International Journal of Modern Physics A

Re-evaluation of Spin-Orbit Dynamics of Polarized e^+e^- Beams in High Energy Circular Accelerators and Storage Rings: An approach based on a Bloch equation *

Klaus Heinemann
Department of Mathematics and Statistics, University of New Mexico
Albuquerque, NM 87131, USA
heineman@math.unm.edu

Daniel Appelö
Department of Applied Mathematics, University of Colorado Boulder
Boulder, CO 80309-0526, USA
Daniel.Appelo@Colorado.edu

Desmond P. Barber
DESY
Hamburg, 22607, Germany
mpybar@mail.desy.de
and: Department of Mathematics and Statistics, University of New Mexico
Albuquerque, NM 87131, USA

Oleksii Beznosov
Department of Mathematics and Statistics, University of New Mexico
Albuquerque, NM 87131, USA
dohter@protonmail.com

James A. Ellison
Department of Mathematics and Statistics, University of New Mexico
Albuquerque, NM 87131, USA
ellison@math.unm.edu

Received Day Month Year
Revised Day Month Year

We give an overview of our current/future analytical and numerical work on the spin polarization in high-energy electron storage rings. Our goal is to study the possibility of polarization for the CEPC and FCC-ee. Our work is based on the so-called Full Bloch equation for the polarization density introduced by Derbenev and Kondratenko in 1975. We also give an outline of the standard approach, the latter being based on the

*This material is based upon work supported by the U.S. Department of Energy, Office of Science, Office of High Energy Physics, under Award Number DE-SC0018008

Derbenev-Kondratenko formulas.

PACS numbers:29.20.db,29.27.Hj,05.10.Gg

1. Introduction

This paper is an update on a talk by K. Heinemann at the IAS Mini-Workshop on Beam Polarization in Future Colliders on January 17, 2019, in Hong Kong.[1] Our ultimate goal is to examine the possibility of polarization for CEPC and FCC-ee.

We will first briefly review the "standard" approach which is based on the Derbenev-Kondratenko formulas.[2] These formulas rely, in part, on plausible assumptions grounded in deep physical intuition. So the following question arises: do the Derbenev-Kondratenko formulas tell full story? In fact there is an alternative approach based on a Bloch-type equation for the polarization density[3] which we call the Full Bloch equation (FBE) and which we believe can deliver more information than the standard approach even if the latter includes potential correction terms.[4] So we aim to determine the domain of applicability of the Derbenev-Kondratenko formulas and the possibility in theory of polarization at the CEPC and FCC-ee energies. Of course both approaches focus on the equilibrium polarization and the polarization time. We use the name "Bloch" to reflect the analogy with equations for magnetization in condensed matter.[5] This paper concentrates on the Bloch approach. The cost of the numerical computations in the Bloch approach is considerable since the polarization density depends on six phase-space variables plus the time variable so that the numerical solution of the FBE, the FBE being a system of three PDEs in seven independent variables, is a nontrivial task which cannot be pursued with traditional approaches like the finite difference method. However we see at least five viable methods:

(1) Approximating the FBE by an effective FBE via the Method of Averaging and solving the effective FBE via spectral phase-space discretization, e.g., a collocation method, plus an implicit-explicit time discretization.
(2) Solving the system of stochastic differential equations (SDEs), which underlies the FBE, via Monte-Carlo spin tracking.
(3) Solving the Fokker-Planck equation, which underlies the FBE, via the Gram-Charlier method.
(4) Solving the FBE via a deep learning method.
(5) Solving the system of SDEs in a way that allows connections with the Derbenev-Kondratenko formulas to be established.

We will dwell on Method 1 in this paper. We plan to validate this method by one of the other four methods. More details on Method 1 can be found in Ref. 6. The method of averaging we use is discussed in Refs. 7-12. One hope tied to Method 1 is the fact that the effective FBE gives analytical insights into the spin-resonance structure of the bunch. Note that Methods 1-4 are independent of the standard approach. In particular they do not rely on the invariant spin field. Note also that

Methods 1-3 and 5 are based on knowing the system of SDEs, which underlies the FBE. For details of this system of SDEs, see the invited ICAP18 paper of Ref. 6. Regarding Method 2 there is a large literature on the numerical solution of SDEs, see Refs. 13, 14 and references in Ref. 15.

By neglecting the spin flip terms and the kinetic-polarization term in the FBE one obtains an equation that we call the reduced Bloch equation (RBE). The RBE approximation is sufficient for computing the radiative depolarization rate due to stochastic orbital effects and it shares the terms with the FBE that are challenging to discretize. For details on our phase-space discretization and time discretization of the RBE, see Refs. 6, 16 and 17.

We proceed as follows. In the second section we sketch the standard approach. In the third section we present, for the laboratory frame, the FBE and its restriction, the RBE. In the fourth section we discuss the RBE in the beam frame and in the fifth section we show how, in the beam frame, the effective RBE is obtained via the method of averaging. In the sixth section we describe ongoing and future work.

2. Sketching the standard approach based on the Derbenev-Kondratenko formulas

We define the "time" $\theta = 2\pi s/C$ where s is the distance around the ring and C is the circumference and we denote by y a position in six-dimensional phase space w.r.t. the closed orbit (CO). In particular, following Ref. 26, y_6 is the relative deviation of the energy from the reference energy. Then if, in the beam frame, $f = f(\theta, y)$ denotes the normalized phase-space density at θ and y and $\vec{P}_{loc} = \vec{P}_{loc}(\theta, y)$ denotes the local polarization vector of the bunch we have

$$\int dy\ f(\theta, y) = 1\ , \quad \int dy\ f(\theta, y)\vec{P}_{loc}(\theta, y) = \vec{P}(\theta)\ , \tag{1}$$

where $\vec{P}(\theta)$ is the polarization vector of the bunch at θ. For a detailed discussion about $\vec{P}_{loc}$, see, e.g., Ref. 18. Here and in the following we use arrows on three-component column vectors.

Central to the standard approach is the invariant spin field $\vec{n} = \vec{n}(\theta, y)$ (ISF) defined as a normalized periodic solution of the Thomas-BMT-equation in phase space, i.e.,

$$\partial_\theta \vec{n} = L_{\text{Liou}}(\theta, y)\vec{n} + \Omega(\theta, y)\vec{n}\ , \tag{2}$$

such that

(1) $\left|\vec{n}(\theta, y)\right| = 1$,
(2) $\vec{n}(\theta + 2\pi, y) = \vec{n}(\theta, y)$,

and where L_{Liou} denotes the Hamiltonian part of the Fokker-Planck operator L_{FP}, the latter being introduced in Section 3 below. The unit vector of the ISF on the CO is denoted by $\hat{n}_0(\theta)$ and it is easily obtained as an eigenvector of the one-turn

spin-transport map on the CO.[26] There are many methods for computing the full invariant spin field but none are trivial (for a recent technique see Ref. 19). In fact the existence, in general, of the invariant spin field is a mathematical issue which is only partially resolved, see, e.g., Ref. 20. The standard approach assumes that a function $P_{\rm DK} = P_{\rm DK}(\theta)$ exists such that

$$\vec{P}_{loc}(\theta, y) \approx P_{\rm DK}(\theta)\vec{n}(\theta, y) \, . \tag{3}$$

Thus, by (1) and (3),

$$\vec{P}(\theta) = \int dy \; f(\theta, y)\vec{P}_{loc}(\theta, y) \approx P_{\rm DK}(\theta) \int dy \; f(\theta, y)\vec{n}(\theta, y) \, . \tag{4}$$

The approximation (3) leads to[2]

$$P_{\rm DK}(\theta) = P_{\rm DK}(\infty)(1 - e^{-\theta/\tau_{\rm DK}}) + P_{\rm DK}(0)e^{-\theta/\tau_{\rm DK}} \, , \tag{5}$$

where $\tau_{\rm DK}$ and $P_{\rm DK}(\infty)$ are given by the Derbenev-Kondratenko formulas

$$P_{\rm DK}(\infty) := \frac{\tau_0^{-1}}{\tau_{\rm DK}^{-1}} \, , \tag{6}$$

$$\tau_{\rm DK}^{-1} := \frac{5\sqrt{3}}{8}\frac{r_e\gamma_0^5\hbar}{m}\frac{C}{4\pi^2}\int_0^{2\pi} d\theta \Big\langle \frac{1}{|\rho|^3}[1 - \frac{2}{9}(\vec{n}\cdot\hat{\beta})^2 + \frac{11}{18}\Big|\partial_{y_6}\vec{n}\Big|^2] \Big\rangle_\theta \, , \tag{7}$$

$$\tau_0^{-1} := \frac{r_e\gamma_0^5\hbar}{m}\frac{C}{4\pi^2}\int_0^{2\pi} d\theta \Big\langle \frac{1}{|\rho|^3}\hat{b}\cdot[\vec{n} - \partial_{y_6}\vec{n}] \Big\rangle_\theta \, , \tag{8}$$

with

- $\Big\langle \cdots \Big\rangle_\theta \equiv \int dy \; f(\theta, y) \cdots$
- $\hat{b} = \hat{b}(\theta, y) \equiv$ normalized magnetic field, $\hat{\beta} = \hat{\beta}(\theta, y) \equiv$ normalized velocity vector, $\gamma_0 \equiv$ Lorentz factor of the reference particle, $\rho(\theta, y) \equiv$ radius of curvature in the external magnetic field, $r_e \equiv$ classical electron radius, $m \equiv$ rest mass of electrons or positrons.

By (4), the equilibrium polarization vector is

$$\vec{P}(\infty) \approx P_{\rm DK}(\infty) \int dy \; f(\theta, y)\vec{n}(\theta, y) \, , \tag{9}$$

where $P_{\rm DK}(\infty)$ is given by (6). Defining

$$\tau_{dep}^{-1} := \frac{5\sqrt{3}}{8}\frac{r_e\gamma_0^5\hbar}{m}\frac{C}{4\pi^2}\int_0^{2\pi} d\theta \Big\langle \frac{1}{|\rho|^3}\frac{11}{18}\Big|\partial_{y_6}\vec{n}\Big|^2 \Big\rangle_\theta \, , \tag{10}$$

we can write (7) as

$$\tau_{\rm DK}^{-1} = \tau_{dep}^{-1} + \frac{5\sqrt{3}}{8}\frac{r_e\gamma_0^5\hbar}{m}\frac{C}{4\pi^2}\int_0^{2\pi} d\theta \Big\langle \frac{1}{|\rho|^3}[1 - \frac{2}{9}(\vec{n}\cdot\hat{\beta})^2] \Big\rangle_\theta \, . \tag{11}$$

For details on (6), (7), (8), (10) and (11) see, e.g., Refs. 25 and 26.

We now briefly characterize the various terms in the Derbenev-Kondratenko formulas. First, τ_{dep}^{-1} is the radiative depolarization rate. Secondly, the term

$\frac{r_e\gamma_0^5\hbar}{m}\frac{C}{4\pi^2}\int_0^{2\pi} d\theta\left\langle \frac{1}{|\rho|^3}\hat{b}\cdot\vec{n}\right\rangle_\theta$ in τ_0^{-1} and the term $\frac{5\sqrt{3}}{8}\frac{r_e\gamma_0^5\hbar}{m}\frac{C}{4\pi^2}\int_0^{2\pi} d\theta\left\langle \frac{1}{|\rho|^3}\right\rangle_\theta$ in $\tau_{\rm DK}^{-1}$ cover the Sokolov-Ternov effect. Lastly, the term $-\frac{r_e\gamma_0^5\hbar}{m}\frac{C}{4\pi^2}\int_0^{2\pi} d\theta\left\langle \frac{1}{|\rho|^3}\hat{b}\cdot[\partial_{y_6}\vec{n}]\right\rangle_\theta$ in τ_0^{-1} covers the kinetic polarization effect and the term in $\tau_{\rm DK}^{-1}$ which is proportional to 2/9 covers the Baier-Katkov correction.

We now sketch three approaches for computing $P_{\rm DK}(\infty)$ via the Derbenev-Kondratenko formulas. All three approaches use (6) but they differ in how τ_0^{-1} and $\tau_{\rm DK}^{-1}$ are computed.

(i) Compute τ_0^{-1} via (8) and $\tau_{\rm DK}^{-1}$ via (7) by computing f and $\vec{n}$ as accurately as needed.

(ii) Approximate τ_0^{-1} by neglecting the usually-small kinetic polarization term in (8) and by approximating the remaining term in (8) by replacing $\vec{n}$ by $\vec{n}_0$. Compute $\tau_{\rm DK}^{-1}$ via (11) where τ_{dep}^{-1} is not computed via (10) but via Monte-Carlo spin tracking and where the remaining terms in (11) are approximated by using the $\vec{n}_0$-axis. [a]

(iii) Compute τ_0^{-1} via (8) and $\tau_{\rm DK}^{-1}$ via (7) by linear approximation in orbit and spin variables via the so-called SLIM formalism.[26]

Approach (ii) is the most practiced while approach (i) is only feasible if one can compute f and $\vec{n}$ as accurately as needed (which is not easy!). Approach (iii), which is historically the first, is very simple and is often used for ballparking $P_{\rm DK}(\infty)$. Since the inception of the Derbenev-Kondratenko formulas one suspects correction terms to the rhs of (10). See Ref. 4 as well as Z. Duan's contribution to this workshop. These correction terms, associated with so-called resonance crossing, in turn associated with large energy spread, are not as well understood as the rhs of (10), partly because of their peculiar form. Nevertheless, careful observation of spin motion during the Monte-Carlo tracking in approach (ii), might provide a way to investigate their existence and form.

3. The Full Bloch equation and the Reduced Bloch equation in laboratory frame

In the previous section we used the beam frame and we will do so later. However the FBE was first presented in Ref. 3 for the laboratory frame and in that frame it also has its simplest form. In this section we focus on the laboratory frame.

[a]Prominent Monte-Carlo spin tracking codes are SLICKTRACK by D.P. Barber,[21] SITROS by J. Kewisch,[21] Zgoubi by F. Meot,[22] PTC/FPP by E. Forest,[23] and Bmad by D. Sagan.[24] This approach provides a useful first impression avoiding the computation of f and $\vec{n}$. For more details on this approach see Ref. 26. Monte-Carlo tracking can also be extended beyond integrable orbital motion to include, as just one example, beam-beam forces. Note that Monte-Carlo tracking just gives an estimate of τ_{dep}^{-1} but it does not provide an explanation. Nevertheless, insights into sources of depolarization can be obtained by switching off terms in the Thomas-BMT equation. In principal such diagnoses can also be applied in approach (i). Such investigations can the systematized under the heading of "spin matching".[26]

In a semiclassical probabilistic description of an electron or positron bunch the spin-orbit dynamics is described by the *spin*-1/2 *Wigner function* ρ (also called the *Stratonovich function*) written as

$$\rho(t,z) = \frac{1}{2}[f_{lab}(t,z)I_{2\times 2} + \vec{\sigma}\cdot\vec{\eta}_{lab}(t,z)] \,, \tag{12}$$

with $z = (\vec{r},\vec{p})$ where $\vec{r}$ and $\vec{p}$ are the position and momentum vectors of the phase space and t is the time, and where f_{lab} is the phase-space density of particles normalized by $\int f_{lab}(t,z)dz = 1$, $\vec{\eta}_{lab}$ is the polarization density of the bunch and $\vec{\sigma}$ is the vector of the three Pauli matrices. As explained in Ref. 18, $\vec{\eta}_{lab}$ is proportional to the spin angular momentum density. In fact it is given by $\vec{\eta}_{lab}(t,z) = f_{lab}(t,z)\vec{P}_{loc,lab}(t,z)$ where $\vec{P}_{loc,lab}$ is the local polarization vector. Thus $f_{lab} = Tr[\rho]$ and $\vec{\eta}_{lab} = Tr[\rho\vec{\sigma}]$. The polarization vector $\vec{P}_{lab}(t)$ of the bunch is $\vec{P}_{lab}(t) = \int \vec{\eta}_{lab}(t,z)dz = \int f_{lab}(t,z)\vec{P}_{loc,lab}(t,z)$.

Then, by neglecting collective effects and after several other approximations, the phase-space density evolves according to Ref. 3 via

$$\partial_t f_{lab} = L^{lab}_{FP}(t,z) f_{lab} \,. \tag{13}$$

Using the units as in Ref. 3 the Fokker-Planck operator L^{lab}_{FP} is defined by

$$L^{lab}_{FP}(t,z) := L^{lab}_{Liou}(t,z) + \vec{F}_{rad}(t,z) + \vec{Q}_{rad}(t,z) + \frac{1}{2}\sum_{i,j=1}^{3} \partial_{p_i}\partial_{p_j}\mathcal{E}_{ij}(t,z) \,, \tag{14}$$

where

$$L^{lab}_{Liou}(t,z) := -\partial_{\vec{r}}\cdot\frac{1}{m\gamma(\vec{p})}\vec{p} - \partial_{\vec{p}}\cdot[e\vec{E}(t,\vec{r}) + \frac{e}{m\gamma(\vec{p})}(\vec{p}\times\vec{B}(t,\vec{r}))] \,, \tag{15}$$

$$\vec{F}_{rad}(t,z) := -\frac{2}{3}\frac{e^4}{m^5\gamma(\vec{p})}|\vec{p}\times\vec{B}(t,\vec{r})|^2\vec{p} \,, \tag{16}$$

$$\vec{Q}_{rad}(t,z) := \frac{55}{48\sqrt{3}}\sum_{j=1}^{3}\frac{\partial[\lambda(t,z)\vec{p}p_j]}{\partial p_j} \,, \tag{17}$$

$$\mathcal{E}_{ij}(t,z) := \frac{55}{24\sqrt{3}}\lambda(t,z)p_ip_j \,, \quad \lambda(t,z) := \hbar\frac{|e|^5}{m^8\gamma(\vec{p})}|\vec{p}\times\vec{B}(t,\vec{r})|^3 \,, \tag{18}$$

$$\gamma(\vec{p}) := \frac{1}{m}\sqrt{|\vec{p}|^2+m^2} \,, \tag{19}$$

and with e being the electric charge of the electron or positron and $\vec{E}$ and $\vec{B}$ being the external fields.

The Fokker-Planck operator L^{lab}_{FP} whose explicit form is taken from Ref. 3 is a linear second-order partial differential operator and, with some additional approximations, is commonly used for electron synchrotrons and storage rings, see Ref. 27 and Section 2.5.4 in Ref. 21. As usual, since it is minuscule compared to all other forces, the Stern-Gerlach effect from the spin onto the orbit is neglected in (13).

The polarization density $\vec{\eta}_{lab}$ evolves via eq. 2 in Ref. 3, i.e., via that which we call the Full Bloch equation (FBE), namely

$$\partial_t \vec{\eta}_{lab} = L^{lab}_{\rm FP}(t,z)\vec{\eta}_{lab} + M(t,z)\vec{\eta}_{lab} + [1+\partial_{\vec{p}}\cdot\vec{p}]\lambda(t,z)\frac{1}{m\gamma(\vec{p})}\frac{\vec{p}\times\vec{a}(t,z)}{|\vec{a}(t,z)|}f_{lab}(t,z)\ , \tag{20}$$

where

$$M(t,z) := \Omega^{lab}(t,z) - \lambda(t,z)\frac{5\sqrt{3}}{8}[I_{3\times 3} - \frac{2}{9m^2\gamma^2(\vec{p})}\vec{p}\vec{p}^T]\ , \tag{21}$$

$$\vec{a}(t,z) := \frac{e}{m^2\gamma^2(\vec{p})}(\vec{p}\times\vec{B}(t,\vec{r}))\ . \tag{22}$$

The FBE was derived in Ref. 3 from the semiclassical approximation of quantum electrodynamics and it is a generalization, to the whole phase space, of the Baier-Katkov-Strakhovenko equation which just describes the evolution of polarization along a single deterministic trajectory.[28] Note also that, while the FBE was new in 1975, the orbital Fokker-Planck equation (13) was already known thanks to research of the 1950s, e.g., Schwinger's paper on quantum corrections to synchrotron radiation.[29] The skew-symmetric matrix $\Omega^{lab}(t,z)$ takes into account the Thomas-BMT spin-precession effect. Thus in the laboratory frame the Thomas-BMT-equation (2) reads as

$$\partial_t \vec{n}_{lab} = L^{lab}_{Liou}(t,z)\vec{n}_{lab} + \Omega^{lab}(t,z)\vec{n}_{lab}\ . \tag{23}$$

The quantum aspect of (13) and (20) is embodied by the factor $\hbar$ in $\lambda(t,z)$. For example $\vec{Q}_{rad}$ is a quantum correction to the classical radiation reaction force $\vec{F}_{rad}$. The terms $-\lambda(t,z)\frac{5\sqrt{3}}{8}\vec{\eta}_{lab}$ and $\lambda(t,z)\frac{1}{m\gamma(\vec{p})}\frac{\vec{p}\times\vec{a}(t,z)}{|\vec{a}(t,z)|}f_{lab}(t,z)$ take into account spin flips due to synchrotron radiation and encapsulate the Sokolov-Ternov effect. The term $\lambda(t,z)\frac{5\sqrt{3}}{8}\frac{2}{9m^2\gamma^2(\vec{p})}\vec{p}\vec{p}^T\vec{\eta}_{lab}$ encapsulates the Baier-Katkov correction, and the term $\partial_{\vec{p}}\cdot\vec{p}\,\lambda(t,z)\frac{1}{m\gamma(\vec{p})}\frac{\vec{p}\times\vec{a}(t,z)}{|\vec{a}(t,z)|}f_{lab}(t,z)$ encapsulates the kinetic-polarization effect. The only terms in (20) which couple the three components of $\vec{\eta}_{lab}$ are the Thomas-BMT term and the Baier-Katkov correction term.

As mentioned above, there exists a system of SDEs underlying (20) (for details, see Ref. 6). In particular, f_{lab} and $\vec{\eta}_{lab}$ are related to a spin-orbit density $\mathcal{P}_{lab} = \mathcal{P}_{lab}(t,z,\vec{s})$ via

$$f_{lab}(t,z) = \int_{\mathbb{R}^3} d\vec{s}\mathcal{P}_{lab}(t,z,\vec{s})\ , \tag{24}$$

$$\vec{\eta}_{lab}(t,z) = \int_{\mathbb{R}^3} d\vec{s}\vec{s}\mathcal{P}_{lab}(t,z,\vec{s})\ , \tag{25}$$

where $\mathcal{P}_{lab}$ satisfies the Fokker-Planck equation corresponding to the system of SDEs in Ref. 6. These SDEs can be used as the basis for a Monte-Carlo spin tracking algorithm, i.e., for Method 2 mentioned in Section 1 above. This would extend the standard Monte-Carlo spin tracking algorithms, which we mentioned in Section 2

above, by taking into account all physical effects described by (20), like the Sokolov-Ternov effect, the Baier-Katkov correction, the kinetic-polarization effect and, of course, spin diffusion.

If we ignore the spin flip terms and the kinetic-polarization term in the FBE then (20) simplifies to the RBE

$$\partial_t \vec{\eta}_{lab} = L^{lab}_{FP}(t,z)\vec{\eta}_{lab} + \Omega^{lab}(t,z)\vec{\eta}_{lab} \,. \tag{26}$$

The RBE models spin diffusion due to the effect of the stochastic orbital motion on the spin and thus contains those terms of the FBE which are related to the radiative depolarization rate τ^{-1}_{dep}. This effect is clearly seen in the SDEs (see, e.g., (28) and (29)).

4. The reduced Bloch equation in the beam frame

In the beam frame, i.e., in the accelerator coordinates y of Section 2, the RBE (26) becomes

$$\partial_\theta \vec{\eta} = L_{\rm FP}(\theta, y)\vec{\eta} + \Omega(\theta, y)\vec{\eta} \,. \tag{27}$$

Because the coefficients of $L_{\rm FP}$ are θ-dependent, the RBE (27) is numerically and analytically quite complex. So we first approximate it by treating the synchrotron radiation as a perturbation. Then, in order to solve it numerically for the large time intervals that we need, we address the system of SDEs underlying (27) and apply the refined averaging technique presented in Ref. 30 (see also 7), for the orbital dynamics, and extend it to include spin. The averaged SDEs are then used to construct an approximate RBE which we call the effective RBE.

The system of SDEs underlying (27) reads as [b]

$$\frac{dY}{d\theta} = (A(\theta) + \epsilon_R \delta A(\theta))Y + \sqrt{\epsilon_R}\sqrt{\omega(\theta)} e_6 \xi(\theta) \,, \tag{28}$$

$$\frac{d\vec{S}}{d\theta} = [\Omega_0(\theta) + \epsilon_S C(\theta, Y)]\vec{S} \,, \tag{29}$$

where the orbital dynamics has been linearized in Y and $\Omega = \Omega_0 + \epsilon_S C$ has been linearized in Y so that

$$C(\theta, Y) = \sum_{j=1}^{6} C_j(\theta) Y_j \,. \tag{30}$$

Also, $A(\theta)$ is the Hamiltonian part of the orbital dynamics, Y has been scaled so that ϵ_R is the size of the orbital effect of the synchrotron radiation. Thus $\epsilon_R \delta A(\theta)$ represents the orbital damping effects due to synchrotron radiation and the cavities, $\sqrt{\epsilon_R}\xi(\theta)$ represents the associated quantum fluctuations, ξ is the white noise process and $e_6 := (0,0,0,0,0,1)^T$. In the spin equation (29), Ω_0 is the closed-orbit contribution to Ω so that $\epsilon_S C(\theta, Y)$ is what remains and $C(\theta, Y)$ is chosen $O(1)$.

[b] We denote the dependent variables by capital letters.

Hence ϵ_S estimates the size of $\Omega - \Omega_0$. Both $\Omega_0(\theta)$ and $C(\theta, Y)$ are, of course, skew-symmetric 3×3 matrices. We are interested in the situation where ϵ_R and ϵ_S are small in some appropriate sense.

Eqs. (28) and (29) can be obtained by transforming the system of SDEs in Ref. 6 from the laboratory frame to the beam frame.[31] However, since in this section we only deal with the RBE (not with the FBE), (28) and (29) can also be found in older expositions on spin in high-energy electron storage rings, e.g., Ref. 32. Note that these expositions make approximations as for example with the linearity of (28) in Y and the linearity of $C(\theta, Y)$ in Y.

With (28) and (29) the evolution equation for the spin-orbit joint probability density $\mathcal{P} = \mathcal{P}(\theta, y, \vec{s})$ is the following Fokker-Planck equation

$$\partial_\theta \mathcal{P} = L_{\mathrm{FP}}(\theta, y)\mathcal{P} - \partial_{\vec{s}} \cdot \left(\left(\Omega(\theta, y)\vec{s} \right) \mathcal{P} \right) , \tag{31}$$

where $\Omega = \Omega_0 + \epsilon_S C$. The phase-space density f and the polarization density $\vec{\eta}$ corresponding to $\mathcal{P}$ are defined by

$$f(\theta, y) = \int_{\mathbb{R}^3} d\vec{s}\, \mathcal{P}(\theta, y, \vec{s}) , \quad \vec{\eta}(\theta, y) = \int_{\mathbb{R}^3} d\vec{s}\, \vec{s}\, \mathcal{P}(\theta, y, \vec{s}) , \tag{32}$$

which are the beam-frame analogs of (24) and (25). The local polarization vector $\vec{P}_{loc}$ from Section 2 above is related to f and $\vec{\eta}$ by

$$\vec{\eta}(\theta, y) = f(\theta, y)\vec{P}_{loc}(\theta, y) . \tag{33}$$

The RBE (27) follows from (31) by differentiating (32) w.r.t. θ and by using the Fokker-Planck equation for $\mathcal{P}$. This proves that (28) and (29) is the system of SDEs which underlie the RBE (27). For (27), see also Ref. 18.

5. The effective reduced Bloch equation in the beam frame

The effective RBE is, by definition, an approximation of the RBE (27) obtained by approximating the system of SDEs (28) and (29) by an effective system of SDEs using the method of averaging, see Refs. 7-12. In other words the system of SDEs underlying the effective RBE is the effective system of SDEs. We now discuss first-order averaging in the case where $\epsilon := \epsilon_S = \epsilon_R$ is small.

To apply the method of averaging to (28) and (29) we must transform them to a standard form for averaging, i.e., we must transform the variables $Y, \vec{S}$ to slowly varying variables. We do this by using a fundamental solution matrix X of the unperturbed $\epsilon = 0$ part of (28), i.e.,

$$X' = A(\theta)X , \tag{34}$$

and a fundamental solution matrix Φ of the unperturbed $\epsilon = 0$ part of (29), i.e.,

$$\Phi' = \Omega_0(\theta)\Phi . \tag{35}$$

We thus transform Y and $\vec{S}$ into the slowly varying U and $\vec{T}$ via

$$Y(\theta) = X(\theta)U(\theta) \ , \quad \vec{S}(\theta) = \Phi(\theta)\vec{T}(\theta) \ . \tag{36}$$

Hence (28) and (29) are transformed to

$$U' = \epsilon\mathcal{D}(\theta)U + \sqrt{\epsilon}\sqrt{\omega(\theta)}X^{-1}(\theta)e_6\xi(\theta) \ , \tag{37}$$

$$\vec{T}' = \epsilon\mathfrak{D}(\theta, U)\vec{T} \ , \tag{38}$$

where $\mathcal{D}$ and $\mathfrak{D}$ are defined by

$$\mathcal{D}(\theta) := X^{-1}(\theta)\delta A(\theta)X(\theta) \ , \tag{39}$$

$$\mathfrak{D}(\theta, U) := \Phi^{-1}(\theta)C(\theta, X(\theta)U)\Phi(\theta) \ . \tag{40}$$

Of course, (37) and (38) carry the same information as (28) and (29). Now, applying the method of averaging to (37) and (38), we obtain the following effective system of SDEs

$$V' = \epsilon\bar{\mathcal{D}}V + \sqrt{\epsilon}\mathcal{B}(\xi_1, ..., \xi_k)^T \ , \tag{41}$$

$$\vec{T}_a' = \epsilon\bar{\mathfrak{D}}(V)\vec{T}_a \ , \tag{42}$$

where the bar denotes θ-averaging, i.e., the operation $\lim_{T\to\infty}(1/T)\int_0^T d\theta \cdots$. Moreover $\xi_1, ..., \xi_k$ are statistically independent versions of the white noise process and $\mathcal{B}$ is a $6 \times k$ matrix which satisfies $\mathcal{B}\mathcal{B}^T = \bar{\mathcal{E}}$ with $k = rank(\bar{\mathcal{E}})$ and where $\bar{\mathcal{E}}$ is the θ-average of

$$\mathcal{E}(\theta) = \omega(\theta)X^{-1}(\theta)e_6e_6^TX^{-T}(\theta) \ . \tag{43}$$

For physically reasonable A and Ω the fundamental matrices X and Φ are quasiperiodic functions whence $\mathcal{D}, \mathfrak{D}(\cdot, U)$ and $\mathcal{E}$ are quasiperiodic functions so that their θ averages $\bar{\mathcal{D}}, \bar{\mathfrak{D}}(V)$ and $\bar{\mathcal{E}}$ exist.

Our derivation of (41) from (37) is discussed in some detail in Ref. 6. We are close to showing that $U = V + O(\epsilon)$ on θ-intervals of length $O(1/\epsilon)$ and it seems likely that this error is valid for $0 \leq \theta < \infty$, because of the radiation damping. This is a refinement of Ref. 30 and assumes a non-resonance condition. Since the sample paths of U are continuous and U is slowly varying it seems likely that $\vec{T}_a$ is a good approximation to $\vec{T}$ and we are working on the error analysis. Spin-orbit resonances will be an important focus in the construction of $\bar{\mathfrak{D}}(V)$ from (40) which contains both the orbital frequencies in X and the spin precession frequency in Φ.

Since, by definition, the effective system of SDEs underly the effective RBE, the latter can be obtained from the former in the same way as we obtained (27) from (28) and (29) (recall the discussion after (32)). Thus the evolution equation for the spin-orbit probability density $\mathcal{P}_V = \mathcal{P}_V(\theta, \mathrm{v}, \vec{t})$ is the following Fokker-Planck equation:

$$\partial_\theta\mathcal{P}_V = L^V_{\mathrm{FP}}(\mathrm{v})\mathcal{P}_V - \epsilon\partial_{\vec{t}}\cdot\left(\left(\bar{\mathfrak{D}}(\mathrm{v})\vec{t}\right)\mathcal{P}_V\right) \ , \tag{44}$$

where

$$L^V_{\rm FP}(\mathrm{v}) = -\epsilon \sum_{j=1}^{6} \partial_{\mathrm{v}_j} (\bar{\mathcal{D}}\mathrm{v})_j + \frac{\epsilon}{2} \sum_{i,j=1}^{6} \bar{\mathcal{E}}_{ij} \partial_{\mathrm{v}_i} \partial_{\mathrm{v}_j} \ . \tag{45}$$

The polarization density $\vec{\eta}_V$ corresponding to $\mathcal{P}_V$ is defined by

$$\vec{\eta}_V(\theta, \mathrm{v}) = \int_{\mathbb{R}^3} d\vec{t}\vec{t}\, \mathcal{P}_V(\theta, \mathrm{v}, \vec{t}) \ , \tag{46}$$

so that, by (44), the effective RBE is

$$\partial_\theta \vec{\eta}_V = L^V_{\rm FP}(\mathrm{v}) \vec{\eta}_V + \epsilon \bar{\mathfrak{D}}(\mathrm{v}) \vec{\eta}_V \ . \tag{47}$$

This then is the focus of our approach in Method 1. For more details on this section, see Refs. 6, 16 and 17.

6. Next steps

- Further development of Bloch equation approach (numerical and theoretical), i.e., of Method 1 and with a realistic lattice.
- Development of validation methods, i.e., Methods 2-4. Note that Method 2 is an extension of the standard Monte-Carlo spin tracking algorithms and for that matter we will study Refs. 13, 14 and 15.
- Comparing the Bloch equation approach with the standard approach using the Derbenev-Kondratenko formulas, in particular the study of potential correction terms[4] to $\tau^{-1}_{\rm DK}$ by using the RBE.

7. Acknowledgement

This material is based upon work supported by the U.S. Department of Energy, Office of Science, Office of High Energy Physics, under Award Number DE-SC0018008.

References

1. K. Heinemann, *Re-evaluation of Spin-Orbit Dynamics of Polarized e^+e^- Beams in High Energy Circular Accelerators and Storage Rings: Bloch equation approach*, Invited talk at IAS Mini-Workshop on Beam Polarization in Future Colliders, Hong Kong, Jan. 17, 2019. See also: `http://iasprogram.ust.hk/hep/2019`.
2. Ya.S. Derbenev, A.M. Kondratenko, *Polarization kinetics of particles in storage rings*, *Sov. Phys. JETP*, vol. 37, p. 968, 1973.
3. Ya.S. Derbenev, A.M. Kondratenko, *Relaxation and equilibrium state of electrons in storage rings*, *Sov. Phys. Dokl.*, vol. 19, p. 438, 1975.
4. Z. Duan, M. Bai, D.P. Barber, Q. Qin, *A Monte-Carlo simulation of the equilibrium beam polarization in ultra-high energy electron (positron) storage rings*, Nucl. Instr. Meth. A793 (2015), pp.81-91. Available also at arXiv.
5. F. Bloch, *Phys. Rev.*, vol. 70, p. 460, 1946.
6. K. Heinemann, D. Appelö, D. P. Barber, O. Beznosov, J.A. Ellison, *The Bloch equation for spin dynamics in electron storage rings: computational and theoretical aspects.* Invited paper for ICAP18. Int. J. Mod. Phys. A, 1942032 (2019).

7. J.A. Ellison, H. Mais, G. Ripken, K. Heinemann, *Details of Orbital Eigen-analysis for Electron Storage Rings*. (An extension of the article in Ref. 30. In preparation).
8. J. A. Ellison and H-Jeng Shih, *The Method of Averaging in Beam Dynamics* in Accelerator Physics Lectures at the Superconducting Super Collider, AIP Conference Proceedings 326, edited by Yiton T. Yan, James P. Naples and Michael J. Syphers (1995).
9. J. A. Ellison, K. A. Heinemann, M. Vogt and M. Gooden, *Planar undulator motion excited by a fixed traveling wave: Quasiperiodic Averaging, normal forms and the FEL pendulum*, Phys. Rev. ST Accel. Beams 16, 090702 (2013). An earlier version is on the archive at arXiv:1303.5797 (2013) and published as DESY report 13-061.
10. J.A. Sanders, F. Verhulst, J. Murdock, *Averaging Methods in Nonlinear Dynamical Systems*, 2nd Edition, Springer, New York (2007).
11. J. Murdock, *Perturbations: Theory and Methods*, SIAM, Philadelphia, 1999.
12. R. Cogburn and J.A. Ellison, *A Stochastic Theory of Adiabatic Invariance*, Communications of Mathematical Physics, Vol. 148 (1992) pp. 97-126. Also: *A four-thirds law for phase randomization of stochastically perturbed oscillators and related phenomena*, Communications of Mathematical Physics, Vol. 166 (1994) pp. 317-336.
13. P.E. Kloeden, E. Platen, *Numerical Solution of Stochastic Differential Equations*, Vol. 23 of Appl. Math., Springer. Third printing, 1999 (first edition (1992)).
14. P.E. Kloeden, E. Platen, H. Schurz, *Numerical Solution of SDEs Through Computer Experiments*, Universitext, Springer. Third corrected printing, 2003 (first edition (1994)).
15. E. Platen, N. Bruti-Liberati, *Numerical Solution of Stochastic Differential Equations with Jumps in Finance*, Springer, 2010.
16. K. Heinemann, D. Appelö, D. P. Barber, O. Beznosov, J. A. Ellison, *Spin dynamics in modern electron storage rings: Computational and theoretical aspects*, ICAP18, Key West, Oct 19–23, 2018.
17. O. Beznosov, J. A. Ellison, K. Heinemann, D. P. Barber, D. Appelö, *Spin dynamics in modern electron storage rings: Computational aspects*, ICAP18, Key West, Oct 19–23, 2018.
18. K. Heinemann, D.P. Barber, *Spin transport, spin diffusion and Bloch equations in electron storage rings*, *Nucl. Instr. Meth. A*, vol. 463, pp. 62–67, 2001.Erratum-ibid.A469:294.
19. D. Sagan, *A Superconvergent Algorithm for Invariant Spin Field Stroboscopic Calculations David Sagan*, IPAC18, Vancouver, April 29-May 4, 2018.
20. J.A. Ellison, K. Heinemann, *Polarization Fields and Phase Space Densities in Storage Rings: Stroboscopic Averaging and the Ergodic Theorem*, Physica D 234, 131 (2007).
21. *Handbook of Accelerator Physics and Engineering*, 1st edition, third printing, edited by A.W. Chao, M. Tigner, 2006. See also the 2nd edition.
22. F. Meot, Zgoubi. `http://sourceforge.net/projects/zgoubi`
23. E. Forest, *From Tracking Code to Analysis*, Springer, 2006.
24. D. Sagan, *Bmad, a subroutine library for relativistic charged-particle dynamics*. See: `https://www.classe.cornell.edu/bmad`
25. S.R. Mane, Yu. M. Shatunov, K. Yokoya, Rep. Prog. Phys. 68, 1997 (2005).
26. D.P. Barber, G. Ripken, *Handbook of Accelerator Physics and Engineering*. Eds. A.W. Chao and M. Tigner, 1st edition, 3rd printing, World Scientific, 2006. See also arXiv:physics/9907034v2.
27. M. Sands, *The physics of electron storage rings*, SLAC-121, 1970. J. Jowett, *Introductory Statistical Mechanics for electron storage rings*, SLAC-PUB-4033, 1986.
28. V.N. Baier, V.M. Katkov, V.M. Strakhovenko, *Kinetics of Radiative Polarization*, *Sov.*

Phys. JETP, vol. 31, p. 908, 1970.
29. J. Schwinger, Proc.Nat.Acad.Sci.,40,132, 1954.
30. J.A. Ellison, H. Mais, G. Ripken, *Orbital Eigen-analysis for Electron Storage Rings*, in: Section 2.1.4 of *Handbook of Accelerator Physics and Engineering*. Eds. A.W. Chao, K.H. Mess, M. Tigner, F. Zimmermann, 2nd edition, World Scientific, 2013.
31. K. Heinemann, D. P. Barber, O. Beznosov, J. A. Ellison. To be published, 2019.
32. D.P. Barber, K. Heinemann, H. Mais, G. Ripken, *A Fokker-Planck treatment of stochastic particle motion*, DESY-91-146, 1991.