I. ORBITAL EIGEN-ANALYSIS FOR ELECTRON STORAGE RINGS

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a. Basic problem A general 6-D formalism is presented for the calculation of the bunch parameters (e.g. 6-D stationary beam-envelope matrix) for electron storage rings including radiation damping and quantum excitation. The problem is formulated in terms of a stochastic differential equation (SDE) and basic to our approach is the orbital eigen-analysis first introduced in [1]. The latter gives a more general framework than that of Courant-Synder. An SDE approach was first introduced in [2] and developed further in [3]. At a later stage the work of [4] was incorporated and the starting point here is the SDE in Frenet-Serret coordinates with respect to a design orbit as described in [5]. The 6D-SDE is

$$\vec{x}' \;=\; \mathbf{A}(s)\vec{x} + \vec{c}(s) + \epsilon\vec{g}(\vec{x},s) + \epsilon^{\frac{1}{2}}\sqrt{b(s)}\xi(s)\vec{e}_6, \quad (1)$$

where $\vec{x} = (x, p_x, y, p_y, z, p_z)^T$. We have expanded up to second order in the dynamical variables and retained only the leading nonlinearities due to sextupoles and due to radiation effects in quadrupoles. All functions except ξ are C-periodic in s where C is the ring circumference. The matrix $\mathbf{A}(s)$ is Hamiltonian and gives rise to the linear symplectic synchro-betatron oscillations. The $\vec{x}$-independent term $\vec{c}$ describes closed orbit distortions induced by dipole field errors and by the fact that the energy losses in the bending magnets and quadrupoles are not replaced at the location in the ring where they occur. Our analysis is perturbative. The parameter ϵ is inserted to indicate the perturbation size and to discuss the perturbation procedure and the nature of the error estimates. In the end, ϵ can be taken to be one in applications of the formulas. The first perturbation term is $\vec{g}(\vec{x},s) = \delta\mathbf{A}(s)\vec{x} + \vec{f}(\vec{x},s)$. Here $\delta\mathbf{A}(s)$ models both the energy losses from radiation and the energy gain in the rf cavities, and $\vec{f}(\vec{x},s)$ takes into account the nonlinear terms due to sextupoles and due to radiation effects in quadrupoles. The last term in (1) simulates the stochastic excitation of the particle motion due to the quantum nature of the radiation. Here ξ is Gaussian white noise, b is an amplitude function proportional to $\hbar$ and $\vec{e}_k$ is the unit vector with 1 in the k-component, thus the stochastic excitation only affects the p_z component directly. The explicit form of these quantities can be found in [5] and details of our analysis below will be given in [6].

The main quantity of interest is the N particle random bunch density

$$\rho_N(\vec{x},s) = \frac{1}{N}\sum_{n=1}^{N}\delta(\vec{x} - \vec{x}_n(s)), \quad (2)$$

where the $\vec{x}_n(s)$ are independent and identically distributed random variables determined by (1). Let p be the single particle probability density defined by (1), then $\langle\rho_N\rangle = p$. Here, and in the following, angular brackets will denote the expected value of stochastic quantities. We will assume for large N that $\rho_N(\vec{x},s) \approx p(\vec{x},s)$, in a coarse grained sense. This article presents an analysis of this single particle probability density.

Eq.(1) will be analyzed in two steps: (i) the (periodic) 6-D closed orbit $\vec{x}_{co}$, satisfying (1) with $b = 0$ will be given by a solution of an integral equation; and (ii) the equation will be linearized around $\vec{x}_{co}$ and the linearized equation analyzed.

b. The equation $\vec{x}' = \mathbf{A}(s)\vec{x}$ and its Eigen-FSM The solutions of the linear periodic Hamiltonian system

$$\vec{x}\,' = \mathbf{A}(s)\vec{x},\ \mathbf{A}^T\mathbf{J} + \mathbf{J}\mathbf{A} = 0,\ \mathbf{A}(s+C) = \mathbf{A}(s), \quad (3)$$

are central to our analysis. Here $\mathbf{J} = \mathrm{diag}(\mathbf{J}_2, \mathbf{J}_2, \mathbf{J}_2)$ is the unit symplectic matrix where

$$\mathbf{J}_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The transfer map, $\mathbf{M}(s, s_0)$, is a fundamental solution matrix (FSM) which satisfies $\mathbf{M}(s_0, s_0) = I$ (often called the principal solution matrix). The basic properties are (P1) $\mathbf{M}(s_2, s_1)\mathbf{M}(s_1, s_0) = \mathbf{M}(s_2, s_0)$ (semigroup property), (P2) $\mathbf{M}^T\mathbf{J}\mathbf{M} = \mathbf{J}$ (symplecticity) and (P3) $\mathbf{M}(s + C, s_0 + C) = \mathbf{M}(s, s_0)$ (periodicity).

We assume solutions of (3) are stable. This is the case if and only if the monodromy matrix, $\mathbf{M}(C, 0)$, has six linearly independent eigenvectors and its eigenvalues (*characteristic multipliers*), ρ_k, have modulus 1 (see e.g. [7]). In addition to stability we assume that the characteristic multipliers are distinct. It follows that ± 1 are not multipliers and that a tune vector $\vec{\nu}$ (vector of *characteristic exponents*) can be chosen such that $\rho_k = \exp(2\pi\imath\nu_k), \nu_k \in (-\frac{1}{2}, 0) \cup (0, \frac{1}{2})$. Because the multipliers are distinct and come in complex conjugate pairs, we can organize them so that the tunes satisfy

$$0 < \nu_1 < \nu_3 < \nu_5 < 1/2,\ \ \nu_{2l} = -\nu_{2l-1},\ l = 1, 2, 3. \quad (4)$$

The matrix $\mathbf{W} := [\vec{w}_1, \cdots, \vec{w}_6]$, of associated eigenvectors, is chosen such that $\vec{w}_{2l} = \vec{w}^*_{2l-1}$. The eigenvectors satisfy $\vec{w}_j^H\mathbf{J}\vec{w}_k = \imath\delta_{jk}\gamma_k$, where H denotes conjugate transpose, δ_{jk} is the Kronecker delta. The γ_k are real with γ_{2l-1} and γ_{2l} of opposite sign; the sign is not known appriori. We assume $\gamma_k \neq 0$ and normalize the eigenvectors so that $\vec{w}^H_{2l-1}\mathbf{J}\vec{w}_{2l-1} = \imath\gamma_{2l-1},\ \gamma_{2l-1} = \pm 1$. Thus $\vec{w}^H_{2l}\mathbf{J}\vec{w}_{2l} = \imath\gamma_{2l}$, where $\gamma_{2l} = -\gamma_{2l-1}$, and in summary

$$\mathbf{W}^H\mathbf{J}\mathbf{W} = \imath\tilde{\mathbf{I}}, (\tilde{\mathbf{I}})_{jk} = \delta_{jk}\gamma_j,\ \gamma_{2l} = -\gamma_{2l-1} = \pm 1. \quad (5)$$

The phase space density of a bunch can be efficiently determined in terms of the Eigen-FSM for (3),

$$\mathbf{\Psi}(s) := \mathbf{M}(s, 0)\mathbf{W}, \quad (6)$$

which was first introduced in [1] and generalizes the Courant-Synder formalism. It follows from (5) and (P2) that

$$\mathbf{\Psi}^H(s)\mathbf{J}\mathbf{\Psi}(s) = \imath\tilde{\mathbf{I}} \text{ and thus } \mathbf{\Psi}^{-1}(s) = -\imath\tilde{\mathbf{I}}\mathbf{\Psi}^H(s)\mathbf{J}. \quad (7)$$

Define $\hat{\mathbf{\Psi}}$ by

$$\mathbf{\Psi}(s) =: \hat{\mathbf{\Psi}}(s)\exp(\imath\omega s\mathbf{N}),\ \omega := 2\pi/C,\ \mathbf{N} := \mathrm{diag}(\vec{\nu}), \quad (8)$$

then $\hat{\mathbf{\Psi}}$ is C-periodic and (8) is a Floquet representation of the Eigen-FSM. Furthermore $\hat{\mathbf{\Psi}}$ satisfies (7). The k-th column of $\hat{\mathbf{\Psi}}$ will be denoted by $\vec{\hat{\psi}}_k$.

c. Closed orbit associated with (1) If no characteristic multiplier of (3) is 1, which we have assumed, then (1), with $b = 0$, has a unique C-periodic solution for ϵ sufficiently small. This is referred to as the closed orbit, is denoted by $\vec{x}_{co}(s,\epsilon)$, and satisfies the integral equation

$$\begin{aligned}\vec{x}_{co}(s,\epsilon) &= \vec{x}_{co}(s,0)\\ &+ \epsilon\int_0^C \mathbf{G}(s,t)\vec{g}(\vec{x}_{co}(s+t,\epsilon),s+t)dt. \quad (9)\end{aligned}$$

Here the Green function is given by $\mathbf{G}(s,t) = (\mathbf{M}(s,s+C)-\mathbf{I})^{-1}\mathbf{M}(s,s+t)$ and $\vec{x}_{co}(s,0) = \int_0^C \mathbf{G}(s,t)\vec{c}(s+t)dt$. The closed orbit can be determined approximately by iterating (9). See Thm.2.1, p.154 of [8].

d. Linearized motion around closed orbit Let $\vec{y}$ be defined by $\vec{x} =: \vec{x}_{co} + \vec{y}$ then for small $\vec{y}$, $\vec{y} \approx \vec{y}_L$ where $\vec{y}_L$ satisfies the linearized equation for $\vec{y}$, namely

$$\vec{y}_L{}' = (\mathbf{A}(s) + \epsilon\mathbf{B}(s))\vec{y}_L + \epsilon^{\frac{1}{2}}\sqrt{b(s)}\xi(s)\vec{e}_6. \quad (10)$$

Here $\mathbf{B}(s) := D_1\vec{g}(\vec{x}_{co}(s),s)$ is the Jacobian matrix of $\vec{g}(\cdot,s)$. The most important information about the bunch is contained in the moment (beam-envelope) matrix $\mathbf{U}(s) = \langle\vec{y}_L(s)\vec{y}_L^T(s)\rangle$. We now determine an approximation to $\mathbf{U}$.

e. Equation for moment matrix and averaging approximation Let $\mathbf{\Psi}$ be an FSM for $\mathbf{A}$, e.g. the Eigen-FSM, then the transformation $\vec{y}_L \to \vec{z}$ via $\vec{y}_L =: \mathbf{\Psi}(s)\vec{z}$ gives

$$\begin{aligned}&\vec{z}\,' = \epsilon\mathbf{D}(s)\vec{z} + \epsilon^{\frac{1}{2}}\xi(s)\vec{d}(s),\ \ \vec{d}(s) = \sqrt{b(s)}\mathbf{\Psi}^{-1}(s)\vec{e}_6,\\ &\mathbf{D}(s) := \mathbf{\Psi}^{-1}(s)\mathbf{B}(s)\mathbf{\Psi}(s), \quad (11)\end{aligned}$$

with initial condition $\vec{z}(0) = \mathbf{\Psi}^{-1}(0)\vec{y}_L(0)$. Now $\mathbf{U} = \mathbf{\Psi}\mathbf{V}\mathbf{\Psi}^H$ where $\mathbf{V} = \langle\vec{z}\,\vec{z}^H\rangle$ and the differential equation for $\mathbf{V}$ is

$$\mathbf{V}' = \epsilon[\mathbf{D}(s)\mathbf{V} + \mathbf{V}\mathbf{D}^H(s) + \mathbf{E}(s)], \quad (12)$$

where $\mathbf{E}(s) = \vec{d}(s)\vec{d}^H(s)$. This equation can be derived by writing the solution of (11) in terms of an FSM for $\mathbf{D}(s)$, calculating $\mathbf{V}$ from this solution, and then noting that it satisfies (12).

Let f be a quasiperiodic function, scalar, vector or matrix, then we define its average $\bar{f}$ by

$$\bar{f} = \lim_{L\to\infty}\frac{1}{L}\int_0^L f(s)ds. \quad (13)$$

We apply averaging methodology (See Remark 2) to (12) and we obtain $\mathbf{V}(s) \approx \mathbf{V}_a(s)$ where

$$\mathbf{V}_a' = \epsilon[\bar{\mathbf{D}}\mathbf{V}_a + \mathbf{V}_a\bar{\mathbf{D}}^H + \bar{\mathbf{E}}]. \quad (14)$$

The nature of the approximation is mentioned in the remarks. If an equilibrium (constant) solution, $\mathbf{V}_{ae}$, of (14) exists then the solution can be written

$$\mathbf{V}_a(s) = \exp(\epsilon\bar{\mathbf{D}}s)(\mathbf{V}_0 - \mathbf{V}_{ae})\exp(\epsilon\bar{\mathbf{D}}^H s) + \mathbf{V}_{ae}, \quad (15)$$

where $\mathbf{V}_0 = \mathbf{V}(0)$. Since $\mathbf{E}$ is proportional to $\hbar$ so is the equilibrium solution $\mathbf{V}_{ae}$.

To determine an equilibrium solution we first assume that $\bar{\mathbf{D}}$ has a full set of linearly independent eigenvectors. Let $\mathbf{X}$ be the matrix of eigenvectors and $\mathbf{\Lambda}$ the diagonal matrix of eigenvalues, λ_j, so that $\bar{\mathbf{D}}\mathbf{X} = \mathbf{X}\mathbf{\Lambda}$. Let $\mathbf{F}$ be defined by $\mathbf{V}_{ae} =: \mathbf{X}\mathbf{F}\mathbf{X}^H$ then (14) gives $\Lambda\mathbf{F} + \mathbf{F}\Lambda^* = -\mathbf{X}^{-1}\bar{\mathbf{E}}\mathbf{X}^{-H}$ from which

$$\mathbf{F}_{jk} = -\frac{1}{\lambda_j + \lambda_k^*}(\mathbf{X}^{-1}\bar{\mathbf{E}}\mathbf{X}^{-H})_{jk}. \quad (16)$$

Thus $\mathbf{V}_{ae}$ is defined and unique as long as $\lambda_j + \lambda_k^* \neq 0$ for all j and k. When $\bar{\mathbf{D}}$ is diagonal the formula simplifies since $\mathbf{X}$ can be chosen to be the identity.

We have thus determined the approximation

$$\mathbf{U}(s) \approx \mathbf{U}_a(s) := \mathbf{\Psi}(s)\mathbf{V}_a(s)\mathbf{\Psi}^H(s), \quad (17)$$

to the moment matrix.

f. Calculation of $\bar{\mathbf{E}}$ and $\bar{\mathbf{D}}$ for the Eigen-FSM under a non-resonance condition From (7), (8), (11), and (12)

$$\begin{aligned}&\mathbf{D}(s) = \exp(-\imath\omega s\mathbf{N})\hat{\mathbf{D}}(s)\exp(\imath\omega s\mathbf{N}),\\ &\hat{\mathbf{D}}(s) = -\imath\tilde{\mathbf{I}}\hat{\mathbf{\Psi}}^H(s)\mathbf{J}\mathbf{B}(s)\hat{\mathbf{\Psi}}(s),\end{aligned}$$

and

$$\begin{aligned}&\mathbf{E}(s) = \vec{d}(s)\vec{d}^H(s) = \exp(-\imath\omega s\mathbf{N})\hat{\mathbf{E}}(s)\exp(\imath\omega s\mathbf{N}),\\ &\hat{\mathbf{E}}(s) = b(s)\tilde{\mathbf{I}}\hat{\mathbf{\Psi}}^H(s)\mathrm{diag}(\vec{e}_5)\hat{\mathbf{\Psi}}(s)\tilde{\mathbf{I}}.\end{aligned}$$

Note that $(\mathbf{D}(s))_{jk} = \exp[-\imath\omega(\nu_j - \nu_k)s]\hat{\mathbf{D}}(s)_{jk}$ and $\hat{\mathbf{D}}$ is C-periodic, the same being true for $\mathbf{E}$.

We have assumed the tune condition (4), thus the averages $\bar{\mathbf{D}}$ and $\bar{\mathbf{E}}$ are diagonal and are given by

$$\bar{\mathbf{D}}_{jj} = \lambda_j = -\imath\gamma_j\overline{\vec{\hat{\psi}}_j^H(s)\mathbf{J}\mathbf{B}(s)\vec{\hat{\psi}}_j(s)}, \quad (18)$$

$$\bar{\mathbf{E}}_{jj} = \overline{b(s)|\hat{\mathbf{\Psi}}(s)_{5j}|^2}. \quad (19)$$

g. Summary and dissipative case The stochastic process $\vec{x}$, defined by (1), is given in a linear approximation by $\vec{x}_L(s) := \vec{x}_{co}(s) + \vec{y}_L(s)$. The moment matrix of $\vec{y}_L$ is $\mathbf{U} = \mathbf{\Psi}\mathbf{V}\mathbf{\Psi}^H$, where $\mathbf{\Psi}$ is defined by (6), with its Floquet representation given in (8), and $\mathbf{V}$ is defined by (12). An averaging approximation gives $\mathbf{V} \approx \mathbf{V}_a$ where $\mathbf{V}_a$ is given by (15) and paragraph f. More specifically

$$\begin{aligned}&\mathbf{V}_a(s)_{jk} =\\ &\exp[\epsilon(\lambda_j + \lambda_k^*)s](\mathbf{V}_{0\,jk} + \delta_{jk}\mathbf{V}_{ae\,jj}) + \delta_{jk}\mathbf{V}_{ae\,jj}, \quad (20)\\ &\mathbf{V}_{ae\,jj} = -\bar{\mathbf{E}}_{jj}/2\Re\lambda_j. \quad (21)\end{aligned}$$

Here λ_j and $\bar{\mathbf{E}}_{jj}$ are given by (18) and (19) respectively and we note that $\lambda^*_{2l} = \lambda_{2l-1}$.

For $\Re\lambda_j < 0$, the dissipative case, $\mathbf{V}_a(s) \to \mathbf{V}_{ae}$ as $s \to \infty$ and the approximate moment matrix in (17) becomes $\mathbf{U}_a(s) = \mathbf{\Psi}(s)\mathbf{V}_{ae}\mathbf{\Psi}^H(s) = \hat{\mathbf{\Psi}}(s)\mathbf{V}_{ae}\hat{\mathbf{\Psi}}^H(s)$. Using (18), (19) and (21), this stationary, C-periodic $\mathbf{U}_a$ can be written

$$\mathbf{U}_a(s)_{mn} = \sum_{l=1,2,3} G_l \Re[(\vec{\hat{\psi}}_{2l-1}(s))_m (\vec{\hat{\psi}}^*_{2l-1}(s))_n],$$

$$G_l = -\frac{2}{\alpha_l}\frac{1}{C}\int_0^C b(s)|\vec{\hat{\psi}}_{2l-1}(s)_5)|^2 ds. \qquad (22)$$

The quantities $\alpha_l := 2\Re\lambda_{2l-1}$ are called the damping constants and are given by

$$\Re\lambda_{2l-1} = \Re\lambda_{2l} = \gamma_{2l-1}\frac{1}{C}\int_0^C \Im[\vec{\hat{\psi}}^H_j(s)\mathbf{J}\mathbf{B}(s)\vec{\hat{\psi}}_j(s)]ds < 0.$$

h. Remarks

1. Since $\mathbf{D}$ and $\mathbf{B}$ are related by the similarity transformation of (11), Tr$\mathbf{D}(s)$ =Tr$\mathbf{B}(s)$ and thus the C-periodicity of $\mathbf{B}$ gives

$$\sum_1^6 \lambda_j = \mathrm{Tr}\bar{\mathbf{D}} = \frac{1}{C}\int_0^C \mathrm{Tr}D_1 g(x_{co}(s), s)ds = 2\frac{U_0}{E_0}, \qquad (23)$$

where U_0 and E_0 are the energy gain in the cavity and beam energy respectively. The λ_j form complex conjugate pairs so that $\sum_1^3 \alpha_l = \sum_1^6 \lambda_j$ and thus (23) is the Robinson sum rule, [9].

2. The averages of $\mathbf{E}$ and $\mathbf{D}$ were computed under the non-resonance condition of (4). However the standard averaging error bound $|\mathbf{V}(s) - \mathbf{V}_a(s)| < O(\epsilon)$ for $0 \le s < O(1/\epsilon)$ requires a sufficient, $O(1)$, separation between $0, \nu_1, \nu_3, \nu_5$, and $1/2$. In the dissipative case, the s-interval of validity of the averaging approximation can be extended to all $s \ge 0$. Details are given in [6]. The resonant case is considered in [10].

3. The mean of $\vec{y}_L$ is easily handled and in the averaging approximation $\langle\vec{y}_L(s)\rangle \approx \mathbf{\Psi}(s)\exp(\epsilon\bar{\mathbf{D}}s)\mathbf{\Psi}^{-1}(0)\langle\vec{y}_L(0)\rangle$ and the covariance matrix is easily computed. Of course, in the dissipative case the mean approaches zero for large s. If $\vec{y}_L(0)$ is a Gaussian random vector, then $\vec{y}_L$ is a Gaussian process. If in addition $\vec{y}_L(0)$ has zero mean then $\vec{y}_L$ is zero mean and the approximate bunch density is

$$(2\pi)^{-3}(\det\mathbf{U}_a(s))^{-1/2}\exp\{-\vec{y}_L^T\mathbf{U}_a(s)^{-1}\vec{y}_L/2\}.$$

4. Spin-orbit motion in electron storage rings and especially spin diffusion due to the quantum fluctuations can be treated approximately by introducing an 8-D matrix formalism. See Secs.2.7.7 and 2.7.8. (which may have changed). The codes SLIM and SLICK discussed in Sec. 2.7.8 are based on the orbital eigen-analysis of this section and can be used to calculate beam polarization, as well as the orbital dynamics of this section.

5. A general formalism for treating the linear electron beam dynamics with radiation effects taken into account is also presented in [11] and [12]. Whereas [11] starts from a kinetic description (Fokker-Planck equation) [12] uses (as we do) the SDE for the particle motion. The second order moments (beam envelopes) are calculated directly (rather than using the orbital-FSM) and these results are used in the computer code SAD. Furthermore generalized radiation integrals are derived which in the limiting case of a completely uncoupled machine reduce to the well known results of Sands, [13] (see Section 3.1.4.1). This is also true in our case if we separate the six dimensional dynamics into fast betatron and slow synchrotron components via the dispersion (see Section 2.7.8, p.181).

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