\mathcal{M}(z) = l_{\rm t}(T(z); x)$ and $p_x[E_{\rm t}, l_{\rm t}, \mathcal{M}]$ has a cross section. To prove the third claim we first note, by the first claim, that $\mathcal{M}$ is an IPTF. Moreover since $\vec{f}$ is the third column of T we find, by (86) and (132), that $\mathcal{M}(z) = l_{\rm t}(T(z); x)$ which implies, by the CST, that $p_x[E_{\rm t}, l_{\rm t}, \mathcal{M}]$ has a cross section.

The above discussion of the IRT and CST, which was made for the choice $(E, l) = (E_{\rm t}, l_{\rm t})$ and $x = x_0$ with $x_0 \in E_{\rm t}$ given by (107), can be generalized to any $x \in E_{\rm t}$ which has two distinct eigenvalues. Recalling from Section 6.3 that in the general case ${\rm Iso}(E_{\rm t}, l_{\rm t}; x)$ is conjugate to ${\rm Iso}(E_{\rm t}, l_{\rm t}; x_0)$ the general case is just a minor modification of its subcase $x = x_0$ and so we leave the remaining details to the reader.

We also have to leave the cases where $x \in E_{\rm t}$ has three distinct eigenvalues or only one eigenvalue to the reader. These cases can be handled in analogy to the case when $x \in E_{\rm t}$ has two distinct eigenvalues by using the information from Section 6.3. Moreover we must leave the $SO(3)$-spaces $(\mathbb{R}^4 \times E_{\rm t}, \overline{l_1})$ and (E_1, l_1) to the reader.

7. Bundle-Theoretic Origins

This work was inspired by bundle theory and we here give evidence for that by following [6, 7, 33]. See also [4, 1].

The "ambient" principal bundle underlying our formalism is a fixed principal $SO(3)$-bundle which is a product principal bundle with base space $\mathbb{T}^d$, i.e., it can be written as the 4-tuple $(\mathcal{E}, p, \mathbb{T}^d, L)$ where $\mathcal{E} := \mathbb{T}^d \times SO(3)$ is the bundle space, $p \in \mathcal{C}(\mathcal{E}, \mathbb{T}^d)$ the bundle projection, i.e., $p(z, r) := z$, and $(\mathcal{E}, L)$ the underlying $SO(3)$-space with $L : SO(3) \times \mathcal{E} \to \mathcal{E}$ defined by $L(r; z, r') := (z, r'r^t)$. Let (E, l) be an $SO(3)$-space (E, l) and let E be Hausdorff and $f \in \mathcal{C}(\mathbb{T}^d, E)$ to be E_x-valued for some $x \in E$. Because E is Hausdorff, ${\rm Iso}(E, l; x)$ is closed, i.e., ${\rm Cl}({\rm Iso}(E, l; x)) = {\rm Iso}(E, l; x)$. By the Reduction Theorem [6, Chapter 6],[33, Chapter 6] of bundle theory, every $(\Sigma_x[E, l, f], p_x[E, l, f], \mathbb{T}^d, L_x[E, l, f])$ is a reduction of the ambient principal bundle and is a principal ${\rm Iso}(E, l; x)$-bundle where $L_x[E, l, f]$ is the restriction of L to ${\rm Iso}(E, l; x) \times \Sigma_x[E, l, f]$. Conversely, every reduction of the ambient principal bundle is of this form. By definition the reductions of the ambient principal $SO(3)$-bundle are those principal H-bundles which are principal subbundles of the ambient bundle such that their bundle space is a closed subset of $\mathcal{E}$ and such that H is a closed subgroup of $SO(3)$. Thus every $\Sigma_x[E, l, f]$ is the bundle space of a reduction if E is Hausdorff.

The bundle-theoretic aspect of the CST follows from the fact that $p_x[E, l, f]$ is the bundle projection of $(\Sigma_x[E, l, f],\ p_x[E, l, f],\ \mathbb{T}^d,\ L_x[E, l, f])$. This implies, by bundle theory, that $p_x[E, l, f]$ has a cross section iff the principal bundle $(\Sigma_x[E, l, f], p_x[E, l, f], \mathbb{T}^d, L_x[E, l, f])$ is trivial, i.e., is isomorphic to a product principal bundle. The proof constructs this isomorphism out of the function $T \in \mathcal{C}(\mathbb{T}^d, SO(3))$ in CST which satisfies $f(z) = l(T(z); x)$.

Given (j, A), bundle theory provides dynamics on reductions by giving us a candidate for a one-turn map on $\Sigma_x[E, l, f]$ by using the restriction $\mathcal{P}[SO(3), l_{SO(3)}, j, A]|_{\Sigma_x[E,l,f]}$ of the one-turn map $\mathcal{P}[SO(3), l_{SO(3)}, j, A]$ to $\Sigma_x[E, l, f]$. However $\mathcal{P}[SO(3), l_{SO(3)}, j, A]|_{\Sigma_x[E,l,f]}$ is a genuine one-turn map, and not just a candidate, only if $\mathcal{P}[SO(3), l_{SO(3)}, j, A]$ is onto $\Sigma_x[E, l, f]$. Luckily it is possible to determine when this is the case because $\mathcal{P}[SO(3), l_{SO(3)}, j, A]$ is a bijection which implies that $\mathcal{P}[SO(3), l_{SO(3)}, j, A]|_{\Sigma_x[E,l,f]}$ is onto $\Sigma_x[E, l, f]$ iff $\Sigma_x[E, l, f]$ is invariant under $\mathcal{P}[SO(3), l_{SO(3)}, j, A]$, i.e., iff

$$\mathcal{P}[SO(3), l_{SO(3)}, j, A](\Sigma_x[E, l, f]) = \Sigma_x[E, l, f] \ . \quad (133)$$

The reduction with bundle space $\Sigma_x[E, l, f]$ is thus called "invariant under (j, A)" if

$$\mathcal{P}[SO(3), l_{SO(3)}, j, A](\Sigma_x[E, l, f]) = \Sigma_x[E, l, f] \ . \quad (134)$$

Therefore, by the IRT, the reduction with bundle space $\Sigma_x[E, l, f]$ is invariant under (j, A) iff f is an invariant (E, l)-field of (j, A). Thus, given (j, A), not every, if any, reduction with bundle space $\Sigma_x[E, l, f]$ is invariant under (j, A). In summary the topological interpretation of the IRT and CST rests on the reductions of the ambient principal bundle.

Moreover we briefly mention that the definitions of $\mathcal{P}[E, l, j, A]$ and $\tilde{\mathcal{P}}[E, l, j, A]$ of Section 3 are borrowed from bundle theory since every $SO(3)$-space (E, l) uniquely determines an "associated bundle" (relative to the ambient principal bundle) which, up to bundle isomorphism, is a 3-tuple of the form $(\mathbb{T}^d \times E, p, \mathbb{T}^d)$ where $p(z, x) := z$. In fact given (j, A), bundle theory manages to merge the functions $\mathcal{P}[SO(3), l_{SO(3)}, j, A]$ and l into the maps $\mathcal{P}[E, l, j, A]$ and $\tilde{\mathcal{P}}[E, l, j, A]$ leading to a topological interpretation of the NFT and the SMT. This completes our sketch of the bundle-theoretic aspects. For analogies with Yang-Mills theory, see [34].

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