Quasiperiodic Method of Averaging Applied to Planar Undulator Motion Excited by a Fixed Traveling Wave

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Introduction

- Topic: Summary of our mathematical study in [1] of planar motion of energetic electrons moving through planar dipole undulator, excited by fixed planar polarized plane wave Maxwell field in X-Ray FEL regime
- Tool: Normal form analysis via first-order Method of Averaging (MoA) which is long time perturbation theory for ODE's
  Normal forms are obtained by averaging over independent variable
- Feature 1: Starting from exact 6D equations of motion, MoA gives explicit error bounds relating exact and normal form solutions
- Feature 2: Near-to-resonant normal form analysis generalizes ponderomotive phase and FEL pendulum system
- Feature 3: Far from resonance Δ-nonresonant normal form is used

The planar undulator motion

- 6D Lorentz equations of motion in SI units with z as the independent variable:

$$\frac{dx}{dz}=\frac{p_x}{p_z}\,,\quad \frac{dy}{dz}=\frac{p_y}{p_z}\,,\quad \frac{dt}{dz}=\frac{m\gamma}{p_z}\,, \tag{1}$$

$$\frac{dp_x}{dz}=-\frac{e}{c}[cB_u\cosh(k_uy)\sin(k_uz) -\frac{p_y}{p_z}cB_u\sinh(k_uy)\cos(k_uz) +E_r(\frac{m\gamma c}{p_z}-1)h(\check{\alpha}(z,t))]\,, \tag{2}$$

$$\frac{dp_y}{dz}=-\frac{ep_x}{cp_z}cB_u\sinh(k_uy)\cos(k_uz)\,, \tag{3}$$

$$\frac{dp_z}{dz}=-\frac{e}{c}[-\frac{p_x}{p_z}cB_u\cosh(k_uy)\sin(k_uz) +E_r\frac{p_x}{p_z}h(\check{\alpha}(z,t))] \tag{4}$$

- x,y,z are Cartesian coordinates
- z distance along undulator
- $t(z)$ arrival time at z
- p_x,p_y,p_z Cartesian momenta
- $\gamma^2=1+\mathbf{p}\cdot\mathbf{p}/m^2c^2$
- m=electron mass; $-e$=electron charge; c=vacuum speed of light
- Undulator magnetic field:

$$\mathbf{B}_u=-B_u\begin{pmatrix}0\\ \cosh(k_uy)\sin(k_uz)\\ \sinh(k_uy)\cos(k_uz)\end{pmatrix}\,, \tag{5}$$

- $B_u>0$ undulator field strength
- $k_u>0$ undulator wave number
- Traveling wave radiation field:

$$\mathbf{E}_r=E_rh(\check{\alpha})\begin{pmatrix}1\\0\\0\end{pmatrix}\,,\quad \mathbf{B}_r=\frac{1}{c}\begin{pmatrix}0\\0\\1\end{pmatrix}\times\mathbf{E}_r$$

- $E_r>0$, $h:\mathbb{R}\to\mathbb{R}$
- $\check{\alpha}(z,t)=k_r(z-ct)$ with $k_r>0$

The 2D System

- We confine to planar motion with no approximation since:
  $y(0)=p_y(0)=0\Longrightarrow y(z)=p_y(z)=0$
  $\Longrightarrow$ the six ODE's (1)-(4) reduce to four ODE's
- Righthand sides of (1)-(4) x-independent $\Longrightarrow$ x equation need not be considered
- $\frac{p_x}{mcK}-\cos(k_uz)-\frac{E_r}{cB_u}\frac{k_u}{k_r}H(\alpha)$, is conserved where H is any antiderivative of h, i.e., $H'=h$
  $\Longrightarrow p_x$ can be eliminated
- Two equations remain $\Longrightarrow$ everything determined from equations for t and p_z
- Natural scaling for z is $z=\zeta/k_u$

- Replace dependent variable t by $\check{\alpha}$ and define

$$\alpha(\zeta)=\check{\alpha}(z,t(z))=k_r(z-ct(z)) \tag{6}$$

Replace p_z by $\gamma\Longrightarrow$ basic 2D system:

$$\frac{d\alpha}{d\zeta}=\frac{k_r}{k_u}(1-\frac{m\gamma c}{p_z})\,, \tag{7}$$

$$\frac{d\gamma}{d\zeta}=-\frac{eE_r}{k_umc^2}\frac{p_x}{p_z}h(\alpha)\,, \tag{8}$$

with p_x and p_z replaced by

$$p_x=p_x(0)+mcK\cdot\Big(\cos(k_uz)-1+\frac{E_r}{cB_u}\frac{k_u}{k_r}[H(\alpha)-H(\alpha(0))]\Big)\,,$$

$$p_z=\sqrt{m^2c^2(\gamma^2-1)-p_x^2}\,,$$

$$K=\frac{eB_u}{mck_u}=\text{undulator parameter}$$

- Transform (7),(8) to standard form for MoA $\Longrightarrow$ introduce normalized energy deviation η and its $O(1)$ counterpart χ via

$$\gamma=\gamma_c(1+\eta)=\gamma_c(1+\varepsilon\chi) \tag{9}$$

- γ_c is characteristic value of γ and ε is characteristic spread of η
- χ new $O(1)$ dependent variable replacing γ
- Do asymptotic analysis for γ_c large and ε small $\Longrightarrow$ γ_c large and η small as in an X-Ray FEL $\Longrightarrow$ (7),(8) become

$$[\alpha+Q(\zeta)]'=\varepsilon K_rq(\zeta)\chi+O(\frac{1}{\gamma_c^2})+O(\varepsilon^2)\,, \tag{10}$$

$$\chi'=-K^2\frac{\mathcal{E}}{\varepsilon\gamma_c^2}(\cos\zeta+\Delta P_{x0})h(\alpha) +O(1/\gamma_c^2)+O(1/\varepsilon\gamma_c^4)\,, \tag{11}$$

where

$$K_r=\frac{k_r}{k_u\gamma_c^2}\,,\ \mathcal{E}=\frac{E_r}{cB_u}\,,\ \Delta P_{x0}=\frac{p_x(0)}{mcK}-1\,,$$

$$q(\zeta)=1+K^2(\cos\zeta+\Delta P_{x0})^2\,,$$

$$Q'(\zeta)=\frac{K_r}{2}q(\zeta)\,,\quad Q(0)=0$$

- $\mathcal{E}$ not necessarily small $\Longrightarrow$ our results may be even relevant in high gain saturation regime
- Transform (10),(11) into standard form for MoA $\Longrightarrow$ need slowly varying dependent variables. Clearly, $\alpha+Q(\zeta)$ is slowly varying and we anticipate that χ will be slowly varying, i.e., $\mathcal{E}/\varepsilon\gamma_c^2$ small
- Thus define $\theta=\alpha+Q(\zeta)\Longrightarrow$ (10),(11) become

$$\theta'=\varepsilon K_rq(\zeta)\chi+O(1/\gamma_c^2)+O(\varepsilon^2)\,, \tag{12}$$

$$\chi'=-K^2\frac{\mathcal{E}}{\varepsilon\gamma_c^2}(\cos\zeta+\Delta P_{x0})h(\theta-Q(\zeta)) +O(1/\gamma_c^2)+O(1/\varepsilon\gamma_c^4) \tag{13}$$

Distinguished case: To obtain pendulum behavior, θ and χ need to interact in (12),(13) first-order ε $\Longrightarrow\varepsilon$ and γ_c related by $\varepsilon=\frac{\mathcal{E}}{\varepsilon\gamma_c^2}\Longrightarrow$

$$\varepsilon=\sqrt{\mathcal{E}}\frac{1}{\gamma_c} \tag{14}$$

$\Longrightarrow$ (12),(13) can be written in standard form:

$$\theta'=\varepsilon K_rq(\zeta)\chi+O(\varepsilon^2)\,, \tag{15}$$

$$\chi'=-\varepsilon K^2(\cos\zeta+\Delta P_{x0})h(\theta-Q(\zeta)) +O(\varepsilon^2) \tag{16}$$

The 2D system in monochromatic case

- Monochromatic case:

$$H(\check{\alpha})=(1/\nu)\sin(\nu\check{\alpha})\,,\quad h(\check{\alpha})=\cos(\nu\check{\alpha})\,, \tag{17}$$

$$K_r=\frac{2}{\bar{q}}\,, \tag{18}$$

where $\nu\ge 1/2$ and

$$\bar{q}=\lim_{T\to\infty}[\frac{1}{T}\int_0^T q(\zeta)d\zeta] =1+\frac{1}{2}K^2+K^2(\Delta P_{x0})^2 \tag{19}$$

- ODE's (15),(16) now become

$$\theta'=\varepsilon f_1(\chi,\zeta)+O(\varepsilon^2)\,, \tag{20}$$

$$\chi'=\varepsilon f_2(\theta,\zeta,\nu)+O(\varepsilon^2)\,, \tag{21}$$

where

$$f_1(\chi,\zeta)=\frac{2q(\zeta)}{\bar{q}}\chi\,,$$

$$f_2(\theta,\zeta,\nu)=-K^2(\cos\zeta+\Delta P_{x0})\cdot\cos\Big(\nu\theta-\nu\zeta-\nu\Upsilon_0\sin\zeta-\nu\Upsilon_1\sin 2\zeta\Big)$$

$$=-\frac{K^2}{2}e^{i\nu\theta}\sum_{n\in\mathbb{Z}}\widehat{jj}(n;\nu,\Delta P_{x0})e^{i(n-\nu)\zeta}+cc\,,$$

and where

$$\Upsilon_0=\frac{2}{\bar{q}}K^2\Delta P_{x0},\quad \Upsilon_1=\frac{\bar{q}K^2}{4} \tag{22}$$

- $f_1(\chi,\zeta)$ and $f_2(\theta,\zeta,\nu)$ are quasiperiodic in ζ
- f_1 is 2π periodic, i.e., has base periodicity, 2π
- f_2 has two base periodicities, 2π and $2\pi/\nu$
- Averages needed for normal form analysis:

$$\bar{f}_1(\chi)=\lim_{T\to\infty}[\frac{1}{T}\int_0^T f_1(\chi,\zeta)d\zeta]=2\chi\,,$$

$$\bar{f}_2(\theta,\nu)=\lim_{T\to\infty}[\frac{1}{T}\int_0^T f_2(\theta,\zeta,\nu)d\zeta]$$

$$=\begin{cases}0 & \text{if }\nu\notin\mathbb{N}\\ -K^2\widehat{jj}(k;k,\Delta P_{x0})\cos(k\theta) & \text{if }\nu=k\in\mathbb{N}\,,\end{cases}$$

where $\mathbb{N}$=set of positive integers

Δ-nonresonant normal form

- Δ-nonresonant case is example of quasiperiodic averaging with a small divisor problem of very simple structure
- Δ-nonresonant case defined by: $\nu\in[k+\Delta,k+1-\Delta]$ with $\Delta\in(0,0.5)$ and $k\in\mathbb{N}$
- Normal form approximation of (20),(21) in Δ-nonresonant case: drop the $O(\varepsilon^2)$ terms and average the $O(\varepsilon)$ terms by holding slowly varying quantities θ,χ fixed
  $\Longrightarrow$ Δ-nonresonant normal form system:

$$v_1'=\varepsilon 2v_2\,,\quad v_2'=0 \tag{23}$$

- Δ-nonresonant case is natural if $|\nu-k|$ "big"
- [1] gives error bounds:

$$|\theta(\zeta,\varepsilon,\nu)-v_1(\zeta,\varepsilon)|\le C(T)\frac{\varepsilon}{\Delta}\,,$$

$$|\chi(\zeta,\varepsilon,\nu)-v_2(\zeta,\varepsilon)|\le C(T)\frac{\varepsilon}{\Delta}\,,$$

for $0\le\zeta\le T/\varepsilon$ with ε sufficiently small and where $C(T)$ is positive constant
- Error bound increases as $\Delta\to 0$, i.e., as ν moves toward resonance

Near-to-resonant normal form

- Near-to-resonant case is an example of periodic averaging. It is defined by: $\nu=k+\varepsilon a$ where $k\in\mathbb{N}$ and $a\in[-1/2,1/2]$
- Near-to-resonant case explores $O(\varepsilon)$ neighborhoods of $\nu=k$ resonances
- Write (20),(21) as:

$$\theta'=\varepsilon f_1(\chi,\zeta)+O(\varepsilon^2)\,, \tag{24}$$

$$\chi'=\varepsilon f_2^R(\theta,\varepsilon\zeta,\zeta,k,a)+O(\varepsilon^2)\,, \tag{25}$$

$$f_2^R(\theta,\tau,\zeta,k,a)=-K^2(\cos\zeta+\Delta P_{x0})\cdot\cos\Big(k[\theta-\zeta-\Upsilon_0\sin\zeta-\Upsilon_1\sin 2\zeta]-a\tau\Big)$$

$$=-\frac{K^2}{2}\exp(i[k\theta-a\tau])\cdot\sum_{n\in\mathbb{Z}}\widehat{jj}(n;k,\Delta P_{x0})e^{i\zeta[n-k]}+cc \tag{26}$$

- $f_1(\chi,\zeta)$, $f_2^R(\theta,\tau,\zeta,k,a)$ are 2π periodic in ζ

- Normal form approximation of (24),(25) in Near-to-resonant case: drop the $O(\varepsilon^2)$ and average the $O(\varepsilon)$ terms by holding slowly varying quantities $\theta,\chi,\varepsilon a\zeta$ fixed:

$$v_1'=2\varepsilon v_2\,, \tag{27}$$

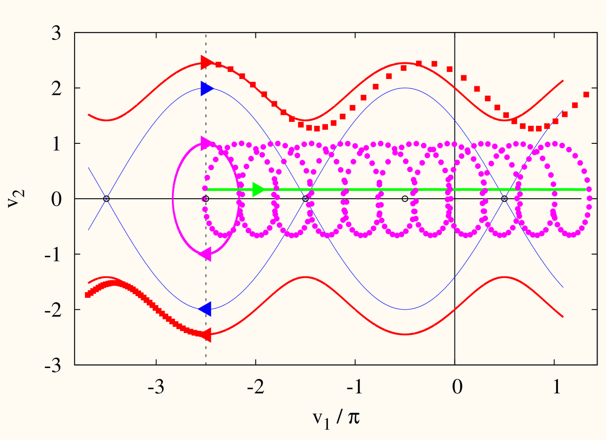
$$v_2'=-\varepsilon K^2\widehat{jj}(k;k,\Delta P_{x0})\cos(kv_1-\varepsilon a\zeta) \tag{28}$$

- Near-to-resonant case is natural if $|\nu-k|$ "small"
- Resonant case is special case when $a=0$
- [1] gives error bounds:

$$|\theta(\zeta,\varepsilon)-v_1(\zeta,\varepsilon)|\le C_R(T)\varepsilon\,,$$

$$|\chi(\zeta,\varepsilon)-v_2(\zeta,\varepsilon)|\le C_R(T)\varepsilon\,,$$

for $0\le\zeta\le T/\varepsilon$ with ε sufficiently small and where $C_R(T)$ is positive constant
- A phase plane portrait for the system (27), (28) is shown in figure below with $k=1$ and $K^2\widehat{jj}(k;k,\Delta P_{x0})=2$
- Phase plane orbits on resonance, i.e., $a=0$ are marked in figure by solid magenta, blue, red curves and five black fixed points
- Near-to-resonant phase plane orbits, for $a=1/3$, are marked in figure by green solid and dotted magenta and red curves and are computed with ode45 solver of Matlab



- θ generalizes so-called ponderomotive phase since, if $a=0,\Delta P_{x0}=0$, it is the ponderomotive phase which in standard treatments is introduced heuristically to maximize energy transfer
- For $\Delta P_{x0}=0$:

$$\widehat{jj}(k;k,0)=\begin{cases}\frac{1}{2}(-1)^n[J_n(x_n)-J_{n+1}(x_n)] & \text{if }k=2n+1\\ 0 & \text{if }k\text{ even}\,,\end{cases}$$

where $x_n=(2n+1)\Upsilon_1$ and $n=0,1,\ldots$ with J_m=m-th-order Bessel function of first kind
$\Longrightarrow$ for $a=0,\Delta P_{x0}=0$, (27),(28) give standard FEL pendulum system for odd k (see references in [1] and [2])
- Remark on non-monochromatic case: If Fourier transform $\tilde{h}(\xi)$ of h is continuous, e.g., narrow Gaussian centered on resonance $\xi=k$, the resonant effect may be washed out and thus FEL pendulum behavior disappears in first order averaging

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References

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