A detailed and unified treatment of spin-orbit systems using tools distilled from the theory of bundles. Part I*

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Abstract

We return to our study [BEH] of invariant spin fields and spin tunes for polarized beams in storage rings but in contrast to the continuous-time treatment in [BEH], we now employ a discrete-time formalism, beginning with the Poincaré maps of the continuous time formalism. As in [BEH] we focus on the spin-vector dynamics which is sufficient for spin-1/2 particles whence again the emphasis is on the notions of invariant spin field, spin tune and invariant frame field. Thet transformations of spin-orbit systems in [BEH] is here extended to a transformation theory involving the notion of H-normal form where H is an arbitrary subgroup of $SO(3)$. Thus the notions of spin tune and invariant frame field can be subsumed under the notion of H-normal form. As in [BEH] we study the impact of the spin tunes on the spectral behavior of the spin motion using the concept of quasiperiodicity. We also show via two examples how the absence of spin tunes impacts the the spectral behavior of the spin motion. As in [BEH], the particle motion is integrable but we now allow for nonlinear particle motion on the torus. Moreover we distinguish between the angle variable z on the torus and the angle variable ϕ on $\mathbb{R}^d$, the latter being used in [BEH]. Since we use many topological properties we here focus on z. This work is inspired by notions from the theory of bundles which will come into play in our follow-up work.

Contents

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1 Introduction

In [BEH] we undertook an extensive study of the concept of spin tune in storage rings on the basis of the Thomas–Bargmann–Michel–Telegdi (T–BMT) equation [Ja] of spin precession. This naturally included a discussion of the invariant spin field and the invariant frame field. This work and the follow-up work are a sequel to [BEH] based largely on mathematical concepts and ideas in the PhD Thesis [He2] of the first author (KH), where a method from Dynamical-Systems Theory is exploited to distil some essential features of particle-spin motion in storage rings. As to be seen in the follow-up work this method clarifies and considerably extends the current theory of [BEH]. In fact it generalizes the concepts of invariant frame field, spin tune, spin-orbit resonances, invariant polarization field and invariant spin field to an arbitrary subgroup H of $SO(3)$ by using the concept of H-normal form. This leads us to the Normal Form Theorem and various theorems which generalize some standard theorems that are also presented in this work. For short versions of the follow-up work, see [HBEV1, HBEV2].

In [BEH] we assumed the particle motion to be independent of the spin, i.e., we neglected the Stern-Gerlach force. Also, the particle motion was described by an integrable Hamiltonian system in action-angle variables, $J, \phi \in \mathbb{R}^d$. We further assumed that the electric and magnetic fields were of class C^1, i.e., continuously differentiable) both in ϕ and θ. Thus the T–BMT equation became a linear system of ordinary differential equations (ODE) for the particle-spin-vector motion with smooth coefficients depending quasiperiodically on θ. This quasiperiodic structure led us to a generalization of the Floquet theorem and a new approach to the spin tune.

Although accelerator physicists tend to concentrate on studying specific models of particle-spin motion in real storage rings, many of the issues surrounding the spin tune and the invariant spin field depend just on the *structure* of the equations of particle-spin motion and can be treated in general terms. This is the strategy to be adopted here and it clears the way for the focus on purely mathematical matters and in particular for the exploitation of methods from Dynamical-Systems Theory and the theory of bundles.

In storage-ring physics there are two main approaches for dealing with the independent variable in the equations of motion (EOM), namely use of the flow formalism or the map formalism. In the flow formalism the EOM is an ODE, whence the independent variable is the continuous variable $\theta \in \mathbb{R}$ describing the distance around the ring. In the map formalism the independent variable in the EOM is the discrete variable $n \in \mathbb{Z}$ labelling the turn number where $\mathbb{Z}$ denotes the set of integers. In Dynamical-Systems Theory it is common practice to refer to the independent variable in the EOM, such as θ, the "time" and that is the convention that we will use here. Thus there is a continuous-time and a discrete-time formalism. In [BEH] we used the former, here the emphasis is on discrete time. Nevertheless it would be possible to present the machinery of this work in the continuous-time formalism.

The external electrodynamic fields inside an accelerator's vacuum chamber are smooth, i.e., of class C^∞. So the C^1 assumption adopted in [BEH] appears to be perfectly reasonable. On the other hand, practical numerical spin–orbit tracking simulations are usually carried out with fields which cut off sharply at the ends of magnets and/or with thin-lens approximations. Thus in [BEH] our formalism involved class C^1 in the time variable θ although numerical calculations cited there in Sec. X had been obtained using hard-edged and thin-lens fields. However, hard-edged and thin-lens ring elements fit naturally into the discrete-time formalism. In particular, for this, we merely require that the fields are continuous (i.e., of class C^0) in the orbital phases and we allow jump discontinuities in θ. Of course, this still allows study of systems with fields smooth in θ and/or the orbital phases. The way that the discrete-time formalism derives from the continuous-time formalism is explained in Section 2.1.

This work is designed so that it can be read independently of [BEH]. However, we wish to avoid repeating the copious contextual material contained in [BEH]. We therefore invite the reader to consult the Introduction and the Summary and Conclusion in [BEH] in order to acquire a better appreciation of the context. In this work, as in [BEH], the particle motion is integrable and we allow the number of angle variables, d, to be arbitrary (but ≥ 1) although for particle-spin motion in storage rings, the case $d = 3$ is the most important. We use the symbols $\phi = (\phi_1, ..., \phi_d)^t$, $J = (J_1, ..., J_d)^t$ and $\omega(J) = (\omega_1(J), ..., \omega_d(J))^t$ respectively for the lists of orbital angles, orbital actions and orbital tunes where ^{t} denotes the transpose and where with continuous time $d\phi/d\theta = \omega(J)$. In the continuous-time formalism, the T–BMT equation is written as $d\mathbf{S}/d\theta = \mathbf{\Omega}(\theta, J, \phi(\theta)) \times \mathbf{S}$ where the 3-vector $\mathbf{S}$ is the spin expectation value ("the spin vector") in the rest frame of a particle and $\mathbf{\Omega}$ is the precession vector obtained as indicated in [BEH] from the electric and magnetic fields on the particle trajectory. For the purposes of this work we don't need to consider the whole (J, ϕ) phase space since it will suffice to confine ourselves to a fixed J-value, i.e., to particle motion on a single torus. Thus the actions J are just parameters. However it is likely that our work can be easily generalized to arbitrary particle motion if one maintains our condition that the particle motion is unaffected by the spin motion.

This work, in which we aim to present particle-spin motions in terms of Dynamical-Systems Theory, is structured as follows. In Section 2.1 we discuss the continuous-time formalism which will motivate, in Section 2.4, the discrete-time concept of the "spin-orbit system" (j, A) which characterizes a given setup by its 1-turn particle map j on the torus $\mathbb{T}^d$. While j characterizes the integrable particle dynamics, A is the 1-turn spin transfer matrix function, the latter being a continuous function from $\mathbb{T}^d$ to $SO(3)$. In the special case of the torus translation we have $j = \mathcal{P}[\omega]$ where ω is the orbital tune and $\mathcal{P}[\omega]$ is the corresponding translation on the torus after one turn. Thus in Section 2.1 we derive the discrete-time Poincaré map formalism from the continuous-time formalism and in Section 2.3 we introduce the torus $\mathbb{T}^d$ as a topological space. For the torus the angle variable ϕ is represented by the angle variable z. Then in Section 2.4 we define the set $\mathcal{SOS}(d, j)$ of spin-orbit systems (j, A) to be considered in this work. In Chapter 3 we define polarization field trajectories and these lead to the definition of the invariant spin field (ISF). A transformation rule, $(j, A) \mapsto (j, A')$, is introduced in Chapter 4. This partitions $\mathcal{SOS}(d, j)$ into equivalence classes and spin-orbit systems belonging to the same equivalence class have similar properties. For the notions of partition and equivalence class, see Appendix A.2. It also leads us to structure-preserving transformations of particle-spin-vector trajectories and to structure-preserving transformations of polarization-field trajectories. In Chapter 5 the partition of $\mathcal{SOS}(d, j)$ leads us to several important subsets of $\mathcal{SOS}(d, j)$ which are denoted by $\mathcal{CB}_H(d, j)$. Each of these subsets of $\mathcal{SOS}(d, j)$ is defined in terms of a simple form of A. In particular a (j, A) in $\mathcal{SOS}(d, j)$ belongs to $\mathcal{CB}_H(d, j)$ iff it can be transformed to a (j, A') such that A' is H-valued where H is a subgroup of $SO(3)$. Then (j, A') is said to be an "H-normal form" of (j, A). The concept of H-normal form is also the driving force which leads us to the general theory of the follow-up work. In Chapter 6 we study H-normal forms in the case $H = SO(2)$ and formulate and prove a standard theorem, which connects the notions of ISF and invariant frame field (IFF).

In Chapter 7 the partition of $\mathcal{SOS}(d, j)$ leads us to the important subset $\mathcal{ACB}(d, j)$ of $\mathcal{SOS}(d, j)$. This subset $\mathcal{ACB}(d, j)$ of $\mathcal{SOS}(d, j)$ is defined in terms of another simple form of A. In particular a (j, A) in $\mathcal{SOS}(d, j)$ belongs to $\mathcal{ACB}(d, j)$ iff it can be transformed to a (j, A') such that A' is constant. On the other hand spin tunes describe constant rates of precession in appropriate reference frames so that one needs special spin-orbit systems which can be reached by transforming from the original spin-orbit systems to such frames. This relates the notions of spin tune and spin-orbit resonance to the notion of H-normal form in the case $H = G_\nu$ and thus to $\mathcal{ACB}(d, j)$. Chapter 8 covers the topic of polarization. In particular in Section 8.1 we derive various formulas which estimate the bunch polarization with special emphasis on the situation where only two ISF's exist. In Section 8.2 we state and prove an important and well-known theorem which provides conditions under which only two ISF's exist. In Appendix A we introduce the basic analytic notions like bijection and continuous function. Appendix B provides results on the quasiperiodicity and spectral properties of spin motions. Appendix C contains some of our proofs.

2 Spin-orbit systems

A central aim of the present work and of its follow-up is a study of the 1-turn particle-spin-vector map $\mathcal{P}[j, A]$ where j is the 1-turn particle map and A is the 1-turn spin transfer matrix function derived from the T-BMT equation, both defined on the d-torus. Thus the 1-turn particle-spin-vector map is the continuous function defined by (2.22), i.e.,

$$\mathcal{P}[j, A](z, S) := \begin{pmatrix} j(z) \\ A(z)S \end{pmatrix} ,$$

where z represents the angle variable of the action-angle variable dynamics on the d-torus.

We will proceed in Chapter 2 as follows. In the pedagogical Section 2.1 we consider the continuous-time T-BMT equation in the case of integrable particle dynamics and by excluding thin-lens magnets. In Section 2.2 we derive the discrete-time particle-spin-vector Poincaré map from the continuous-time particle-spin-vector motion of Section 2.1. Also in Section 2.2 we generalize the Poincaré map to the general 1-turn map $\hat{\mathcal{P}}[\hat{j}, \hat{A}]$ in (2.12) in order to capture thin-lens magnets like Siberian snakes. Thus from Section 2.2 onwards we use the language of topological dynamical systems and no more restrict ourselves to the assumptions underlying the continuous-time formalism of Section 2.1. In Section 2.3 we introduce the d-torus and its topology, i.e., we replace the angle variable ϕ of Sections 2.1 and 2.2 by the angle variable z. Thus in Section 2.3 we derive the 1-turn map $\mathcal{P}[j, A]$ from the 1-turn map $\hat{\mathcal{P}}[\hat{j}, \hat{A}]$ and we show their "equivariance".

2.1 The continuous-time particle-spin-vector motion

We assume the particle dynamics is integrable and that there exist action-angle variables (J, ϕ) so that the particle dynamics is governed by

$$\frac{d\phi}{d\theta} = \omega(J) \ \text{ and } \ \frac{dJ}{d\theta} = 0. \tag{2.1}$$

Here ϕ is on the torus where, in dynamical systems, the torus, $\mathbb{T}^d$, is often considered to be the quotient space $\mathbb{R}^d/\mathbb{Z}^d$. Here we will take $\phi \in \mathbb{R}^d$ and then at a later stage introduce the torus variable z of (2.22).

We begin our study by deriving our discrete-time particle-spin-vector motion from a continuous-time initial value problem (IVP) which has the form

$$\frac{d\phi}{d\theta} = \omega \ , \qquad \phi(0) = \phi_0 \in \mathbb{R}^d \ , \tag{2.2}$$

$$\frac{dS}{d\theta} = \mathcal{A}(\theta, \phi)S \ , \qquad S(0) = S_0 \in \mathbb{R}^3 \ , \tag{2.3}$$

where $\omega \in \mathbb{R}^d$ and where the matrix-valued function $\mathcal{A} : \mathbb{R}^{d+1} \to \mathbb{R}^{3\times3}$ is continuous, i.e., $\mathcal{A} \in \mathcal{C}(\mathbb{R}^{d+1}, \mathbb{R}^{3\times3})$ where $\mathcal{C}(\mathbb{R}^{d+1}, \mathbb{R}^{3\times3})$ is the set of continuous functions from $\mathbb{R}^{d+1}$ into $\mathbb{R}^{3\times3}$. See Appendix A.4 too. Moreover we assume that $\mathcal{A}$ is 2π-periodic in each of its $d+1$ arguments and that it is skew–symmetric, i.e., $\mathcal{A}^t(\theta, \phi) = -\mathcal{A}(\theta, \phi)$. Without loss of generality and for simplicity of notation we choose $\theta = 0$ as the initial time. We denote the set of $\mathcal{A}$, where $\mathcal{A}$ satisfies the above conditions, by $\mathcal{BMT}$.

As is clear from the above and the Introduction, the above IVP and the assumptions on $\mathcal{A}$ are motivated by our underlying interest in particle-spin-vector motion in storage rings. In the application to particle-spin-vector motion in storage rings, S is a column vector of components of the spin $\mathbf{S}$ and $\mathcal{A}(\theta, \phi) \equiv \mathcal{A}_J(\theta, \phi)$ represents the rotation rate vector $\mathbf{\Omega}(\theta, J, \phi)$ of the T–BMT equation [BEH]. Note that $\mathcal{A}(\theta, \phi)$ is 2π-periodic in θ because we deal with storage rings and $\mathcal{A}(\theta, \phi)$ is 2π-periodic in the d components of ϕ since the latter are angle variables. Moreover $\mathcal{A}$ is skew-symmetric by its origin in the T–BMT equation, thus preserving the norm of S. We suppress the J, except for a few occasions where we need it, since we work mainly on a fixed-J torus. The set $\mathcal{BMT}$ includes standard particle-spin-vector motion but need not, and is only restricted by the above mentioned constraints on $\mathcal{A}$, in keeping with our wish to investigate the properties of any system defined by (2.2) and (2.3).

Since the system (2.2),(2.3) is periodic in θ the 1-turn map it defines is identical with the Poincaré map (PM) which will be studied and generalized in Section 2.2. The PM is convenient for studying the behavior of solutions of (2.2),(2.3) [AP, HK2]. To derive a convenient representation for the PM we solve (2.2), resulting in

$$\phi(\theta) = \phi_0 + \omega\theta \ , \tag{2.4}$$

whence (2.3) reads as

$$\frac{dS}{d\theta} = \mathcal{A}(\theta, \phi_0 + \omega\theta)S \ , \qquad S(0) = S_0 \in \mathbb{R}^3 \ . \tag{2.5}$$

Since $\mathcal{A}$ is continuous and (2.5) is linear in S the general solution of (2.5) can be written as

$$S(\theta) = \Phi(\theta; \phi_0)S_0 \ , \tag{2.6}$$

where the function $\Phi : \mathbb{R} \times \mathbb{R}^d \to SO(3)$ is of class C^1 and satisfies

$$\partial_\theta \Phi(\theta; \phi_0) = \mathcal{A}(\theta, \phi_0 + \omega\theta)\Phi(\theta; \phi_0) \ , \quad \Phi(0; \phi_0) = I_{3\times 3} \ , \tag{2.7}$$

and where $I_{3\times 3}$ is the 3×3 unit matrix [Am, Ha]. Since the values of $\mathcal{A}$ are real skew–symmetric 3×3 matrices, Φ is $SO(3)$-valued, i.e., $\Phi(\theta; \phi_0) \in SO(3)$ where $SO(3)$ is the set of real 3×3–matrices R for which $R^t R = I_{3\times 3}$ and $\det(R) = 1$. Furthermore $\Phi(\theta, \phi)$ is 2π-periodic in the components of ϕ. Using (2.4) and (2.6), the solution of the IVP (2.2),(2.3) can now be written

$$\begin{pmatrix} \phi(\theta) \\ S(\theta) \end{pmatrix} = \varphi(\theta; \phi_0, S_0) \ , \tag{2.8}$$

where the function $\varphi : \mathbb{R}^{d+4} \to \mathbb{R}^{d+3})$ is of class C^1 and is defined by

$$\varphi(\theta; \phi, S) := \begin{pmatrix} \phi + \omega\theta \\ \Phi(\theta; \phi)S \end{pmatrix} \ . \tag{2.9}$$

2.2 Deriving the particle-spin-vector 1-turn map from the continuous-time formalism. Defining the general 1-turn map

The 1-turn map of the DS of Section 2.1 is the PM $\varphi(2\pi; \cdot)$ on $\mathbb{R}^{d+3}$ whence it reads, by (2.9), as

$$\varphi(2\pi; \phi, S) = \begin{pmatrix} \hat{\mathcal{P}}[\omega](\phi) \\ \Phi(2\pi; \phi)S \end{pmatrix} \ , \tag{2.10}$$

where $\hat{\mathcal{P}}[\omega] : \mathbb{R}^d \to \mathbb{R}^d$ being defined by $\hat{\mathcal{P}}[\omega](\phi) := \phi + 2\pi\omega$. With this the study of the non-autonomous continuous-time Dynamical System (DS) of (2.2),(2.3) has been replaced by a study of an autonomous discrete-time DS given by the PM (2.10).

We now have to generalize the 1-turn map (2.10). In fact not every $\mathcal{A}$ used in practice belongs to $\mathcal{BMT}$. For example if thin-lens Siberian snakes are involved, as in Section 3.3 below, then $\mathcal{A} \notin \mathcal{BMT}$. Thus, instead of generalizing $\mathcal{BMT}$ and as it is quite common, we generalize the 1-turn map (2.10) by replacing $\Phi(2\pi;\cdot)$ by an arbitrary continuous function $\hat{A}:\mathbb{R}^d\to SO(3)$ which is 2π-periodic in each of its d arguments. This more general 1-turn map can capture the situations mentioned above, e.g., when thin-lens Siberian snakes are involved.

It is less important to also generalize the particle maps $\hat{\mathcal{P}}[\omega]$ but it is nevertheless convenient because most notions and results of this work are very general. Thus it would be an unnecessary restriction, in particular for future theoretical work on beam polarization, to confine ourselves to the above translation maps. Before generalizing $\hat{\mathcal{P}}[\omega]$ we make some comments on $\hat{\mathcal{P}}[\omega]$. We first note that it is a homeomorphism on $\mathbb{R}^d$, i.e., $\hat{\mathcal{P}}[\omega]\in \mathrm{Homeo}(\mathbb{R}^d)$ which means that it is continuous and invertible and that the inverse is continuous. In fact $\hat{\mathcal{P}}[\omega]$ is continuous and it has the continuous inverse $\hat{\mathcal{P}}[-\omega]$. For the notion of "homeomorphism" see also Appendix A.4. We will see in the following section that the $\hat{\mathcal{P}}[\omega]$ correspond to certain homeomorphisms on the d-torus thus our task of generalizing $\hat{\mathcal{P}}[\omega]$ is to generalize them to all *all* functions on $\mathbb{R}^d$ which correspond to homeomorphisms on the d-torus. The key insight which solves this problem is the peculiar property that, for fixed but arbitrary $N\in\mathbb{Z}^d$, $\hat{\mathcal{P}}[\omega](\phi+2\pi N)=\hat{\mathcal{P}}[\omega](\phi)+2\pi N$ as can be easily checked. Thus $\hat{\mathcal{P}}[\omega]$ belongs to the set, Fun_d, of functions $\hat{f}\in\mathcal{C}(\mathbb{R}^d,\mathbb{R}^d)$ with the property that, for fixed but arbitrary $N\in\mathbb{Z}^d$, there exists a $\tilde{N}\in\mathbb{Z}^d$ such that $\hat{f}(\phi+2\pi N)=\hat{f}(\phi)+2\pi\tilde{N}$. We will see in the following section (see Theorems 2.5d and 2.6a) that the $\hat{f}$ in Fun_d correspond to the continuous functions on the d-torus whence we look for an appropriate subset of Fun_d. In fact in the next section (see Theorems 2.5e and 2.6b) we will show that Map_d, defined by

$$\mathrm{Map}_d := \{\hat{j}\in \mathrm{Homeo}(\mathbb{R}^d) : \hat{j},\hat{j}^{-1}\in \mathrm{Fun}_d\}\subset \mathrm{Fun}_d\ , \tag{2.11}$$

is the set of those functions on $\mathbb{R}^d$ which correspond to the homeomorphisms on the d-torus. Thus we have generalized the $\hat{\mathcal{P}}[\omega]$ to the $\hat{j}$ in Map_d. Clearly every $\hat{\mathcal{P}}[\omega]$ belongs to Map_d since, by the above $\hat{\mathcal{P}}[\omega]$ is an homeomorphism for which $\hat{\mathcal{P}}[\omega]$ and $\hat{\mathcal{P}}[\omega]^{-1}=\hat{\mathcal{P}}[-\omega]$ belong to Fun_d. All physical applications we have in mind have $\hat{j}=\hat{\mathcal{P}}[\omega]$ and so in this case $\hat{j}$ is just a shorthand.

With the above generalizations of $\Phi(2\pi;\cdot)$ to $\hat{A}$ and $\hat{\mathcal{P}}[\omega]$ to $\hat{j}$ we have generalized the 1-turn map $\varphi(2\pi;\cdot)$ in (2.10) to the 1-turn map $\hat{\mathcal{P}}[\hat{j},\hat{A}]$ defined by

$$\hat{\mathcal{P}}[\hat{j},\hat{A}](\phi,S) := \begin{pmatrix} \hat{j}(\phi) \\ \hat{A}(\phi)S \end{pmatrix}\ , \tag{2.12}$$

where $\hat{A}\in\mathcal{C}(\mathbb{R}^d,SO(3))$ is 2π-periodic in its arguments and where $\hat{j}\in\mathrm{Map}_b$. Whenever appropriate we abbreviate $\hat{\mathcal{P}}[\hat{j},\hat{A}]$ by $\hat{\mathcal{P}}$. Clearly $\hat{\mathcal{P}}$ is a homeomorphism on $\mathbb{R}^d\times\mathbb{R}^3$ since it is continuous and its inverse $\hat{\mathcal{P}}^{-1}$ is continuous, because the latter reads as

$$\hat{\mathcal{P}}[\hat{j},\hat{A}]^{-1}(\phi,S) = \begin{pmatrix} \hat{j}^{-1}(\phi) \\ \hat{A}^t(\hat{j}^{-1}(\phi))S \end{pmatrix}\ , \tag{2.13}$$

as can be easily checked. Note also that (2.10) is a special case of (2.12) since $\varphi(2\pi;\cdot)=\hat{\mathcal{P}}[\hat{\mathcal{P}}[\omega],\Phi(2\pi;\cdot)]$.

It is very common in polarized beam physics to use the angle variable ϕ and so the structure of the 1-turn map (2.12) is well-known in the Beam Polarization community, especially in the case where $\hat{j} = \hat{\mathcal{P}}[\omega]$. In the following section we will see how (2.12) can be expressed in terms of the torus variable z to be defined below. In fact in this work, and in our follow-up work, we use so many topological properties that it is natural and convenient to express most of our results in terms of z. However this is no restriction because ϕ and z are of the same expressive power. In fact it is a simple exercise to express any of our results in terms of ϕ. For the particle spin-vector motion this can be done by means of Theorems 2.5 and 2.6 below and for the field motion this can be done by means of Remark 1 in Chapter 3. An example of the latter are the statement and proof of the Uniqueness Theorem in Section 8.2 which involve the topological notions of topological transitivity and path-connectedness. Both notions are most easily defined in terms of z but it is possible to prove and state the Uniqueness Theorem in terms of ϕ (see [He2]). In fact this theorem was introduced in [Yo1] in terms of ϕ. Note also that the results in [BEH] are expressed in terms of ϕ.

A reader familiar with the torus variable z can imagine (2.12) on the d-torus and safely move to Section 2.4 on a first reading.

2.3 Expressing the general particle-spin-vector 1-turn map in terms of $\mathbb{T}^d$

To express the 1-turn map $\hat{\mathcal{P}}[\hat{j}, \hat{A}]$ in (2.12) in terms of z we first consider the case of most interest, $\hat{j} = \hat{\mathcal{P}}[\omega]$. Then (2.12) gives

$$\begin{pmatrix} \phi' \\ S' \end{pmatrix} := \hat{\mathcal{P}}(\phi, S) = \begin{pmatrix} \phi + 2\pi\omega \\ \hat{A}(\phi)S \end{pmatrix} , \tag{2.14}$$

where $\omega \in \mathbb{R}^d$, $\phi \in \mathbb{R}^d$ and $\hat{A} \in \mathcal{C}(\mathbb{R}^d, SO(3))$ is 2π-periodic in its arguments. Recall from the remarks after (2.12) that $\hat{\mathcal{P}} \in \text{Homeo}(\mathbb{R}^d \times \mathbb{R}^3)$. Clearly, since $\hat{A}$ is 2π-periodic in its arguments it is uniquely defined by its values for ϕ in $(-\pi, \pi]^d$.

As mentioned at the beginning of Section 2.1, it is common in Dynamical Systems Theory to take the torus to be $\mathbb{R}^d/\mathbb{Z}^d$. Here we take the more geometrical and equally common approach and define it as the subset of $\mathbb{R}^{2d}$ given by:

Definition 2.1 *(d-torus $\mathbb{T}^d$)*
The 1-torus is defined by $\mathbb{T} := \{(z_1, z_2)^t : \ z_1^2 + z_2^2 = 1\} \subset \mathbb{R}^2$. The d-torus is defined as the d-fold product $\mathbb{T}^d$ of $\mathbb{T}$, i.e.,

$$\mathbb{T}^d := \{(z_1, z_2, \cdots, z_{2d})^t : \ z_{2i-1}^2 + z_{2i}^2 = 1, i = 1, \cdots, d\} \subset \mathbb{R}^{2d}. \tag{2.15}$$

For more details on d-tori, see [wiki2] and the remarks after Theorem 2.6 below. □

Because continuity is central to our work we define a metric on $\mathbb{T}^d$ given by

Definition 2.2 *(Metric on $\mathbb{T}^d$)*
The metric μ_d on $\mathbb{T}^d$ is the function $\mu_d : \mathbb{T}^d \times \mathbb{T}^d \to [0, \infty)$, defined by

$$\mu_d(z_1, z_2, \cdots, z_{2d}; u_1, u_2, \cdots, u_{2d}) := \sqrt{(z_1 - u_1)^2 + (z_2 - u_2)^2 + \cdots (z_{2d} - u_{2d})^2} \, . \tag{2.16}$$

Note that the rhs of (2.16) is known from the Euclidean metric on $\mathbb{R}^{2d}$ whence μ_d is a metric since it is a restriction of the Euclidean metric on $\mathbb{R}^{2d}$. □

We now introduce a 1-turn map on $\mathbb{T}^d \times \mathbb{R}^3$ which we will show is essentially equivalent to the 1-turn map (2.14) on $\mathbb{R}^d \times \mathbb{R}^3$. For $z \in \mathbb{T}^d$, $\mathcal{S} \in \mathbb{R}^3$, we define

$$\begin{pmatrix} z' \\ \mathcal{S}' \end{pmatrix} := \begin{pmatrix} \exp(2\pi\mathcal{J}_\omega)z \\ \hat{A}(\mathrm{Arg}(z))\mathcal{S} \end{pmatrix} , \tag{2.17}$$

where Arg is defined in Definition 2.3 below and where the $2d \times 2d$-matrix $\mathcal{J}_\omega$ is defined by

$$\mathcal{J}_\omega(z_1, z_2, \cdots, z_{2d})^t := (u_1, u_2, \cdots, u_{2d})^t , \quad \begin{pmatrix} u_{2i-1} \\ u_{2i} \end{pmatrix} := \begin{pmatrix} 0 & -\omega_i \\ \omega_i & 0 \end{pmatrix} \begin{pmatrix} z_{2i-1} \\ z_{2i} \end{pmatrix} ,$$

i.e., $z' = (z'_1, z'_2, \cdots, z'_{2d})^t$ with $\begin{pmatrix} z'_{2i-1} \\ z'_{2i} \end{pmatrix} = \begin{pmatrix} \cos 2\pi\omega_i & -\sin 2\pi\omega_i \\ \sin 2\pi\omega_i & \cos 2\pi\omega_i \end{pmatrix} \begin{pmatrix} z_{2i-1} \\ z_{2i} \end{pmatrix}$.

Definition 2.3 *(Arg)*
We define the function $\mathrm{Arg} : \mathbb{T}^d \to \mathbb{R}^d$ *by*

$$\mathrm{Arg}(z) := \Theta , \tag{2.18}$$

where $\Theta = (\Theta_1, \cdots, \Theta_d)^t$ *with* $\Theta_i \in (-\pi, \pi]$ *being uniquely defined by* $(\cos\Theta_i, \sin\Theta_i) = (z_{2i-1}, z_{2i})$. □

Arg is continuous for every z such that no $\Theta_i = \pi$ and has a 2π jump discontinuity in Θ_i at $(z_{2i-1}, z_{2i}) = (1, 0)$. However, since $\hat{A}$ is 2π-periodic in its arguments and continuous, A is in $\mathcal{C}(\mathbb{T}^d, SO(3))$ where $A(z) := \hat{A}(\mathrm{Arg}(z))$. To see this, let's examine $d = 1$ as follows. Consider u and v in $\mathbb{T}$ such that $\mathrm{Arg}(u) = \pi - \delta$ and $\mathrm{Arg}(v) = -\pi + \delta$ then $\hat{A}(\mathrm{Arg}(u)) - \hat{A}(\mathrm{Arg}(v)) = \hat{A}(\pi - \delta) - \hat{A}(-\pi + \delta) = \hat{A}(\pi - \delta) - \hat{A}(\pi + \delta)$ which is small by continuity if δ is small. (For the full proof see Theorem 2.5c below)

To show the relation between (2.14) and (2.17) we need the function π_d:

Definition 2.4 *(π_d)*
We define the function $\pi_d : \mathbb{R}^d \to \mathbb{T}^d$ *by*

$$\pi_d(\phi) := (\cos\phi_1, \sin\phi_1, ..., \cos\phi_d, \sin\phi_d)^t = \exp(\mathcal{J}_\phi)(1, 0, 1, 0, \cdots, 1, 0)^t . \tag{2.19}$$

Clearly $z = \pi_d(\mathrm{Arg}(z))$ *whence* π_d *is surjective and Arg is injective, i.e., one-one. Conversely, for every* $\phi \in \mathbb{R}^d$ *and since* π_d *is* 2π*-periodic in its arguments, there exists an* $N(\phi) \in \mathbb{Z}^d$ *such that* $\phi + 2\pi N(\phi) = \mathrm{Arg}(\pi_d(\phi))$. *In Theorem 2.5a below we will also show that* π_d *is continuous.* □

With the function π_d we can show the following relation between (2.14) and (2.17):
Claim:
Let ϕ and S be given so that ϕ' and S' are defined by (2.14). Let also $z = \pi_d(\phi)$ and $\mathcal{S} = S$ in (2.17). Then $z' = \pi_d(\phi')$ and $\mathcal{S}' = S'$.

Proof of Claim: We prove the claim in the case $d = 1$. The proof for general d is a straight forward extension and done in Theorem 2.5e below. First,

$$z' = \begin{pmatrix} \cos 2\pi\omega & -\sin 2\pi\omega \\ \sin 2\pi\omega & \cos 2\pi\omega \end{pmatrix} \begin{pmatrix} \cos\phi \\ \sin\phi \end{pmatrix} = \begin{pmatrix} \cos(\phi + 2\pi\omega) \\ \sin(\phi + 2\pi\omega) \end{pmatrix} = \pi_1(\phi + 2\pi\omega) = \pi_1(\phi').$$

Second, we note that if $z = \pi_d(\phi)$ then $\mathrm{Arg}(z) = \Theta$ where Θ is uniquely defined by $(\cos(\Theta_1), \sin(\Theta_1)) = (\cos(\phi_1), \sin(\phi_1))$. Thus $\mathcal{S}' = \hat{A}(\Theta)S = \hat{A}(\phi)S = S'$. QED

In particular if $(\phi^{(n)}, S^{(n)})$ and $(z^{(n)}, \mathcal{S}^{(n)})$ denote the n-turn iterates of (ϕ', S') and $(z', \mathcal{S}')$ iterates, defined by (2.14) and (2.17) with $z^0 = \pi_d(\phi^0)$ and $\mathcal{S}^0 = S^0$, then, by the above claim, $(z^{(n)}, \mathcal{S}^{(n)}) = (\pi_d(\phi^{(n)}), S^{(n)})$. In Theorem 2.5e below we will prove the above claim in the general case, i.e., for arbitrary d and by generalizing $\hat{\mathcal{P}}[\omega]$ to $\hat{j}$.

To generalize (2.17) from the case $\hat{j} = \hat{\mathcal{P}}[\omega]$ in (2.12), we rewrite (2.17) in terms of π_d by noting that $z = \pi_d(\mathrm{Arg}(z))$ so we compute, by (2.17),(2.19),

$$\begin{aligned} z' &= \exp(2\pi\mathcal{J}_\omega)z = \exp(2\pi\mathcal{J}_\omega)\pi_d(\mathrm{Arg}(z)) = \exp(2\pi\mathcal{J}_\omega)\pi_d(\Theta) \\ &= \exp(2\pi\mathcal{J}_\omega)\exp(\mathcal{J}_\Theta)(1,0,1,0,\cdots,1,0)^t \\ &= \exp(\mathcal{J}_{2\pi\omega+\Theta})(1,0,1,0,\cdots,1,0)^t = \pi_d(\Theta+2\pi\omega) = \pi_d(\mathrm{Arg}(z)+2\pi\omega)\ , \end{aligned} \tag{2.20}$$

where $\Theta = \mathrm{Arg}(z)$ whence (2.17) can be written as $\begin{pmatrix} z' \\ \mathcal{S}' \end{pmatrix} := \begin{pmatrix} \pi_d(\mathrm{Arg}(z)+2\pi\omega) \\ \hat{A}(\mathrm{Arg}(z))\mathcal{S} \end{pmatrix} = \begin{pmatrix} (\pi_d \circ \hat{\mathcal{P}}[\omega] \circ \mathrm{Arg})(z) \\ ((\hat{A} \circ \mathrm{Arg})(z))\mathcal{S} \end{pmatrix}$ which is readily generalized, via replacing $\hat{\mathcal{P}}[\omega]$ by $\hat{j}$, to

$$\begin{pmatrix} z' \\ \mathcal{S}' \end{pmatrix} := \begin{pmatrix} (\pi_d \circ \hat{j} \circ \mathrm{Arg})(z) \\ ((\hat{A} \circ \mathrm{Arg})(z))\mathcal{S} \end{pmatrix} . \tag{2.21}$$

We will show in Theorem 2.5e below that $\pi_d \circ \hat{j} \circ \mathrm{Arg}$ belongs to $\mathrm{Homeo}(\mathbb{T}^d)$ and that $\hat{A} \circ \mathrm{Arg}$ belongs to $\mathcal{C}(\mathbb{T}^d, SO(3))$ whence (2.21) can be written as $\begin{pmatrix} z' \\ S' \end{pmatrix} = \mathcal{P}(z,S)[j,A]$ where the function $\mathcal{P}[j,A] : \mathbb{T}^d \times \mathbb{R}^3 \to \mathbb{T}^d \times \mathbb{R}^3$ is defined by

$$\mathcal{P}[j,A](z,S) := \begin{pmatrix} j(z) \\ A(z)S \end{pmatrix} , \tag{2.22}$$

with $j \in \mathrm{Homeo}(\mathbb{T}^d)$ and $A \in \mathcal{C}(\mathbb{T}^d, SO(3))$. Clearly $\mathcal{P}[j,A]$ is a homeomorphism on $\mathbb{T}^d \times \mathbb{R}^3$, i.e., $\mathcal{P}[j,A] \in \mathrm{Homeo}(\mathbb{R}^d \times \mathbb{R}^3)$ since it is continuous and its inverse $\mathcal{P}^{-1}[j,A]$ is continuous, because the latter reads as

$$\mathcal{P}[j,A]^{-1}(z,S) := \begin{pmatrix} j^{-1}(z) \\ A^t(j^{-1}(z))S \end{pmatrix} , \tag{2.23}$$

as can be easily checked. Whenever appropriate we abbreviate $\mathcal{P}[j,A]$ by $\mathcal{P}$. With (2.22) we are ready to express the 1-turn map $\hat{\mathcal{P}}[\hat{j},\hat{A}]$ in (2.12) in terms of z by mapping it to the 1-turn map $\mathcal{P}[j,A]$ as will be shown in Theorem 2.5e below.

Theorems 2.5e and 2.6b below show the main features of $\mathcal{P}[j,A]$ and of its relation to $\hat{\mathcal{P}}[\hat{j},\hat{A}]$ and they demonstrate that ϕ and z are of the same expressive power for the particle spin-vector dynamics (for the field dynamics this is demonstrated in Remark 1 of Chapter 3 below).

For our work here, it is convenient to prove continuity with the metric topology τ_d from μ_d of Definition 2.2. The topology τ_d is simply the collection of open sets where $B \subset \mathbb{T}^d$ is open iff for all $z \in B$ there exists an open μ_d-ball B' around z such that $B' \subset B$. Thus, by the open-set definition of continuity, an $f : \mathbb{T}^d \to \mathbb{T}^d$ is continuous iff $f^{-1}(B)$ is open whenever B is open. This generalizes to continuity for functions $f : \mathbb{T}^d \to X$ where X is a topological space as discussed in Appendix A.4. It turns out that $\tau_d = \tau_d^{fin}$ where the "final topology" τ_d^{fin} is defined in terms of π_d. In fact by this topology a subset $B \subset \mathbb{T}^d$ is open, i.e., $B \in \tau_d^{fin}$ iff the inverse image $\pi_d^{-1}(B)$ is an open in $\mathbb{R}^d$ w.r.t. the natural topology $\tau_{\mathbb{R}^d}$ of $\mathbb{R}^d$. The topology τ_d^{fin} is very convenient in the proof of Theorem 2.5b below.

Theorem 2.5 *a)* $\tau_d = \tau_d^{fin}$. *Moreover* π_d *is continuous, i.e.,* $\pi_d \in \mathcal{C}(\mathbb{R}^d, \mathbb{T}^d)$.

b) Let X *be a topological space and let* $F \in \mathcal{C}(\mathbb{R}^d, X)$ *be* 2π*-periodic in its arguments. Then* $F = F \circ Arg \circ \pi_d$ *and* $f \in \mathcal{C}(\mathbb{T}^d, X)$ *where* $f(z) := F(\mathrm{Arg}(z))$*. Also* $f(\pi_d(\phi)) = F(\phi)$*. Moreover if* $g \in \mathcal{C}(\mathbb{T}^d, X)$ *such that* $g(\pi_d(\phi)) = F(\phi)$ *then* $f = g$*. Furthermore if* $G \in \mathcal{C}(\mathbb{R}^d, X)$ *is* 2π*-periodic in its arguments and* $F(\mathrm{Arg}(z)) = G(\mathrm{Arg}(z))$ *then* $F = G$.

If conversely $h \in \mathcal{C}(\mathbb{T}^d, X)$*, then* $H \in \mathcal{C}(\mathbb{R}^d, X)$ *is* 2π*-periodic in its arguments where* $H(\phi) := h(\pi_d(\phi))$*. Furthermore* $H(\mathrm{Arg}(z)) = h(z)$.

c) (Baby Lift Theorem for π_d*) Let* $F \in \mathcal{C}(\mathbb{R}^k, \mathbb{T}^d)$*. Then there exists a* $\hat{f} \in \mathcal{C}(\mathbb{R}^k, \mathbb{R}^d)$ *such that* $F = \pi \circ \hat{f}$*. Moreover* $\hat{f}$ *is unique up to a constant in the following sense: if* $\hat{f}_1, \hat{f}_2 \in \mathcal{C}(\mathbb{R}^k, \mathbb{R}^d)$ *and* $\pi_d \circ \hat{f}_1 = F = \pi_d \circ \hat{f}_2$ *then there exists a constant* $N \in \mathbb{Z}^d$ *such that* $\hat{f}_1(\phi) = \hat{f}_2(\phi) + 2\pi N$*.*
Remark: We will apply the Baby Lift Theorem in the present chapter to the case $k = d$ *and in Chapter 6 to the case* $k = 1$.

d) Fun_d *is equal to the set of functions* $\hat{f}$ *in* $\mathcal{C}(\mathbb{R}^d, \mathbb{R}^d)$ *for which* $\pi_d \circ \hat{f}$ *are* 2π*-periodic in their arguments. Moreover* Map_d *is the set of* $\hat{j} \in \mathrm{Homeo}(\mathbb{R}^d)$ *for which* $\pi_d \circ \hat{j}$ *and* $\pi_d \circ \hat{j}^{-1}$ *are* 2π*-periodic in their arguments. Furthermore if* $\hat{f} \in \mathrm{Fun}_d$ *then* $\pi_d \circ \hat{f} \circ \mathrm{Arg}$ *belongs to* $\mathcal{C}(\mathbb{T}^d, \mathbb{T}^d)$.
e) Let $\hat{A} \in \mathcal{C}(\mathbb{R}^d, SO(3))$ *be* 2π*-periodic in its arguments and let* $\hat{j} \in \mathrm{Map}_d$*, i.e.,* $\hat{j} \in \mathrm{Homeo}(\mathbb{R}^d)$ *with* $\pi_d \circ \hat{j}$ *and* $\pi_d \circ \hat{j}^{-1}$ *being* 2π*-periodic in their arguments. Then* $j := \pi_d \circ \hat{j} \circ \mathrm{Arg}$ *belongs to* $\mathrm{Homeo}(\mathbb{T}^d)$ *and* $A := \hat{A} \circ \mathrm{Arg}$ *belongs to* $\mathcal{C}(\mathbb{T}^d, SO(3))$.

Moreover the claim after Definition 2.4 generalizes to the following claim: If $z = \pi_d(\phi)$ *and* $\mathcal{S} = S$ *then* $z' = \pi_d(\phi')$ *and* $\mathcal{S}' = S'$ *where* $\begin{pmatrix} \phi' \\ S' \end{pmatrix} := \hat{\mathcal{P}}[\hat{j}, \hat{A}](\phi, S) = \begin{pmatrix} \hat{j}(\phi) \\ \hat{A}(\phi) S \end{pmatrix}$ *and where* $\begin{pmatrix} z' \\ \mathcal{S}' \end{pmatrix} := \mathcal{P}[j, A](z, \mathcal{S}) = \begin{pmatrix} j(z) \\ A(z)\mathcal{S} \end{pmatrix}$.
Remark: The above claim means, in the language of Dynamical Systems Theory, that $\hat{\mathcal{P}}[\hat{j}, \hat{A}]$ *and* $\mathcal{P}[j, A]$ *are equivariant as well as that* $\hat{j}$ *and* j *are equivariant, see for example [HK1]. The notion of equivariance will be very important in our follow-up work.*

Proof of Theorem 2.5: The claims are proved in Appendix C.1. □

We now make some comments on Theorems 2.5b and 2.5c. We use Theorem 2.5b repeatedly in this work (for example in the proofs of Theorems 2.5d,2.5e and 2.6 below). The situation of Theorem 2.5b is nicely summarized by the commuting diagram in Figure 1 if we add that $\mathrm{Arg} : \mathbb{T}^d \to \mathbb{R}^d$ and note that $\mathrm{Arg}(\pi_d(\phi)) = \Theta$ and $\pi_d(\mathrm{Arg}(z)) = z$. That the diagram "commutes" means that $F = f \circ \pi_d$. Standard readings of the diagram are given F, there exists an f which "completes" the diagram and given f, there exists an F which "completes" the diagram. We also use Theorem 2.5c repeatedly in this work (for example in Theorem 6.3 below).

$$\begin{tikzcd} & \mathbb{R}^d \arrow[dl, "F"'] \arrow[d, "\pi_d"] \\ X & \mathbb{T}^d \arrow[l, "f"] \end{tikzcd}$$

Figure 1: Commutative diagram for Theorem 2.5b

Theorem 2.5e above shows how every $\hat{\mathcal{P}}[\hat{j}, \hat{A}]$ is mapped to a unique $\mathcal{P}[j, A]$ and that both are equivariant. Part b) of the following theorem answers a remaining question: how many $\hat{\mathcal{P}}[\hat{j}, \hat{A}]$ are mapped to a fixed but arbitrary $\mathcal{P}[j, A]$ and are there any?

Theorem 2.6 *a) Let $f \in \mathcal{C}(\mathbb{T}^d, \mathbb{T}^d)$. Then there exists a $\hat{f} \in \mathrm{Fun}_d$, i.e., $\hat{f} \in \mathcal{C}(\mathbb{R}^d, \mathbb{R}^d)$ such that $\pi_d \circ \hat{f}$ is 2π-periodic in its arguments and with the property that $f = \pi_d \circ \hat{f} \circ \mathrm{Arg}$. Moreover $\hat{f}$ is uniquely determined by f up to a constant in the following sense: if $\hat{f}_1, \hat{f}_2 \in \mathrm{Fun}_d$ and such that $\pi_d \circ \hat{f}_1 \circ \mathrm{Arg} = f = \pi_d \circ \hat{f}_2 \circ \mathrm{Arg}$ then there exists a constant $N \in \mathbb{Z}^d$ such that $\hat{f}_1(\phi) = \hat{f}_2(\phi) + 2\pi N$.*
Remark: Thus, and by Theorem 2.5d, $\hat{f} \mapsto \pi_d \circ \hat{f} \circ \mathrm{Arg}$ maps Fun_d onto $\mathcal{C}(\mathbb{T}^d, \mathbb{T}^d)$.
b) Let $j \in \mathrm{Homeo}(\mathbb{T}^d)$. Then there exists a $\hat{j} \in \mathrm{Map}_d$ such that $\hat{j}$ is related to j as in Theorem 2.5e, i.e., $j = \pi_d \circ \hat{j} \circ \mathrm{Arg}$. Also if $\hat{j} \in \mathrm{Map}_d$ and $j = \pi_d \circ \hat{j} \circ \mathrm{Arg}$ and if $N \in \mathbb{Z}^d$ then the function $\hat{g} \in \mathcal{C}(\mathbb{R}^d, \mathbb{R}^d)$, defined by $\hat{g}(\phi) := \hat{j}(\phi) + 2\pi N$ belongs to Map_d and satisfies $j = \pi_d \circ \hat{g} \circ \mathrm{Arg}$. Conversely $\hat{j}$ is uniquely determined by j up to a constant in the following sense: if $\hat{j}_1, \hat{j}_2 \in \mathrm{Map}_d$ such that $\pi_d \circ \hat{j}_1 \circ \mathrm{Arg} = j = \pi_d \circ \hat{j}_2 \circ \mathrm{Arg}$ then there exists a constant $N \in \mathbb{Z}^d$ such that $\hat{j}_1(\phi) = \hat{j}_2(\phi) + 2\pi N$.

Let $A \in \mathcal{C}(\mathbb{T}^d, SO(3))$. Then there exists an $\hat{A} \in \mathcal{C}(\mathbb{R}^d, SO(3))$ which is 2π-periodic in its arguments and is related to A as in Theorem 2.5e, i.e., $A = \hat{A} \circ \mathrm{Arg}$. Moreover $\hat{A}$ is uniquely determined by A. Remark: Thus $\hat{j} \mapsto \pi_d \circ \hat{j} \circ \mathrm{Arg}$ maps Map_d onto $\mathrm{Homeo}(\mathbb{T}^d)$. Also $\hat{A} \mapsto \hat{A} \circ \mathrm{Arg}$ maps the periodic $\hat{A} \in \mathcal{C}(\mathbb{R}^d, SO(3))$ bijectively onto $\mathcal{C}(\mathbb{T}^d, SO(3))$.

Proof of Theorem 2.6: The claims are proved in Appendix C.2. □

We first make some comments on the important case $\hat{j} = \hat{\mathcal{P}}[\omega]$. Then the homeomorphism j in Theorem 2.5e reads as $\pi_d \circ \hat{\mathcal{P}}[\omega] \circ \mathrm{Arg}$ which we denote by $\mathcal{P}[\omega]$, i.e., we define $\mathcal{P}[\omega] \in \mathrm{Homeo}(\mathbb{T}^d)$ by

$$\mathcal{P}[\omega] := \pi_d \circ \hat{\mathcal{P}}[\omega] \circ \mathrm{Arg} \,. \tag{2.24}$$

Note that

$$\mathcal{P}[\omega](z) = \pi_d(\hat{\mathcal{P}}[\omega](\mathrm{Arg}(z))) = \pi_d(\hat{\mathcal{P}}[\omega](\Theta)) = \pi_d(\Theta + 2\pi\omega) = \pi_d(\phi + 2\pi\omega) \,, \tag{2.25}$$

with $\Theta := \mathrm{Arg}(z)$ and $\phi \in \mathbb{R}^d$ such that $\pi_d(\phi) = z$ and where in the fourth equality we used that π_d is 2π-periodic in its arguments. It is easy to show, for $n \in \mathbb{Z}$ and by (2.25), that

$$\mathcal{P}[\omega]^n = \mathcal{P}[n\omega] \ . \tag{2.26}$$

Note that (2.26) justifies the terminology "torus translation" for $\mathcal{P}[\omega]$. It also follows from (2.26) that $\mathcal{P}[-\omega]$ is the inverse of $\mathcal{P}[\omega]$ whence $\mathcal{P}[\omega]$ and its inverse are continuous which confirms that $\mathcal{P}[\omega]$ is a homeomorphism. It is amusing to see, by Theorem 2.6b, that the 1-turn maps of the form $\hat{\mathcal{P}}[\omega+2\pi N]$ with $N \in \mathbb{Z}^d$ belong to Map_d and that they are the only ones in Map_d which are related to $\mathcal{P}[\omega]$ as in Theorem 2.5e, i.e., $\mathcal{P}[\omega] = \pi_d \circ \hat{\mathcal{P}}[\omega+2\pi N] \circ \mathrm{Arg}$. Note also that with (2.24) the generalization from $\mathcal{P}[\omega]$ to j comes from the generalization from $\hat{\mathcal{P}}[\omega]$ to $\hat{j}$. Whence, by the remarks after (2.10), the generalization from $\mathcal{P}[\omega]$ to j is a matter of convenience and a decision w.r.t. future theoretical work on beam polarization.

We now make some comments on the topology $\tau_d = \tau_d^{fin}$ of $\mathbb{T}^d$. It is easy to show that τ_d^{fin} is the largest topology on $\mathbb{T}^d$ for which π_d is continuous. For the notion of final topology, see for example [wiki1] and Appendix A.5. Note that, in an older terminology, π_d is called an "identification" and τ_d^{fin} is called the "identification topology" w.r.t. π_d [Du, Hu]. For the topology τ_d^{fin} the open-set definition of continuity is very convenient so we use this approach in the proof of Theorem 2.5b above and in analogous computations of this work. Thus in this work we don't use the "$\epsilon - \delta$" definition of continuity which is given in terms of the metric μ_d. In other words μ_d is unimportant in our computations. For the open-set definition of continuity, see for example Appendix A.4. Note that our choice of $\mathbb{T}^d$ and its topology is very common in Theoretical Physics and Mathematics because τ_d makes $\mathbb{T}^d$ a smooth submanifold of $\mathbb{R}^{2d}$. Since on $\mathbb{T}^d$ always the topology τ_d is used, one also calls it the"natural" topology. We will introduce in Section 2.4 the important notion of topological transitivity for j. We will also use the fact that $\mathbb{T}^d$ is path-connected (see, e.g., the proof of the Uniqueness Theorem in Section 8.2). These notions involve $\mathbb{T}^d$ and its topology and they make the use of $\mathbb{T}^d$ very natural. Nevertheless the use of $\mathbb{T}^d$ is justified rather by convenience than by necessity so we could confine ourselves to $\mathbb{R}^d$ as pointed out in great detail in this section. However the above mentioned topological notions, when expressed in terms of $\mathbb{R}^d$, are inconvenient and unnatural (imagine to formulate the path-connectedness of $\mathbb{T}^d$ in terms of $\mathbb{R}^d$!). In our follow-up work we will use even more notions which involve $\mathbb{T}^d$ and its topology. Of course there are other important definitions of the d-torus, e.g., $\mathbb{R}^d/\mathbb{Z}^d$ [wiki2] and all these d-tori are equipped with a "natural" topology which makes them homeomorphic to $\mathbb{T}^d$. For the notion of "homeomorphic" see Appendix A.4. The choice of the particular d-torus to be used is a matter of convenience, i.e., it depends on the application in mind. Another popular definition exists for the case $d = 2$, which is the "doughnut", e.g., the doughnut of major radius 2 and minor radius 1, i.e., $\{(x, y, z) \in \mathbb{R}^3 : (x^2 + y^2 + z^2 + 3)^2 = 16(x^2 + y^2)\}$ and which is equipped with the subspace topology from $\mathbb{R}^3$ [wiki2]. Note that the doughnut is homeomorphic to $\mathbb{T}^2$.

To show that the $\hat{j}$ are more general than the $\hat{\mathcal{P}}[\omega]$ and that the j are more general than the $\mathcal{P}[\omega]$ we first make some comments on Fun_d and Map_d so let $\hat{f} \in \mathrm{Fun}_d$ and $\hat{j} \in \mathrm{Map}_d$. By the definition of Fun_d and for every $N \in \mathbb{Z}^d$, a unique $\tilde{N} \in \mathbb{Z}^d$ exists such that $\hat{f}(\phi+2\pi N) = \hat{f}(\phi) + 2\pi\tilde{N}$. Also if $N_1, \tilde{N}_1, N_2, \tilde{N}_2 \in \mathbb{Z}^d$ and $\hat{f}(\phi + 2\pi N_1) = \hat{f}(\phi) + 2\pi\tilde{N}_1$, $\hat{f}(\phi + 2\pi N_2) = \hat{f}(\phi) + 2\pi\tilde{N}_2$ then $\hat{f}(\phi+2\pi N_1+2\pi N_2) = \hat{f}(\phi+2\pi N_1)+2\pi\tilde{N}_2 = \hat{f}(\phi)+2\pi\tilde{N}_1+2\pi\tilde{N}_2$ whence the dependence of $\tilde{N}$ on N is linear so that a matrix $M \in \mathbb{Z}^{d\times d}$ exists such that $\hat{f}(\phi+2\pi N) = \hat{f}(\phi) + 2\pi MN$. Since the latter equality holds for every $N \in \mathbb{Z}^d$, the matrix M is uniquely determined by $\hat{f}$. With M we define the function $\hat{g} \in \mathcal{C}(\mathbb{R}^d, \mathbb{R}^d)$ by $\hat{g}(\phi) =: \hat{f}(\phi) - M\phi$ and immediately oberve that $\hat{g}$ is 2π-periodic in its arguments. It follows that Fun_d is equal to the set of functions $\hat{f} : \mathbb{R}^d \to \mathbb{R}^d$ of the form $\hat{f}(\phi) = M\phi + \hat{g}(\phi)$ where $M \in \mathbb{Z}^{d\times d}$ and where $\hat{g}$ is 2π-periodic in its arguments and belongs to $\mathcal{C}(\mathbb{R}^d, \mathbb{R}^d)$. We can now discuss $\hat{j}$ since it belongs to the subset Map_d of Fun_d whence, by the above, $\hat{j}(\phi) = M\phi+\hat{g}(\phi)$ where $M \in \mathbb{Z}^{d\times d}$ and where $\hat{g} \in \mathcal{C}(\mathbb{R}^d, \mathbb{R}^d)$ is 2π-periodic in its arguments. Moreover since $\hat{j} \in \mathrm{Map}_d$ and by (2.11), also $\hat{j}^{-1}$ belongs to Fun_d. Thus M has an inverse M^{-1} and M^{-1} belongs to $\mathbb{Z}^{d\times d}$ as can be easily checked as well as $\hat{j}^{-1}(\phi) = M^{-1}\phi + \hat{h}(\phi)$ where $\hat{h} \in \mathcal{C}(\mathbb{R}^d, \mathbb{R}^d)$ is 2π-periodic in its arguments. Clearly $\hat{\mathcal{P}}[\omega]$ is the special case of $\hat{j}$ for which $\hat{g}(\phi) = \omega$ and for which M is the unit $d \times d$-matrix. Choosing M to be the negative of the unit matrix we arrive at the function $\hat{j}_0 \in \mathcal{C}(\mathbb{R}^d, \mathbb{R}^d)$, defined by $\hat{j}_0(\phi) := -\phi$. We see that $\hat{j}_0^{-1} = \hat{j}_0$ whence $\hat{j}_0$ has a continuous inverse so that $\hat{j}_0 \in \mathrm{Homeo}(\mathbb{R}^d)$. Moreover $\pi_d \circ \hat{j}_0$ and $\pi_d \circ \hat{j}_0^{-1}$ are 2π-periodic in their arguments whence, by Theorem 2.5d, $\hat{j}_0 \in \mathrm{Map}_d$. Thus, and since $\hat{j}_0$ is different from every $\hat{\mathcal{P}}[\omega]$, we see that the $\hat{j}$ are more general than the $\hat{\mathcal{P}}[\omega]$. Of course, by Theorems 2.5e and 2.6b this implies that the j are more general than the $\mathcal{P}[\omega]$.

2.4 The set $\mathcal{SOS}(d, j)$ of spin-orbit systems

At the core of this work, in Chapters 4-8, the key technique is to transform, for fixed but arbitrary $j \in \mathrm{Homeo}(\mathbb{T}^d)$, every $A \in \mathcal{C}(\mathbb{T}^d, SO(3))$ into a certain set of $A' \in \mathcal{C}(\mathbb{T}^d, SO(3))$. Thus the transformation theory, introduced in Chapter 4, defines a set of transformation rules, each of which is labelled by a $j \in \mathrm{Homeo}(\mathbb{T}^d)$. It is thus convenient to define

$$\mathcal{SOS}(d, j) := \{(j, A) : A \in \mathcal{C}(\mathbb{T}^d, SO(3))\} , \tag{2.27}$$

where $j \in \mathrm{Homeo}(\mathbb{T}^d)$ and so the transformation theory of this work is based on a transformation rule on each $\mathcal{SOS}(d, j)$.

We call every pair (j, A) in $\mathcal{SOS}(d, j)$ a “spin-orbit system”. We call A the “1-turn spin transfer matrix function” of a spin-orbit system (j, A). In the special case $j = \mathcal{P}[\omega]$ we call ω the “orbital tune vector” of a spin-orbit system $(\mathcal{P}[\omega], A)$. Recalling Section 2.3, the homeomorphism $\mathcal{P}[j, A]$ in (2.22) is defined for every (j, A) in $\mathcal{SOS}(d, j)$ and we call $\mathcal{P}[j, A]$ the “1-turn particle-spin-vector map of (j, A)”.

It follows from Sections 2.2 and 2.3 that the set of spin-orbit systems which are derived from the continuous-time non-autonomous DS (2.2),(2.3) are of the form (j, A) where $j = \mathcal{P}[\omega]$ and $A(z) = \Phi(2\pi; \mathrm{Arg}(z))$. We denote this set of spin-orbit systems by $\mathcal{SOS}_{CT}(d, \omega)$. Clearly

$$\mathcal{SOS}_{CT}(d, \omega) \subset \mathcal{SOS}(d, \mathcal{P}[\omega]) , \tag{2.28}$$

and it will be shown after (2.29) that the inclusion in (2.28) is proper, i.e., that $\mathcal{SOS}_{CT}(d, \omega) \neq \mathcal{SOS}(d, \mathcal{P}[\omega])$. All physical applications we have in mind have $j = \mathcal{P}[\omega]$ and so in this case j is just a shorthand. However, since for most notions and results of this work a general j is perfectly applicable, we do not confine ourselves to $j = \mathcal{P}[\omega]$.

As is clear from Section 2.2, we are interested in $\mathcal{SOS}(d,j)$ rather than $\mathcal{SOS}_{CT}(d,\omega)$ because we want to capture situations where thin-lens magnets are involved as in Section 3.3 below. In particular we expect that $\mathcal{SOS}_{CT}(d,\omega) \neq \mathcal{SOS}(d,\mathcal{P}[\omega])$. This is an analogue of the following question from beam dynamics: given a symplectic map, can it be generated as the 1-turn map of a Hamiltonian system? We do not deal with this question. Returning to the question if $\mathcal{SOS}_{CT}(d,\omega) \neq \mathcal{SOS}(d,\mathcal{P}[\omega])$, one may try to show this inequality by proving that the spin-orbit system $(\mathcal{P}[1/2], A_{2S})$ in Section 3.3 does not belong to $\mathcal{SOS}_{CT}(1,1/2)$. In fact $(\mathcal{P}[1/2], A_{2S})$ is derived from the continuous-time formalism with an $\mathcal{A}$ involving thin-lens Siberian snakes which cannot be captured, via Section 2.1, i.e., $\mathcal{A} \notin \mathcal{BMT}$ so chances are that $(\mathcal{P}[1/2], A_{2S})$ does not belong to $\mathcal{SOS}_{CT}(1,1/2)$. However, we can address our question much more easily as follows so let $\omega \in \mathbb{R}$ and $m \in \mathbb{Z}$ and let $\hat{A} \in \mathcal{C}(\mathbb{R}, SO(3))$ be defined by

$$\hat{A}(\phi) := \begin{pmatrix} \cos m\phi & -\sin m\phi & 0 \\ \sin m\phi & \cos m\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} . \tag{2.29}$$

Clearly $\hat{A}$ is 2π-periodic in ϕ whence, by Theorem 2.5e, $A \in \mathcal{C}(\mathbb{T}, SO(3))$ where $A(z) := \hat{A}(\mathrm{Arg}(z))$ so that $(\mathcal{P}[\omega], A) \in \mathcal{SOS}(1,\mathcal{P}[\omega])$. It was shown in [He2, Section 7.2], by using the "quaternion formalism" of $SO(3)$ and simple tools from Algebraic Topology, that $(\mathcal{P}[\omega], A) \in \mathcal{SOS}_{CT}(1,\mathcal{P}[\omega])$ iff m is even. Note that this argumentation uses the mathematical analogy of the z-ϕ correspondence of Chapter 2 with an r-ρ correspondence between $r \in SO(3)$ and a "quaternion" variable ρ (for more details, see [He2]). Thus for m odd we have a very simple example showing that the inclusion (2.28) is proper.

Note that in the case $m = 0$, the function A in (2.29) is constant and has the following $I_{3\times 3}$ and this case plays an important role on so-called spin-orbit resonances (see Chapter 7 below). The function A in (2.29) also plays the role of a so-called transfer field, e.g., in the proof of Theorem 7.3f.

2.5 The particle-spin-vector trajectories of the spin-orbit systems

We now discuss the DS defined by (2.22) by computing the n-th iterate $\mathcal{P}[j,A]^n$ of $\mathcal{P}[j,A]$. We call $\mathcal{P}[j,A]^n$ the "n-turn particle-spin-vector map of (j,A)". If $z_0 \in \mathbb{T}^d$ and $S_0 \in \mathbb{R}^3$ we define the functions $Z : \mathbb{Z} \to \mathbb{T}^d$ and $S : \mathbb{Z} \to \mathbb{R}^3$ by

$$\begin{pmatrix} Z(n) \\ S(n) \end{pmatrix} := \mathcal{P}[j,A]^n(z_0, S_0) . \tag{2.30}$$

We call Z a "particle trajectory of (j,A)" and S a "spin-vector trajectory of (j,A)". Also we call (Z,S) a "particle-spin-vector trajectory of (j,A)". Clearly, by (2.22),(2.30),

$$\begin{pmatrix} Z(n+1) \\ S(n+1) \end{pmatrix} = \mathcal{P}[j,A] \begin{pmatrix} Z(n) \\ S(n) \end{pmatrix} = \begin{pmatrix} j(Z(n)) \\ A(Z(n))S(n) \end{pmatrix} , \tag{2.31}$$

whence

$$Z(n) = j^n(z_0) . \tag{2.32}$$

It follows from (2.31),(2.32) that

$$S(n+1) = A(Z(n))S(n) = A(j^n(z_0))S(n) \ , \tag{2.33}$$

and $S(-n) = A^t(j^{-n}(z_0))S(-n+1)$ whence

$$\begin{aligned} S(n) &= A(j^{n-1}(z_0)) \cdots A(j(z_0))A(z_0)S_0 \ , \qquad (n = 1, 2, ...) \\ S(n) &= A^t(j^n(z_0)) \cdots A^t(j^{-1}(z_0))S_0 \ , \qquad (n = -1, -2, ...) \ , \end{aligned} \tag{2.34}$$

where we also used the fact that $A^t(z)A(z) = I_{3\times 3}$. It follows from (2.35) that

$$S(n) = \Psi[j, A](n; z_0)S_0 \ , \tag{2.35}$$

where the function $\Psi[j, A] : \mathbb{Z} \times \mathbb{T}^d \to SO(3)$ is defined by

$$\begin{aligned} &\Psi[j, A](0; z) := I_{3\times 3} \ , \\ &\Psi[j, A](n; z) := A(j^{n-1}(z)) \cdots A(j(z))A(z) \ , \qquad (n = 1, 2, ...) \ , \\ &\Psi[j, A](n; z) := A^t(j^n(z)) \cdots A^t(j^{-1}(z)) \ , \ \ (n = -1, -2, ...) \ . \end{aligned} \tag{2.36}$$

We call $\Psi[j, A]$ the "spin transfer matrix function" of (j, A) and we call $\Psi[j, A](n; \cdot)$ the "n-turn spin transfer matrix function" of (j, A). Clearly

$$\Psi[j, A](1; z) = A(z) \ , \tag{2.37}$$

which justifies calling A the 1-turn spin transfer matrix function. We use the standard topology on $\mathbb{Z}$ (see Appendix A.3) whence the function $\Psi[j, A]$ is continuous since it is continuous in the second argument. It follows from (2.32),(2.35) that

$$\begin{pmatrix} Z(n) \\ S(n) \end{pmatrix} = \begin{pmatrix} j^n(z_0) \\ \Psi[j, A](n; z_0)S_0 \end{pmatrix} \ ,$$

whence, by (2.30),

$$\mathcal{P}[j, A]^n(z, S) = \begin{pmatrix} j^n(z) \\ \Psi[j, A](n; z)S \end{pmatrix} \ . \tag{2.38}$$

The behavior of the spin-vector trajectories in (2.33) depends on the values of A reached by the particle motion $Z(n)$ in its argument, which in turn depends on j. In the case $j = \mathcal{P}[\omega]$ the argument $Z(n)$ of A in (2.33) will remain in a confined subset of the torus for some values of ω and for other values it will cover the torus densely. To be more precise we define resonance. We say $\chi \in \mathbb{R}^n$ is *resonant* if there exists a non-zero integer vector $k \in \mathbb{Z}^n$ such that $k \cdot \chi = 0$ and nonresonant if not resonant. If $j = \mathcal{P}[\omega]$ and $(1, \omega)$ is nonresonant then the argument $Z(n)$ of A in (2.33) covers the torus densely and since A is continuous all values of A affect the spin-vector trajectory whereas if $(1, \omega)$ is resonant the values of A are only sampled by its values on a sub-torus. The spin-orbit system $(\mathcal{P}[\omega], A)$ is said to be "off orbital resonance" if $(1, \omega)$ is nonresonant and "on orbital resonance" if $(1, \omega)$ is resonant.

Thus spin-vector trajectories may exhibit significantly different qualitative behaviors on and off orbital resonance. We will now generalize the notion "off orbital resonance". One says that $j \in \mathrm{Homeo}(\mathbb{T}^d)$ is "topologically transitive" if a $z_0 \in \mathbb{T}^d$ exists such that the set $B := \{j^n(z_0) : n \in \mathbb{Z}\}$ is dense in $\mathbb{T}^d$, i.e., $\overline{B} = \mathbb{T}^d$ where $\overline{B}$ denotes the topological closure of B, see Appendix A.3. An important special case is when $j = \mathcal{P}[\omega]$: then j is topologically transitive iff $(1, \omega)$ is nonresonant.

Since $\mathcal{P}[j, A]^{n+m} = \mathcal{P}[j, A]^n \circ \mathcal{P}[j, A]^m$ we get from (2.38)

$$\Psi[j, A](n + m; z) = \Psi[j, A](n; j^m(z)) \, \Psi[j, A](m; z) \; . \tag{2.39}$$

Since $\Psi[j, A]$ is continuous and $SO(3)$-valued and due to (2.39) it is common in Dynamical Systems Theory to call $\Psi[j, A]$ an "$SO(3)$-cocycle" over the homeomorphism j. While this aspect does not play a role in this work, it inspires to use the terminologies "almost coboundary" and "coboundary" in Chapter 7. The cocycle aspect of $\Psi[j, A]$ is of great interest if one studies the existence problem of the ISF in terms of the so-called "algebraic hull" (for more details on the algebraic hull, see the remarks at the end of Chapter 5 below). For literature on cocycles, see, e.g., [KR, Zi1] and Chapter 1 in [HK1].

3 Polarization-field trajectories and invariant polarization fields

In this chapter we introduce the notions of polarization field, invariant polarization field, spin field and invariant spin field and we present their most basic properties.

3.1 Generalities

High precision measurements of the spin-dependent properties of the interactions of colliding particles in storage rings depend on the equilibrium spin polarization being maximized. This, in turn, is facilitated by an understanding of the meaning of the term equilibrium, both as it applies to the value of the polarization and to its direction at each point in phase space. We will return to these matters in Section 8.1 but continue now with a definition and an exploration of the effects of maps.

Suppose therefore that $(j, A) \in \mathcal{SOS}(d, j)$ and that, at time $n = 0$, a spin vector S_{z_0} has been assigned to every point $z_0 = \pi_d(\phi_0)$ of the "particle torus" and consider their evolution according to (2.35). Let S_{z_0} denote the spin-vector trajectory with the initial value $S_0 = S_{z_0}(0)$. We define the field trajectory $\mathcal{S} = \mathcal{S}(n, z)$ by $\mathcal{S}(n, j^n(z)) = S_z(n)$ where n and z vary over $\mathbb{Z}$ and $\mathbb{T}^d$ respectively. Clearly $\mathcal{S}(n, \cdot)$ is the distribution of spins which started at $n = 0$ with the assignments S_{z_0} and evolved under the dynamics of (2.35). Since (2.35) gives us $S_z(n + 1) = A(j^n(z))S_z(n)$, we have

$$\mathcal{S}(n + 1, z) = A\Big(j^{-1}(z)\Big)\mathcal{S}\Big(n, j^{-1}(z)\Big) \; . \tag{3.1}$$

Definition 3.1 *(Polarization-field trajectory)*
Let $(j, A) \in \mathcal{SOS}(d, j)$. We call a function $\mathcal{S} \in \mathcal{C}(\mathbb{Z} \times \mathbb{T}^d, \mathbb{R}^3)$ a "polarization-field trajectory

of (j,A)", if it satisfies the evolution equation (3.1). Clearly $\mathcal{S}(n,\cdot)\in\mathcal{C}(\mathbb{T}^d,\mathbb{R}^3)$ and we call $\mathcal{S}(0,\cdot)$ the "initial value of $\mathcal{S}$". A polarization-field trajectory $\mathcal{S}$ is also called a "spin-field trajectory" if $|\mathcal{S}|=1$. □

It follows from the remarks before Definition 3.1 that if $\mathcal{S}$ is a polarization-field trajectory and if Z is a particle trajectory of (j,A) then (Z,S) is a particle-spin-vector trajectory of (j,A) where $S(n):=\mathcal{S}(n,Z(n))$. Every such trajectory we call a "characteristic particle-spin-vector trajectory". Thus the particle-spin motion can be viewed as a characteristic motion of the field motion and this plays a key role in the spin-vector tracking of the stroboscopic averaging method [EH, HH] (for more details on stroboscopic averaging, see Section 7.2). For an application of the characteristic motion in the present work, see Section 4.2.

At (2.22) we defined the function $\mathcal{P}[j,A]$ for transporting particles and their spin vectors. We now define the 1-turn field map, i.e., the function $\tilde{\mathcal{P}}[j,A]:\mathcal{C}(\mathbb{T}^d,\mathbb{R}^3)\to\mathcal{C}(\mathbb{T}^d,\mathbb{R}^3)$ for the field evolution by

$$\tilde{\mathcal{P}}[j,A](f):=(Af)\circ j^{-1}\,, \tag{3.2}$$

i.e., $(\tilde{\mathcal{P}}[j,A](f))(z):=A(j^{-1}(z))f(j^{-1}(z))$ where $f\in\mathcal{C}(\mathbb{T}^d,\mathbb{R}^3)$. To show that $\tilde{\mathcal{P}}[j,A]$ is a bijection we note, by (3.2) and for $(j,A)\in\mathcal{SOS}(d,j)$ and $(j',A')\in\mathcal{SOS}(d,j')$, that

$$\tilde{\mathcal{P}}[j',A']\circ\tilde{\mathcal{P}}[j,A]=\tilde{\mathcal{P}}[j'\circ j,A'']\,, \tag{3.3}$$

where $A''\in\mathcal{C}(\mathbb{T}^d,SO(3))$ is defined by $A'':=(A'\circ j)A$, whence

$$\tilde{\mathcal{P}}[j,A]=\tilde{\mathcal{P}}[j,A_{d,0}]\circ\tilde{\mathcal{P}}[id_{\mathbb{T}^d},A]\,, \tag{3.4}$$

where $A_{d,0}\in\mathcal{C}(\mathbb{T}^d,SO(3))$ is defined by $A_{d,0}(z):=I_{3\times3}$. Thus $\tilde{\mathcal{P}}[j,A]$ is a bijection since it has the inverse, $\tilde{\mathcal{P}}[j,A]^{-1}$, given by

$$\tilde{\mathcal{P}}[j,A]^{-1}=\tilde{\mathcal{P}}[id_{\mathbb{T}^d},A^t]\circ\tilde{\mathcal{P}}[j^{-1},A_{d,0}]=\tilde{\mathcal{P}}[j^{-1},A^t\circ j^{-1}]\,. \tag{3.5}$$

Amusingly, (3.3),(3.4),(3.5) would even hold if one would replace $\tilde{\mathcal{P}}$ by $\mathcal{P}$ as can be easily checked.

With (3.2) the evolution equation (3.1) can be written as $\mathcal{S}(n+1,\cdot)=\tilde{\mathcal{P}}[j,A](\mathcal{S}(n,\cdot))$ whence, for every polarization-field trajectory $\mathcal{S}$,

$$\mathcal{S}(n,\cdot)=\tilde{\mathcal{P}}[j,A]^n(\mathcal{S}(0,\cdot))\,. \tag{3.6}$$

We now compute n-th iterate $\tilde{\mathcal{P}}[j,A]^n$ of $\tilde{\mathcal{P}}[j,A]$ which we call the "n-turn field map" of (j,A). In fact, by (2.36), (3.2) and via induction in n,

$$\tilde{\mathcal{P}}[j,A]^n(f)=\Big(\Psi[j,A](n;\cdot)f\Big)\circ j^{-n}\,, \tag{3.7}$$

i.e., $(\tilde{\mathcal{P}}[j,A]^n(f))(z)=\Psi[j,A](n;j^{-n}(z))f(j^{-n}(z))$.

In Section 2.2 we demonstrated, in case of the particle spin-vector dynamics, that ϕ and z are of the same expressive power. In fact this is also the case for field dynamics as the following remark shows.

Remark:

(1) Let, as in Theorem 2.5e, $\hat{A} \in \mathcal{C}(\mathbb{R}^d, SO(3))$ be 2π-periodic in its arguments and $\hat{j} \in \mathrm{Map}_d$, i.e., $\hat{j} \in \mathrm{Homeo}(\mathbb{R}^d)$ such that $\pi_d \circ \hat{j}$ and $\pi_d \circ \hat{j}^{-1}$ are 2π-periodic in their arguments. We thus define the function $\tilde{\hat{\mathcal{P}}}[\hat{j}, \hat{A}] : \mathcal{C}_{per}(\mathbb{R}^d, \mathbb{R}^3) \to \mathcal{C}_{per}(\mathbb{R}^d, \mathbb{R}^3)$ for the field evolution in the ϕ variable by

$$\tilde{\hat{\mathcal{P}}}[\hat{j}, \hat{A}](\hat{f}) := (\hat{A}\hat{f}) \circ \hat{j}^{-1} \ , \tag{3.8}$$

i.e., $(\tilde{\hat{\mathcal{P}}}[\hat{j}, \hat{A}](\hat{f}))(z) := \hat{A}(\hat{j}^{-1}(\phi))\hat{f}(\hat{j}^{-1}(\phi))$ where $\hat{f} \in \mathcal{C}_{per}(\mathbb{R}^d, \mathbb{R}^3)$ and where $\mathcal{C}_{per}(\mathbb{R}^d, \mathbb{R}^3)$ denotes the set of functions in $\mathcal{C}(\mathbb{R}^d, \mathbb{R}^3)$ which are 2π-periodic in their arguments. Then, by the first part of Theorem 2.5e, $j := \pi_d \circ \hat{j} \circ \mathrm{Arg}$ belongs to $\mathrm{Homeo}(\mathbb{T}^d)$ and $A := \hat{A} \circ \mathrm{Arg}$ belongs to $\mathcal{C}(\mathbb{T}^d, SO(3))$. In analogy to the second part of Theorem 2.5e, we arrive at the following claim:
If $\hat{f} \in \mathcal{C}_{per}(\mathbb{R}^d, \mathbb{R}^3)$ is defined by $\hat{f} := f \circ \pi_d$ with $f \in \mathcal{C}(\mathbb{T}^d, \mathbb{R}^3)$ then $\hat{f}' = f' \circ \pi_d$ where $f' \in \mathcal{C}(\mathbb{T}^d, \mathbb{R}^3)$ and $\hat{f} \in \mathcal{C}_{per}(\mathbb{R}^d, \mathbb{R}^3)$ are defined by $f' := \tilde{\mathcal{P}}[j, A](f)$ and $\hat{f}' := \tilde{\hat{\mathcal{P}}}[\hat{j}, \hat{A}](\hat{f})$.

The claim means, in the language of Dynamical Systems Theory, that $\tilde{\hat{\mathcal{P}}}[\hat{j}, \hat{A}]$ and $\tilde{\mathcal{P}}[j, A]$ are equivariant, see for example [HK1].

Proof of the claim: We compute, by (3.2), (3.8) and Definition 2.4, $\hat{f}' \circ \hat{j} \circ \mathrm{Arg} = (\hat{A}\hat{f}) \circ \mathrm{Arg} = (A \circ \pi_d)(f \circ \pi_d) \circ \mathrm{Arg} = Af$ and $f' \circ \pi_d \circ \hat{j} \circ \mathrm{Arg} = f' \circ j = (Af) \circ j^{-1} \circ j = Af$ whence $\hat{f}' \circ \hat{j} \circ \mathrm{Arg} = f' \circ \pi_d \circ \hat{j} \circ \mathrm{Arg}$. Thus, and since $\hat{f}' \circ \hat{j}$ and $f' \circ \pi_d \circ \hat{j}$ are 2π-periodic in their arguments and continuous, we conclude from Theorem 2.5b that $\hat{f}' \circ \hat{j} = f' \circ \pi_d \circ \hat{j}$ whence, and since $\hat{j}$ is a bijection, we indeed get $\hat{f}' = f' \circ \pi_d$.
Clearly $\tilde{\hat{\mathcal{P}}}[\hat{j}, \hat{A}]$ is the "field" version of $\hat{\mathcal{P}}[\hat{j}, \hat{A}]$ and it would thus be easy to define the notion of polarization-field trajectory in terms of the angle variable ϕ (however in this work we focus on the variable z). □

In our follow-up work we will take a deeper look into polarization fields by generalizing the spin vector variable and thus generalizing the notion of polarization field.

3.2 Invariant polarization fields and invariant spin fields

We first need a definition.

Definition 3.2 *(Invariant polarization field, ISF)*
Let $(j, A) \in \mathcal{SOS}(d, j)$. A function $f \in \mathcal{C}(\mathbb{T}^d, \mathbb{R}^3)$ is called an "invariant polarization field of (j, A)" if it satisfies

$$f \circ j = Af \ . \tag{3.9}$$

Note, by (3.2),(3.9), that an $f \in \mathcal{C}(\mathbb{T}^d, \mathbb{R}^3)$ is an invariant polarization field of (j, A) iff

$$f = \tilde{\mathcal{P}}[j, A](f) \ . \tag{3.10}$$

Thus an $f \in \mathcal{C}(\mathbb{T}^d, \mathbb{R}^3)$ with $|f| = 1$ is an ISF of (j, A) iff (3.10) holds. An invariant polarization field f is called an "invariant spin field (ISF)" if $|f| = 1$. Thus an $f \in \mathcal{C}(\mathbb{T}^d, \mathbb{R}^3)$ with $|f| = 1$ is an ISF of (j, A) iff (3.10) holds. We denote the set of invariant spin fields of (j, A) by $\mathcal{ISF}(j, A)$. □

We now take a closer look at invariant polarization fields and we first recall from (3.1) that if $\mathcal{S}$ is a polarization-field trajectory of (j, A) then

$$\mathcal{S}(n+1, j(z)) = A(z)\mathcal{S}(n, z) \ , \tag{3.11}$$

whence if $\mathcal{S}$ is also time-independent then

$$\mathcal{S}(n, j(z)) = \mathcal{S}(n+1, j(z)) = A(z)\mathcal{S}(n, z) \ . \tag{3.12}$$

It follows from (3.11) and (3.12) and Definitions 3.1,3.2 and by induction in n that if $\mathcal{S}$ is a polarization-field trajectory of (j, A) then $\mathcal{S}$ is time-independent iff its initial value $\mathcal{S}(0, \cdot)$ is an invariant polarization field of (j, A).

Invariant polarization fields play an important role in polarized beam physics since they can be used to estimate the maximum attainable polarization of a bunch as we explain in Section 8.1, and since they are closely tied to the notions of spin tune and spin-orbit resonance (see Chapter 7). In fact as indicated in the Introduction invariant polarization fields are central to this work. This view will be confirmed in our follow-up work where we will generalize the notion of invariant polarization field.

We now make some comments on the question of the existence of the ISF for spin-orbit systems in $\mathcal{SOS}(d, j)$. It should be clear that the constraints involved in the definition of the ISF are nontrivial. However, if a spin-orbit system (j, A) has an ISF f then $-f$ is also an ISF of (j, A). So since $f \neq -f$, if (j, A) has a finite number of ISF's, then this number is even. The important subcase where (j, A) has exactly two ISF's is dealt with in Chapter 8.

It is also known [BV1] and to be examined in the following section, that spin-orbit systems exist which are on orbital resonance and which have no continuous ISF of the kind that we treat here. At the same time there are some indications, mainly from numerical computations on ISF's, that practically relevant spin-orbit systems which have no ISF are "rare". Thus we state the following conjecture, which we call the "ISF-conjecture": If (j, A) is a spin-orbit system such that j is topologically transitive then (j, A) has an ISF. Note that a special case of this conjecture is: If a spin-orbit system $(\mathcal{P}[\omega], A)$ is off orbital resonance, then it has an ISF.

The ISF-conjecture is, at least to our knowledge, unresolved. The question of the existence of the ISF is widely considered important both as a theoretical matter and as it relates to the practical matter of deciding whether a beam can have stable, non-vanishing polarization. Our follow-up work will present a new framework for discussing it.

Remark:

(2) Since we work in the framework of topological dynamical systems, A, j are continuous functions and we therefore require our fields to be continuous, in particular the invariant polarization fields. Thus every polarization-field trajectory $\mathcal{S}$ fulfills two different conditions: the "dynamical" condition (3.1) and the "regularity" condition that $\mathcal{S}$ is continuous. However, in contrast to the dynamical condition, the regularity condition is a matter of choice. While in this work, and in [He2], we choose continuity as the regularity property, this property can basically vary between the extremes "Borel measurable" and "of class C^∞". □

Since the ISF-conjecture deals with topologically transitive j we state the following theorem whose part b) considers this situation.

Theorem 3.3 *a) Let $j \in \mathrm{Homeo}(\mathbb{T}^d)$ be topologically transitive and let $g \in \mathcal{C}(\mathbb{T}^d, \mathbb{R})$ satisfy, for all $z \in \mathbb{T}^d$,*

$$g(j(z)) = g(z) \ . \tag{3.13}$$

Then g is constant, i.e., $g(z)$ is independent of z.
b) Let $(j, A) \in \mathcal{SOS}(d, j)$ where j is topologically transitive. If f is an invariant polarization field of (j, A) then $|f|$ is constant, i.e., $|f(z)|$ is independent of z. Also (j, A) has an ISF iff it has an invariant polarization field which is not identically zero.

Proof of Theorem 3.3a: We pick a $z_0 \in \mathbb{T}^d$ such that the set $B := \{j^n(z_0) : n \in \mathbb{Z}\}$ is dense in $\mathbb{T}^d$, i.e., $\overline{B} = \mathbb{T}^d$ and we define $g_0 := g(z_0)$. It follows from (3.13) that $B \subset C := \{z \in \mathbb{T}^d : g(z) = g_0\}$. On the other hand, the singleton $\{g_0\}$ is a closed subset of $\mathbb{R}$ whence, because g is continuous, C is a closed subset of $\mathbb{T}^d$. Therefore $\mathbb{T}^d = \overline{B} \subset \overline{C} = C$ so that $\mathbb{T}^d = C$. Thus, by the definition of C, we conclude that $g(z) = g_0$ for all $z \in \mathbb{T}^d$ which proves the claim. Note that C being closed means that its complement, say C', is open, i.e., $C' \in \tau_d$. $\square$
Proof of Theorem 3.3b: Let f be an invariant polarization field of (j, A). Then, by Definition 3.2, $f \in \mathcal{C}(\mathbb{T}^d, \mathbb{R}^3)$ and

$$|f(j(z))| = |f(z)| \ . \tag{3.14}$$

Defining $g \in \mathcal{C}(\mathbb{T}^d, \mathbb{R})$ by $g(z) := |f(z)|$ it follows from (3.14) that g satisfies (3.13). It thus follows from Theorem 3.3a that $|f(z)|$ is independent of z.

To prove the second claim, let f be an invariant polarization field of (j, A) which is not identically zero. Clearly by the first claim $|f|$ is constant and takes a nonzero value because $|f|$ is not identically zero. Thus we define $h \in \mathcal{C}(\mathbb{T}^d, \mathbb{R}^3)$ by $h := f/|f|$ whence, by Definition 3.2, h is an ISF of (j, A). Conversely every ISF of (j, A) is an invariant polarization field of (j, A) which is not identically zero. $\square$

We will apply Theorem 3.3a in the proof of Theorem 7.6 below. Moreover, Theorem 3.3a is of practical importance as will be explained in Section 7.2 when we discuss the computer code SPRINT. In Section 8.1 below we apply Theorem 3.3b and explain its practical importance.

In the special case when $j = \mathcal{P}[\omega]$ with $(1, \omega)$ nonresonant one can prove Theorem 3.3 alternatively by the machinery of Appendix B (see also [He2]). With Theorem 3.3b, the ISF conjecture is equivalent to the following statement: If j is topologically transitive then (j, A) has an invariant polarization field which is not identically zero. Note also that Theorem 3.3b will be generalized in our follow-up work.

A less formal picture surrounding Theorem 3.3b is as follows. When j is topologically transitive, the whole of $\mathbb{T}^d$ can effectively be reached from any starting position z_0 by repeated application of j. Moreover, by a corresponding repeated application of A, $f(z_0)$ generates $f(z)$ at effectively all points on $\mathbb{T}^d$. So the $f(z)$ on $\mathbb{T}^d$ are all "connected". Also, since A is $SO(3)$-valued all the $|f(z)|$ are the same. On the other hand, if j is not transitive, the $f(z)$ at different z need not be connected. For example the $f(z)$ at adjacent z could have opposite signs. We will encounter this situation in the following section but we first need the following remark on the notion of n-turn ISF.

Remark:

(3) Let $(j, A) \in \mathcal{SOS}(d, j)$ and let n be an integer. Then $j^{-n} = (j^n)^{-1}$ whence, by (3.7), $\tilde{\mathcal{P}}[j, A]^n(f) = \Big(\Psi[j, A](n; \cdot) f\Big) \circ (j^n)^{-1}$ so that, by (3.2),

$$\tilde{\mathcal{P}}[j, A]^n = \tilde{\mathcal{P}}\Big[j^n, \Psi[j, A](n; \cdot)\Big] . \tag{3.15}$$

Thus the n-turn field map of the spin-orbit system (j, A) is equal to the 1-turn field map of the spin-orbit system $(j^n, \Psi[j, A](n; \cdot))$. In particular if f is an ISF of (j, A) then, by Definition 3.2, $\tilde{\mathcal{P}}[j, A]f = f$ whence $\tilde{\mathcal{P}}[j, A]^n f = f$ so that, by (3.15), f is an ISF of $(j^n, \Psi[j, A](n; \cdot))$. An ISF of $(j^n, \Psi[j, A](n; \cdot))$ is sometimes called an "n-turn ISF" of (j, A), see, e.g., [HBEV3]. Thus every ISF of (j, A) is an n-turn ISF of (j, A) but the converse is not true (see, e.g., the spin-orbit system $(\mathcal{P}[1/2], A_{2S})$ in Section 3.3 below). It also follows from the above that, when $j^n = id_{\mathbb{T}^d}$ (e.g., if $j = \mathcal{P}[\omega]$ with $\omega \in \mathbb{Q}^d$), every ISF of (j, A) has a rather simple form if $n \neq 0$. In fact if f is an ISF of (j, A) and if $j^n = id_{\mathbb{T}^d}$ then f is an n-turn ISF of (j, A), i.e., an ISF of $(id_{\mathbb{T}^d}, \Psi[j, A](n; \cdot))$ whence, by (3.9), $f(z) = \Psi[j, A](n; z) f(z)$, i.e., $f(z)$ is an eigenvector in $\mathbb{R}^3$ so that the ISF is the solution of infinitely many eigenvalue problems for the eigenvalue 1. Thus if $j^n = id_{\mathbb{T}^d}$ with $n \neq 0$, the existence problem of the ISF of (j, A) is rather simple and in Section 3.3 below we will use this fact in a situation where $j^2 = id_{\mathbb{T}^d}$. Note that in the case $n = 0$ the eigenvalue problems read as $f(z) = f(z)$ so in this case they carry no interesting information. The above also suggests the following approach to the ISF conjecture: If $(1, \omega)$ is nonresonant then in order to find an ISF for $(\mathcal{P}[\omega], A)$ one approximates this spin-orbit system by a spin-orbit system $(\mathcal{P}[\chi], A)$ where $\chi \in \mathbb{Q}^d$ approximates ω and where existence problem of the ISF of $(\mathcal{P}[\chi], A)$ is rather simple due to the above. Note also that it can be easily shown, in analogy to (3.15), that $\mathcal{P}[j, A]^n = \mathcal{P}[j^n, \Psi[j, A](n; \cdot)]$. □

3.3 The 2-snake model

In this section we consider a model describing the spin-orbit system of a flat storage ring which has two thin-lens Siberian Snakes with mutually perpendicular axes of spin rotation placed at $\theta = 0$ and $\theta = \pi$. With this layout, the spin tune, ν_0, on the design orbit, of the ring is 1/2. Here we are interested in the situation where, in the absence of snakes, the spin motion is dominated by the effect of a single harmonic in the Fourier expansion of the radial component of the $\mathbf{\Omega}(\theta, J, \phi(\theta))$, mentioned in the Introduction, and due to vertical betatron motion. This case is often called the "single resonance model". The combination of the single resonance model and two snakes considered in this section has been studied extensively. See for example [BV1, Vo] and the references therein. The interest in this model stems from the effect on the polarization of the so-called "snake resonances". These occur at vertical betatron tunes of $1/2, 1/6, 5/6, 1/10, 3/10 \ldots$ Note that the term snake resonance is a misnomer since it does not refer to the proper definition of spin-orbit resonance given in (7.23). Our main interest here is in the fact that at snake resonance, there is no ISF of the kind that we define in this paper. We have already mentioned this situation in Section 7.2. For further background material see [BV1].

Here we focus on the simplest case, namely that with vertical betatron tune, $\omega = 1/2$, and we denote the resulting spin-orbit system by $(\mathcal{P}[1/2], A_{2S})$. Of course a real bunch is not stable at $\omega = 1/2$ but this does not play a role in the present section. In this section we will prove that $(\mathcal{P}[1/2], A_{2S})$ has no ISF and, as a byproduct, we will construct a "discontinuous" ISF $\hat{k}$, i.e., a normalized and piecewise continuous solution of (3.9) (see Remark 4 below).

We first define $(\mathcal{P}[1/2], A_{2S})$. For this we define the function $A_{2S} \in \mathcal{C}(\mathbb{T}, SO(3))$, for $\epsilon \in (\mathbb{R} \setminus \mathbb{Z})$ with [BV1, Vo] by

$$A_{2S}(z) := \begin{pmatrix} 1 - 2c^2(\phi) & 2b(\phi)c(\phi) & 2a(\phi)c(\phi) \\ 2b(\phi)c(\phi) & 1 - 2b^2(\phi) & -2a(\phi)b(\phi) \\ -2a(\phi)c(\phi) & 2a(\phi)b(\phi) & 2a^2(\phi) - 1 \end{pmatrix} , \tag{3.16}$$

where $\phi \in \mathbb{R}^d$ such that $\pi_d(\phi) = z$ and where the functions $a, b, c \in \mathcal{C}(\mathbb{R}, \mathbb{R})$ are defined by

$$a(\phi) := -2\sin^2(\pi\epsilon/2)\sin(\phi)\cos(\phi) , \quad b(\phi) := -2\sin(\pi\epsilon/2)\cos(\pi\epsilon/2)\cos(\phi) ,$$
$$c(\phi) := 2\sin^2(\pi\epsilon/2)\cos^2(\phi) - 1 . \tag{3.17}$$

Due to Theorem 2.5b, A_{2S} is well-defined by (3.16) and continuous since the functions a, b, c are 2π-periodic and continuous. Note also that

$$a^2 + b^2 + c^2 = 1 , \tag{3.18}$$

and that we exclude ϵ from being an integer because in that case $(\mathcal{P}[1/2], A_{2S})$ would have an ISF [He1].

It follows from (2.25),(2.26) that $\mathcal{P}[1/2]^2 = \mathcal{P}[1] = id_{\mathbb{T}}$ whence the existence problem of the ISF of $(\mathcal{P}[1/2], A_{2S})$ can be solved along the lines of Remark 3 above, i.e., by solving the eigenvalue problem $h(z) = \Psi\Big[\mathcal{P}[1/2], A_{2S}\Big](2; z)h(z)$ at every z. Note that $\mathcal{P}[1/2]^2 = id_{\mathbb{T}}$ means that a particle returns to the same z over two turns. We thus will prove that $(\mathcal{P}[1/2], A_{2S})$ has no ISF as follows. In the first step we show, by solving the eigenvalue problem $h(z) = \Psi\Big[\mathcal{P}[1/2], A_{2S}\Big](2; z)h(z)$ at every z, that $(\mathcal{P}[1/2], A_{2S})$ has just two 2-turn ISF's namely k and $-k$ defined below. Thus, recalling Remark 3, if $(\mathcal{P}[1/2], A_{2S})$ has an ISF then this ISF is also a 2-turn ISF of $(\mathcal{P}[1/2], A_{2S})$ whence it must be equal to k or $-k$. Therefore in the second step we will show that neither k nor $-k$ is an ISF of $(\mathcal{P}[1/2], A_{2S})$ which will finish the proof.

To perform the first step we recall from Remark 3 that a function $h \in \mathcal{C}(\mathbb{T}, \mathbb{R}^3)$ is a 2-turn ISF of $(\mathcal{P}[1/2], A_{2S})$ iff it satisfies, for every $z \in \mathbb{T}$,

$$h(z) = \Psi\Big[\mathcal{P}[1/2], A_{2S}\Big](2; z)h(z) , \tag{3.19}$$
$$|h(z)| = 1 . \tag{3.20}$$

To address the eigenvalue problem (3.19) we need to compute the 2–turn spin transfer matrix function $\Psi[\mathcal{P}[1/2], A_{2S}](2;\cdot)$. In fact, by (3.16) and (3.17),

$$A_{2S}(\mathcal{P}[1/2](z)) = A_{2S}(\pi_d(\phi+\pi)) = \begin{pmatrix} 1-2c^2(\phi) & -2b(\phi)c(\phi) & 2a(\phi)c(\phi) \\ -2b(\phi)c(\phi) & 1-2b^2(\phi) & 2a(\phi)b(\phi) \\ -2a(\phi)c(\phi) & -2a(\phi)b(\phi) & 2a^2(\phi)-1 \end{pmatrix}, \tag{3.21}$$

where $z = \pi_d(\phi)$ and where in the first equality we used (2.25). We conclude from (2.36), (3.16) and (3.21) that the 2–turn spin transfer matrix function reads as

$$\begin{aligned} &\Psi\Big[\mathcal{P}[1/2], A_{2S}\Big](2;z) = A_{2S}(\mathcal{P}[1/2](z))A_{2S}(z) \\ &= \begin{pmatrix} 1-8c^2(\phi)+8c^4(\phi) & 4b(\phi)c(\phi)(1-2c^2(\phi)) & 4a(\phi)c(\phi)(1-2c^2(\phi)) \\ -4b(\phi)c(\phi)(1-2c^2(\phi)) & 1-8b^2(\phi)c^2(\phi) & -8a(\phi)b(\phi)c^2(\phi) \\ -4a(\phi)c(\phi)(1-2c^2(\phi)) & -8a(\phi)b(\phi)c^2(\phi) & 1-8a^2(\phi)c^2(\phi) \end{pmatrix}, \end{aligned} \tag{3.22}$$

where $z = \pi_d(\phi)$. Since ϵ is not an integer, $|\sin(\pi\epsilon/2)|$ equals neither 0 or 1, and so we define the 2π-periodic function $K \in \mathcal{C}(\mathbb{R}, \mathbb{R}^3)$ by

$$K(\phi) := \frac{\cos(\pi\epsilon/2)}{|\cos(\pi\epsilon/2)|\sqrt{1-\sin^2(\pi\epsilon/2)\cos^2(\phi)}}\Big(0, \sin(\pi\epsilon/2)\sin(\phi), -\cos(\pi\epsilon/2)\Big)\,. \tag{3.23}$$

By Theorem 2.5b and since K is continuous and 2π-periodic, a unique function $k \in \mathcal{C}(\mathbb{T}, \mathbb{R}^3)$ exists such that

$$K = k \circ \pi_1\,. \tag{3.24}$$

It is easy to show that (3.19) and (3.20) are fullfilled for $h = k$, i.e.,

$$k(z) = \Psi[\mathcal{P}[1/2], A_{2S}](2;z)k(z)\,, \tag{3.25}$$

$$|k(z)| = 1\,. \tag{3.26}$$

Thus indeed k and $-k$ are 2-turn ISF's of $(\mathcal{P}[1/2], A_{2S})$.

To complete the first step of our proof we need to show that k and $-k$ are the only 2-turn ISF's of $(\mathcal{P}[1/2], A_{2S})$ so let $h \in \mathcal{C}(\mathbb{T}, \mathbb{R}^3)$ be an arbitrary 2-turn ISF of $(\mathcal{P}[1/2], A_{2S})$, i.e., let h satisfy (3.19) and (3.20). To show that either $h = k$ or $h = -k$ let $R \neq I_{3\times 3}$ be a matrix in $SO(3)$. Then R has a real eigenvector $v \in \mathbb{R}^3$ with eigenvalue 1 and such that $|v| = 1$ whence $r \in SO(3)$ exists such that $v = r(0,0,1)^t$. Thus $r^tRr(0,0,1)^t = (0,0,1)^t$ whence, by Theorem 6.2a in Chapter 6 below, $r^tRr \in SO(2)$ so that a $\nu \in [0,1)$ exists such that $R = r\exp(2\pi\nu\mathcal{J})r^t$. This implies, since $R \neq I_{3\times 3}$, that $\nu \neq 0$. Thus if $w, w' \in \mathbb{R}^3$ are real eigenvectors of r^tRr with the eigenvalue 1 and $|w| = |w'| = 1$ then $|w\cdot w'| = 1$ whence if $v, v' \in \mathbb{R}^3$ are real eigenvectors of R with the eigenvalue 1 and $|v| = |v'| = 1$ then $|v\cdot v'| = 1$.

Defining the set

$$M := \{z \in \mathbb{T} : \Psi[\mathcal{P}[1/2], A_{2S}](2;z) = I_{3\times 3}\}\,, \tag{3.27}$$

we observe that, if $z \in (\mathbb{T}\setminus M)$, then $\Psi[\mathcal{P}[1/2], A_{2S}](2;z) \neq I_{3\times 3}$. Thus, and since by (3.19), (3.20), (3.25) and (3.26), $h(z), k(z)$ are real eigenvectors of $\Psi[\mathcal{P}[1/2], A_{2S}](2;z)$ with eigenvalue 1 and $|h(z)| = |k(z)| = 1$ we conclude that, if $z \in (\mathbb{T} \setminus M)$, then $\lambda(z) = 1$ where the function $\lambda : \mathbb{T} \to \mathbb{R}$ is defined by $\lambda(z) := |h(z) \cdot k(z)|$. To show that $\lambda(z) = 1$ for all $z \in \mathbb{T}$ we only have to show that λ is a constant function. We thus compute, by (3.17) and (3.22),

$$M = \{\pi_1(\phi) : \phi \in \mathbb{R}, c(\phi)(c^2(\phi) - 1) = 0\} = \{\pi_1(\phi) : \phi \in \mathbb{R}, \cos^2(\phi) = \frac{1}{2\sin^2(\pi\epsilon/2)}\} , \tag{3.28}$$

whence M consists of only finitely many points. Since $\lambda(z) = 1$ on $\mathbb{T} \setminus M$ and since M has only finitely many points we conclude that λ is a continuous function with only finitely many values. Since $\mathbb{T}$ is path-connected and λ is continuous we use the same argument as in the proof of Theorem 8.1b and conclude that the range of λ is an interval whence λ is constant so that $\lambda(z) = |h(z) \cdot k(z)| = 1$ holds for *every* $z \in \mathbb{T}$. Thus, and since $|h(z)| = |k(z)| = 1$, either $h = k$ or $h = -k$. So we have shown that the only 2-turn ISF's are $h = k$ and $h = -k$. This completes the first step of our proof.

In the second step we now show that neither k nor $-k$ is an ISF so we compute, by (3.16) and (3.23),

$$A_{2S}(\pi_1(\phi))K(\phi) = -K(\phi + \pi) , \tag{3.29}$$

whence, by (2.25) and (3.24), $A_{2S}(z)k(z) = -k(\mathcal{P}[1/2](z))$ so that, by (3.2),

$$\tilde{\mathcal{P}}[\mathcal{P}[1/2], A_{2S}](k) = -k , \tag{3.30}$$

which implies, by Definition 3.2, that k is not an ISF of $(\mathcal{P}[1/2], A_{2S})$. Thus $-k$ is not an ISF of $(\mathcal{P}[1/2], A_{2S})$ either. Therefore the only two 2-turn ISF's of $(\mathcal{P}[1/2], A_{2S})$ are not ISF's of $(\mathcal{P}[1/2], A_{2S})$. This completes the second and final step of our proof. We thus conclude that $(\mathcal{P}[1/2], A_{2S})$ has no ISF.

Remark:

(4) While $(\mathcal{P}[1/2], A_{2S})$ has no ISF, it is easy to construct a normalized, piecewise continuous solution of (3.9) for the spin-orbit system $(\mathcal{P}[1/2], A_{2S})$ (see also [BV2]). In fact defining $\hat{K} : \mathbb{R} \to \mathbb{R}^3$ by

$$\hat{K}(\phi) := \begin{cases} K(\phi) & \text{if } \phi \in \bigcup_{n\in\mathbb{Z}}[2\pi n, 2\pi n + \pi) \\ -K(\phi) & \text{if } \phi \in \bigcup_{n\in\mathbb{Z}}[2\pi n + \pi, 2\pi n + 2\pi) , \end{cases} \tag{3.31}$$

we are led, by Section 2.2, to define the function $\hat{k} : \mathbb{T} \to \mathbb{R}^3$ by $\hat{k}(z) := \hat{K}(\mathrm{Arg}(z))$. It is a simple exercise to show that $\hat{k}$ is a normalized piecewise continuous solution of (3.9) for the spin-orbit system $(\mathcal{P}[1/2], A_{2S})$. Of course, $\hat{k}$ is not an ISF of $(\mathcal{P}[1/2], A_{2S})$ since $(\mathcal{P}[1/2], A_{2S})$ has no ISF. In fact it is an easy exercise to show, by (3.23) and (3.31), that $\hat{k}$ is discontinuous at $z = \pi_1(0)$ and $z = \pi_1(\pi)$. This is an example of a consequence of a lack of topological transitivity of j mentioned just after Theorem 3.3.

As mentioned in Remark 2 above, since A, j are continuous we require that invariant fields be continuous. However this requirement is a matter of choice. In fact if one would impose the weaker condition of Borel measurability then $\hat{k}$ would be an ISF. In fact, as mentioned in Section 7.2 below, the requirement of continuity was relaxed in [BV1]. □

For more details on $(\mathcal{P}[1/2], A_{2S})$, see Appendix B.6 below.

4 Transforming spin-orbit systems and the partition of $\mathcal{SOS}(d, j)$

Transformations of spin-orbit systems are important since they underly the important notions of IFF and spin tune as illustrated in Chapters 5,6,7. Thus in this chapter, in order to provide the basic machinery needed in Chapters 5,6,7, we introduce the transformation of any $(j, A) \in \mathcal{SOS}(d, j)$ under any $T \in \mathcal{C}(\mathbb{T}^d, SO(3))$, i.e., we state a transformation formula which is well-known in the Beam Polarization community in the rigorous terms of the formalism developed in the previous chapters. In fact (j, A) is transformed into (j, A') where A' is given by (4.2) below, i.e., $A'(z) := T^t(j(z))A(z)T(z)$. Thanks to our formalism, we find that the above transformations partition every $\mathcal{SOS}(d, j)$ into "equivalence classes" where two spin-orbits systems are "equivalent", i.e., belong to the same class, if they are connected by one of these transformations. The dynamics of equivalent spin-orbit systems can be considered as essentially the same.

A central idea in Dynamical Systems Theory is, for a given dynamical system, to find an "equivalent" one which is simpler to analyze. The particular notion of equivalence mentioned above and used in this chapter is an example and thus it is of interest to find transformations T which simplify a given (j, A). More precisely we want to find a spin-orbit system (j, A'), where $A'(z) := T^t(j(z))A(z)T(z)$, such that the function $S'(n) := T^t(Z(n))S(n)$ is as simple as possible.

We use the transpose of T as a matter of convention (this convention has the side effect that under certain conditions on T, the third column, not the third row, of T is an ISF - see the IFF Theorem in Chapter 6). Also note that if $T \in \mathcal{C}(\mathbb{T}^d, SO(3))$ then $T^t = T^{-1} \in \mathcal{C}(\mathbb{T}^d, SO(3))$.

In Section 4.1 we make precise our transformation law and define the corresponding partition of $\mathcal{SOS}(d, j)$. In addition we relate in Section 4.1 the particle-spin-vector trajectories of equivalent spin-orbit systems by relating $\mathcal{P}[j, A]$ and $\mathcal{P}[j, A']$ etc. Analogously we relate in Section 4.2 the polarization-field trajectories of equivalent spin-orbit systems. In Section 4.3 we put Sections 4.1 and 4.2 in a more general "tranformation theoretical context" a context we will not consider in this work. In Chapter 5 we use the transformation law of the present chapter by considering spin-orbit systems (j, A) which have an H-normal form, that is (j, A) is equivalent to (j, A') where, for all $z \in \mathbb{T}^d$, $A'(z) \in H$ with H being a subgroup of $SO(3)$ w.r.t. matrix multiplication. The spin-orbit system (j, A') is considered as "simple" if A' is H-valued where H is a subgroup of $SO(3)$ which is "small" (for more details of this philosophy, see Chapter 5). This is the case of the subgroups $SO(2)$ and G_ν which we consider in Chapters 6 and 7.

4.1 The transformation of spin-orbit systems and of particle-spin-vector trajectories

Consider $(j, A) \in \mathcal{SOS}(d, j)$ and let (Z, S) be a particle-spin-vector trajectory of (j, A), i.e., let (2.31) hold so that S is a spin-vector trajectory of (j, A) and thus $S(n+1) = A(Z(n))S(n)$. For arbitrary $T \in \mathcal{C}(\mathbb{T}^d, SO(3))$, the function $S' : \mathbb{Z} \to \mathbb{R}^3$, defined by

$$S'(n) := T^t(Z(n))S(n) \;, \tag{4.1}$$

satisfies $S'(n+1) = T^t(Z(n+1))A(Z(n))T(Z(n))S'(n)$. So (Z, S') is a particle-spin-vector trajectory of a new spin-orbit system, namely of $(j, A') \in \mathcal{SOS}(d, j)$ which is defined by

$$A'(z) := T^t(j(z))A(z)T(z) \;. \tag{4.2}$$

Eq. (4.2) gives rise to a partition of $\mathcal{SOS}(d, j)$ as we formalize in the next two definitions.

Definition 4.1 *(Transformation of spin-orbit systems, transfer field)*
Let (j, A) and (j, A') be in $\mathcal{SOS}(d, j)$. Then a T in $\mathcal{C}(\mathbb{T}^d, SO(3))$ is called a "transfer field from (j, A) to (j, A')" iff (4.2) holds. We also say that "(j, A') is the transform of (j, A) under T". We denote the collection of all transfer fields from (j, A) to (j, A') by $\mathcal{TF}(A, A'; d, j)$. Note that if $T \in \mathcal{TF}(A, A'; d, j)$ then $T^t \in \mathcal{TF}(A', A; d, j)$, i.e., (j, A) is the transform of (j, A') under T^t. □

It follows from the remarks before Definition 4.1 that if T is any transfer field from (j, A) to (j, A') and if (Z, S) is a particle-spin-vector trajectory of (j, A) then (Z, S') is a particle-spin-vector trajectory of (j, A') where S' is defined by (4.1), i.e., $S'(n) = T^t(Z(n))S(n)$.

Following Appendix A.2 we make the definition:

Definition 4.2 *Let (j, A) and (j, A') be in $\mathcal{SOS}(d, j)$. Then we write $(j, A) \sim (j, A')$ and say that (j, A) and (j, A') are "equivalent" iff (j, A') is a transform of (j, A) under some $T \in \mathcal{C}(\mathbb{T}^d, SO(3))$. The relation $\sim$ is reflexive, symmetric, and transitive and thus is an equivalence relation on $\mathcal{SOS}(d, j)$, see Remark 0 below. Let $\overline{(j, A)} := \{(j, A') : (j, A') \sim (j, A)\}$, i.e., the equivalence class of (j, A) under $\sim$. As outlined in Appendix A.2, the sets $\overline{(j, A)}$ partition $\mathcal{SOS}(d, j)$.* □

Two spin-orbit systems which are equivalent share many important properties, e.g., the existence or nonexistence of an ISF (see Remark 3 below). We will see other properties shared by equivalent spin-orbit systems throughout this work and in our follow-up work. Thus the dynamics of equivalent spin-orbit systems can be considered as essentially the same. Therefore if (j, A) can be transformed into a "simple" (j, A') then we consider the particle-spin-vector motions, the polarization field motions and the invariant polarization fields of *all* spin-orbit systems in $\overline{(j, A)}$ as "simple". Of course, for checking those shared properties it can be convenient to check them for a "simple" element of $\overline{(j, A)}$ (see Chapters 5,6 and 7).

We now study how the transformation formula (4.2) affects $\mathcal{P}[j, A]$ and $\Psi[j, A]$. Under the transformation $(j, A) \longrightarrow (j, A')$, given by (4.2), $\mathcal{P}[j, A]$ becomes $\mathcal{P}[j, A']$ and the latter is given by

$$\mathcal{P}[j, A'] = \mathcal{P}[id_{\mathbb{T}^d}, T]^{-1} \circ \mathcal{P}[j, A] \circ \mathcal{P}[id_{\mathbb{T}^d}, T] \;, \tag{4.3}$$

where T is any transfer field from (j, A) to (j, A'). Eq. (4.3) is easily checked since, by (2.22), $\mathcal{P}[j, A](z, S) = \begin{pmatrix} j(z) \\ A(z)S \end{pmatrix}$ and $\mathcal{P}[j, A'](z, S) = \begin{pmatrix} j(z) \\ A'(z)S \end{pmatrix}$ as well as $\mathcal{P}[id_{\mathbb{T}^d}, T](z, S) = \begin{pmatrix} z \\ T(z)S \end{pmatrix}$. Amusingly, even the converse holds, i.e., (4.3) implies (4.2). Of course (4.3) also implies

$$\mathcal{P}[j, A']^n = \mathcal{P}[id_{\mathbb{T}^d}, T]^{-1} \circ \mathcal{P}[j, A]^n \circ \mathcal{P}[id_{\mathbb{T}^d}, T] \ , \tag{4.4}$$

Finally under the transformation $(j, A) \longrightarrow (j, A')$, given by (4.2), $\Psi[j, A]$ becomes $\Psi[j, A']$ and the latter is given by

$$\Psi[j, A'](n; z) = T^t(j^n(z))\Psi[j, A](n; z)T(z) \ , \tag{4.5}$$

where T is any transfer field from (j, A) to (j, A'). Eq. (4.5) is easily checked via induction in n and by (2.39) and (4.2). Recall from Section 2.5 that $\Psi[j, A]$ is a cocycle in the terminology of Dynamical Systems Theory and in this terminology the equality (4.5) means that the cocycles $\Psi[j, A]$ and $\Psi[j, A']$ are "cohomologous". For this notion, see, e.g., [He2, KR, Zi1] and Chapter 1 in [HK1]. The transformation behavior displayed in (4.1), (4.3),(4.5) will be generalized in our follow-up work by generalizing the spin vector variable.

We now make some comments on Definitions 4.1 and 4.2. First of all $\mathcal{TF}(A, A'; d, j) \neq \emptyset$ iff (j, A') is a transform of (j, A) as in (4.2). In the case, where $j = \mathcal{P}[\omega]$, the equivalence relation $\sim$ on $\mathcal{SOS}(d, j)$ has more than one equivalence class whence not every set $\mathcal{TF}(A, A'; d, j)$ is nonempty. In fact it is shown after Remark 6 in Chapter 7 that the equivalence relation $\sim$ on $\mathcal{SOS}(d, j)$ has infinitely many equivalence classes in the case where $j = \mathcal{P}[\omega]$. It is likely that this holds not only in the case $j = \mathcal{P}[\omega]$. Finally the following remark shows that $\sim$, defined in Definition 4.2, is an equivalence relation.

Remark:

(0) We here make some comments on the relation $\sim$ defined in Definition 4.2. First of all $\sim$ is reflexive since the constant $I_{3\times3}$-valued function on $\mathbb{T}^d$ is a transfer field from (j, A) to (j, A). Secondly $\sim$ is symmetric since if T is a transfer field from (j, A) to (j, A') then T^t is a transfer field from (j, A') to (j, A) (note that (4.2) implies $A(z) = T(j(z))A'(z)T^t(z)$). Thirdly $\sim$ is transitive since if T is a transfer field from (j, A) to (j, A') and T' is a transfer field from (j, A') to (j, A'') then TT' is a transfer field from (j, A) to (j, A''). This completes the proof that $\sim$ is an equivalence relation on $\mathcal{SOS}(d, j)$ (see also [wiki4]). $\square$

4.2 Transforming polarization-field trajectories

We now study how the transformation formula (4.2) affects $\tilde{\mathcal{P}}[j, A]$ and we will also find, correspondingly to (4.1), transformation formulas for polarization field trajectories and invariant polarization fields.

First of all under the transformation $(j, A) \longrightarrow (j, A')$, given by (4.2), $\tilde{\mathcal{P}}[j, A]$ becomes $\tilde{\mathcal{P}}[j, A']$ and the latter is given by

$$\tilde{\mathcal{P}}[j, A'] = \tilde{\mathcal{P}}[id_{\mathbb{T}^d}, T]^{-1} \circ \tilde{\mathcal{P}}[j, A] \circ \tilde{\mathcal{P}}[id_{\mathbb{T}^d}, T] \ , \tag{4.6}$$

where T is any transfer field from (j, A) to (j, A'). Eq. (4.6) is easily checked by (3.3),(4.2). Of course (4.6) implies that

$$\tilde{\mathcal{P}}[j, A']^n = \tilde{\mathcal{P}}[id_{\mathbb{T}^d}, T]^{-1} \circ \tilde{\mathcal{P}}[j, A]^n \circ \tilde{\mathcal{P}}[id_{\mathbb{T}^d}, T] \ .?? \tag{4.7}$$

We now transform polarization-field trajectories. Recall from the remarks after Definition 4.1 that if T is a transfer field from (j, A) to (j, A') and if (Z, S) is a particle-spin-vector trajectory of (j, A) then (Z, S') is a particle-spin-vector trajectory of (j, A') where S is defined by (4.1), i.e., $S'(n) := T^t(Z(n))S(n)$. We now apply this transformation to the characteristic particle-spin-vector trajectories so let $\mathcal{S}$ be a polarization-field trajectory of (j, A). Defining $S(n) := \mathcal{S}(n, Z(n))$ we recall from the remarks after Definition 3.1 that (Z, S) is a particle-spin-vector trajectory of (j, A) whence $S'(n) := T^t(Z(n))\mathcal{S}(n, Z(n))$ is a particle-spin-vector trajectory of (j, A'). This suggests that $\mathcal{S}'$, defined by

$$\mathcal{S}'(n, z) := T^t(z)\mathcal{S}(n, z) \ , \tag{4.8}$$

is a polarization-field trajectory of (j, A') which is easily checked by (3.1),(4.2).

With (4.8), and by the remarks after (3.12), we have the following transformation formula of invariant polarization fields:

$$f'(z) = T^t(z)f(z) \ , \tag{4.9}$$

where f is an invariant polarization field of (j, A) and f' is an invariant polarization field of (j, A') with T being any transfer field from (j, A) to (j, A'). For an application of (4.9), see the proof of the IFF Theorem in Chapter 6.

The transformation behavior displayed in (4.6),(4.8),(4.9) will be generalized in our follow-up work where it will be derived from an $SO(3)$-gauge transformation.

Remarks:

(1) The transformation formulas (4.1),(4.3),(4.5) and (4.6),(4.8),(4.9) are no strangers to the polarized-beam community. In fact when researchers deal with the topics of spin tune, spin frequency, spin resonances, resonance strengths etc. then they often appeal more or less directly to these transformation formulas. In those applications the aim, typically, is to transform (j, A) to a "simple" (j, A'). In the present work these transformation formulas are applied to the notions of IFF and spin tune in Chapters 6,7.

(2) Let (Z, S) be a particle-spin-vector trajectory of a spin-orbit system (j, A). Then the transformation formula (4.1) could be generalized to

$$S'(n) := R^t(n, Z(n))S(n) \ , \tag{4.10}$$

where $R : \mathbb{Z} \times \mathbb{T}^d \to SO(3)$ is an arbitrary continuous function generalizing the notion of transfer field. However in general (Z, S'), with S' from (4.10), is not a particle-spin-vector trajectory of any spin-orbit system. This is evident in the case where $j = \mathcal{P}[\omega]$ as follows. If Z is a particle trajectory of $(\mathcal{P}[\omega], A)$ with $Z(0) = \pi_d(\phi_0)$ then, by Appendix B.1, the equations of motion for the spin-vector trajectory S read as $S(n+1) = A(\pi_d(\phi_0 + 2\pi n\omega))S(n)$. In the language of Appendix B.1, the matrix elements of $A(\pi_d(\phi_0 + 2\pi n\omega))$ are quasiperiodic functions of n. Also, the equations of motion for the spin-vector trajectory S' in (4.10) read as $S'(n+1) = R^t(n+1, \pi_d(\phi_0 + 2\pi(n+1)\omega))A(\pi_d(\phi_0 + 2\pi n\omega))R(n, \pi_d(\phi_0 + 2\pi n\omega))S'(n)$. However, and using again the language of Appendix B.1, in general the matrix elements of $R^t(n+1, \pi_d(\phi_0 + 2\pi(n+1)\omega))A(\pi_d(\phi_0 + 2\pi n\omega))R(n, \pi_d(\phi_0 + 2\pi n\omega))$ are not quasiperiodic functions of n. Thus in general (Z, S'), with S' from (4.10), is not a particle-spin-vector trajectory of any spin-orbit system.

(3) It is clear that (4.8) maps the polarization-field trajectories of (j, A) bijectively onto the set of polarization-field trajectories of (j, A'). It is equally clear that (4.9) maps the set of invariant polarization fields of (j, A) bijectively onto the set of invariant polarization fields of (j, A') and that it maps $\mathcal{ISF}(j, A)$ bijectively onto $\mathcal{ISF}(j, A')$. In particular, $\sim$-related spin-orbit systems have the same number of ISF's. $\square$

4.3 Remarks on conjugate 1-turn particle-spin-vector maps and structure preserving homeomorphisms

Note that $\mathrm{Homeo}(\mathbb{T}^d \times \mathbb{R}^3)$ forms a group, where the group multiplication is understood to be the composition of functions. Thus, since $\mathcal{P}[j, A] \in \mathrm{Homeo}(\mathbb{T}^d \times \mathbb{R}^3)$, it follows from (4.3) and Definition 4.2 and Appendix A.6 that if $(j, A) \sim (j, A')$ then $\mathcal{P}[j, A]$ and $\mathcal{P}[j, A']$ are conjugate elements of the group $\mathrm{Homeo}(\mathbb{T}^d \times \mathbb{R}^3)$, i.e., a $\mathcal{T} \in \mathrm{Homeo}(\mathbb{T}^d \times \mathbb{R}^3)$ exists such that $\mathcal{P}[j, A'] = \mathcal{T}^{-1} \circ \mathcal{P}[j, A] \circ \mathcal{T}$. In fact $\mathcal{T} = \mathcal{P}[id_{\mathbb{T}^d}, T]$ with $T \in \mathcal{TF}(A, A'; d, j)$ is an example. We call a $\mathcal{T} \in \mathrm{Homeo}(\mathbb{T}^d \times \mathbb{R}^3)$ "structure preserving for a $\mathcal{SOS}(d, j)$" if, for every $(j, A) \in \mathcal{SOS}(d, j)$, the homeomorphism $\mathcal{T}^{-1} \circ \mathcal{P}[j, A] \circ \mathcal{T}$ in $\mathrm{Homeo}(\mathbb{T}^d \times \mathbb{R}^3)$ is of the form $\mathcal{P}[j, A']$ for some $(j, A') \in \mathcal{SOS}(d, j)$. As we discovered in Section 4.1, every $\mathcal{P}[id_{\mathbb{T}^d}, T]$ with $T \in \mathcal{C}(\mathbb{T}^d, SO(3))$ is structure preserving for $\mathcal{SOS}(d, j)$ (for every $j \in \mathrm{Homeo}(\mathbb{T}^d)$). Thus the natural question arises: Which $\mathcal{T} \in \mathrm{Homeo}(\mathbb{T}^d \times \mathbb{R}^3)$ are structure preserving for a given $\mathcal{SOS}(d, j)$? While this question from Dynamical-Systems Theory will not be fully addressed in this work we now give a brief glimpse. Let $(j, A) \in \mathcal{SOS}(d, j)$ and $(j, A') \in \mathcal{SOS}(d, j)$ and let $\mathcal{T} \in \mathrm{Homeo}(\mathbb{T}^d \times \mathbb{R}^3)$. Writing $\mathcal{T}$ in terms of components $\mathcal{T} = (\mathcal{T}_{part}, \mathcal{T}_v)$, where $\mathcal{T}_{part} \in \mathcal{C}(\mathbb{T}^d, \mathbb{T}^d)$ and $\mathcal{T}_v \in \mathcal{C}(\mathbb{T}^d \times \mathbb{R}^3, \mathbb{R}^3)$, we compute

$$
\begin{aligned}
&(\mathcal{T} \circ \mathcal{P}[j, A'])(z, S) = \mathcal{T}(j(z), A'(z)S) = (\mathcal{T}_{part}(j(z)), \mathcal{T}_v(j(z), A'(z)S)) \, , \\
&(\mathcal{P}[j, A] \circ \mathcal{T})(z, S) = \mathcal{P}[j, A](\mathcal{T}_{part}(z), \mathcal{T}_v(z, S)) \\
&\quad = (j(\mathcal{T}_{part}(z)), A(\mathcal{T}_{part}(z))\mathcal{T}_v(z, S)) \, ,
\end{aligned}
$$

whence $\mathcal{P}[j, A'] = \mathcal{T}^{-1} \circ \mathcal{P}[j, A] \circ \mathcal{T}$ iff

$$\mathcal{T}_{part}(j(z)) = j(\mathcal{T}_{part}(z)) \, , \tag{4.11}$$

$$\mathcal{T}_v(j(z), A'(z)S) = A(\mathcal{T}_{part}(z))\mathcal{T}_v(z, S) \, . \tag{4.12}$$

The system of equations (4.11),(4.12) plays a central role when one addresses the aforementioned questions. Of course in the special case $\mathcal{T} = \mathcal{P}[id_{\mathbb{T}^d}, T]$ with $T \in \mathcal{C}(\mathbb{T}^d, SO(3))$ we see that $\mathcal{T}_{part} = id_{\mathbb{T}^d}$ and $\mathcal{T}_v(z,S)) = T(z)S$ so that in that case we recover the fact from Section 4.1 that $\mathcal{P}[j,A'] = \mathcal{T}^{-1} \circ \mathcal{P}[j,A] \circ \mathcal{T}$ iff $T \in \mathcal{TF}(A,A';d,j)$. We finally mention that there are structure preserving $\mathcal{T} \in \mathrm{Homeo}(\mathbb{T}^d \times \mathbb{R}^3)$ which are different from any $\mathcal{P}[id_{\mathbb{T}^d}, T]$. To give a simple example, we define $\mathcal{T} \in \mathrm{Homeo}(\mathbb{T}^d \times \mathbb{R}^3)$ by $\mathcal{T} = (\mathcal{T}_{part}, \mathcal{T}_v)$ where $\mathcal{T}_{part}(z) = z, \mathcal{T}_v(z,S) = T(z)\hat{\mathcal{J}}S$ with $T \in \mathcal{C}(\mathbb{T}^d, SO(3))$ and $\hat{\mathcal{J}} := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$. Note that $\mathcal{T} \in \mathrm{Homeo}(\mathbb{T}^d \times \mathbb{R}^3)$ because its inverse is the continuous function $\mathcal{T}^{inv}$ which is defined by $\mathcal{T}^{inv} := (\mathcal{T}^{inv}_{part}, \mathcal{T}^{inv}_v)$, where $\mathcal{T}^{inv}_{part}(z) := z, \mathcal{T}^{inv}_v(z,S) := \hat{\mathcal{J}}T^t(z)S$. One easily sees that $\mathcal{T}$ is structure preserving and is different from any $\mathcal{P}[id_{\mathbb{T}^d}, T]$. The latter follows from the fact that $\hat{\mathcal{J}}$ has determinant -1. Note that (4.11),(4.12) read for this example as $j(z) = j(z), T(j(z))\hat{\mathcal{J}}A'(z)S = A(z)T(z)\hat{\mathcal{J}}S$.

5 H-normal forms and the subsets $\mathcal{CB}_H(d,j)$ of $\mathcal{SOS}(d,j)$

In Chapter 4 we introduced the fundamental transformation formula (4.2) which partitions every $\mathcal{SOS}(d,j)$ into equivalence classes $\overline{(j,A)}$. Clearly the spin-orbit systems in an equivalence class are related by a transformation and the underlying motivation is to find a simple spin-orbit system in every $\overline{(j,A)}$. Dictated by the notions of IFF and spin tune and suggested by our formalism, our way to address this search goes by formulating the notion of "H-normal form" of (j,A). Precisely, (j,A') is an H-normal form of (j,A) if $(j,A') \in \overline{(j,A)}$ and if A' is H-valued, i.e., $A'(z) \in H$ for all $z \in \mathbb{T}^d$ where H is a subgroup of $SO(3)$. Thus we introduce the notation $\mathcal{CB}_H(d,j)$ for the collection of all (j,A) in $\mathcal{SOS}(d,j)$ which have an H-normal form. As explained in Section 4.1, the dynamics of equivalent spin-orbit systems can be considered as essentially the same whence the spirit here is that if H is "small" then the dynamics of *all* spin-orbit systems in $\overline{(j,A)}$ are considered as "simple" (see also the remarks after Remark 3). More precisely, if $(j,A) \in \mathcal{CB}_H(d,j)$ and H is "small" then the particle-spin-vector trajectories of (j,A) are "simple" and the particle-spin-vector trajectories of $(j,A') \in \overline{(j,A)}$ are even manifestly "simple" if A' is H-valued. In fact in Chapter 6, for the purpose of studying the IFF, we will consider the case of $H = SO(2)$ which is substantially smaller than $SO(3)$ and in Chapter 7 for the purpose of studying the spin tune and spin-orbit resonance, we will consider the case of $H = G_\nu \subset SO(2)$ which is even smaller than $SO(2)$. However in the present chapter we focus on the general H. Note that the notion of H-normal form is different from the usual definition of normal form for spin [Yo2] but it is inspired by the $SO(2)$-normal forms studied in [Yo1].

Definition 5.1 *(H-normal form, $\mathcal{CB}_H(d,j)$)*
Consider a subgroup, H, of $SO(3)$ and let (j,A) be in $\mathcal{SOS}(d,j)$. Then we call a (j,A') in $\mathcal{SOS}(d,j)$ an "H-normal form of (j,A)" if A' is H-valued and $(j,A) \sim (j,A')$, i.e., $(j,A') \in \overline{(j,A)}$. We denote by $\mathcal{CB}_H(d,j)$ the set of all spin-orbit systems in $\mathcal{SOS}(d,j)$ which have an H-normal form. Thus $(j,A) \in \mathcal{CB}_H(d,j)$ iff $T \in \mathcal{C}(\mathbb{T}^d, SO(3))$ exists such that

$$T^t(j(z))A(z)T(z) \in H \,, \tag{5.1}$$

holds for every $z \in \mathbb{T}^d$. The acronym $\mathcal{CB}$ will be explained in Remark 4 of Chapter 7. Clearly $\mathcal{CB}_H(d,j)$ is the union of all $\overline{(j,A)}$ for which A is H-valued.

We also define

$$\mathcal{TF}_H(j,A) := \Big\{ T \in \mathcal{C}(\mathbb{T}^d, SO(3)) : (\forall\ z \in \mathbb{T}^d)\ T^t(j(z))A(z)T(z) \in H \Big\} . \quad (5.2)$$

Thus $(j,A) \in \mathcal{CB}_H(d,j)$ iff $\mathcal{TF}_H(j,A)$ is nonempty. Note that the elements of $\mathcal{TF}_H(j,A)$ are the transfer fields from (j,A) to those (j,A') for which A' is H-valued. □

In our follow-up work we will take a deeper look into the notion of H-normal form.

We now make some remarks on Definition 5.1.

Remarks:

(1) Definition 5.1 gives us another property shared by equivalent spin-orbit systems since it implies that if (j,A) belongs to $\mathcal{CB}_H(d,j)$ then every spin-orbit system in $\overline{(j,A)}$ belongs to $\mathcal{CB}_H(d,j)$. Thus every $\mathcal{CB}_H(d,j)$ is a union of equivalence classes of our partition of $\mathcal{SOS}(d,j)$.

(2) Let (j,A) be in $\mathcal{SOS}(d,j)$ and let H' and H be subgroups of $SO(3)$ such that $H \subset H'$. Then, by Definition 5.1, $\mathcal{TF}_H(j,A) \subset \mathcal{TF}_{H'}(j,A)$. We will use this fact in the proof of Theorem 7.5 and it will also give us (5.3). In fact, by Definition 5.1, if $(j,A) \in \mathcal{CB}_H(d,j)$ then $\mathcal{TF}_H$ is nonempty whence $\mathcal{TF}_{H'}$ is nonempty so that, by Definition 5.1, $(j,A) \in \mathcal{CB}_{H'}(d,j)$. Thus

$$\mathcal{CB}_H(d,j) \subset \mathcal{CB}_{H'}(d,j) . \quad (5.3)$$

This fact implies that the "larger H" the more likely it is that a given (j,A) belongs to $\mathcal{CB}_H(d,j)$. Also, (5.3) is true under more general conditions than $H \subset H'$ as explained after Remark 3 below.

(3) Let (j,A) be in $\mathcal{SOS}(d,j)$, let H be a subgroup of $SO(3)$ and $r \in SO(3)$. Then it is an easy exercise to show, by Definition 5.1, that $\mathcal{TF}_{rHr^t}(j,A) = \{Tr^t : T \in \mathcal{TF}_H(j,A)\}$. Thus, and by applying Definition 5.1 once more, $\mathcal{CB}_{rHr^t}(d,j) = \mathcal{CB}_H(d,j)$. □

To relate H-normal forms for different H the following definition is useful, so let H and H' be subsets of $SO(3)$. We write $H \trianglelefteq H'$ if an $r \in SO(3)$ exists such that $rHr^t \subset H'$. For the notation rHr^t see Appendix A.6. If H, H' are subgroups of $SO(3)$ then $H \trianglelefteq H'$ iff H is conjugate to a subgroup of H'. Recalling Appendix A.2, $\trianglelefteq$ is a relation on the set of subsets of $SO(3)$ and it is easy to show that $\trianglelefteq$ is reflexive and transitive but not symmetric. Thus $\trianglelefteq$ is a preorder [wiki3] but not an equivalence relation.

If $H \subset H'$ then $rHr^t \subset H'$ with $r = I_{3\times 3}$ whence $H \trianglelefteq H'$. Thus the relation $\trianglelefteq$ is as least as fine as $\subset$ (in fact $\trianglelefteq$ is finer than $\subset$, see Remark 3 in Chapter 6). If H, H' are subgroups of $SO(3)$ such that $H \trianglelefteq H'$ then an $r \in SO(3)$ exists such that $rHr^t \subset H'$ whence, by Remark 2 above, $\mathcal{CB}_{rHr^t}(d,j) \subset \mathcal{CB}_{H'}(d,j)$ so that, by Remark 3 above, $\mathcal{CB}_H(d,j) \subset \mathcal{CB}_{H'}(d,j)$. Thus (5.3) holds whenever H, H' are subgroups of $SO(3)$ such that $H \trianglelefteq H'$ (this strengthens Remark 2). Therefore, via $\trianglelefteq$, spin-orbit tori are sorted in terms of their normal forms since according to our philosophy mentioned at the beginning of this chapter, the smaller H is, the simpler is the behavior of the (j, A) which belong to $\mathcal{CB}_H(d, j)$ and the less it is likely that a given spin-orbit system has an H-normal form. Conversely, the "larger H" is w.r.t. $\trianglelefteq$ the more likely it is that a given spin-orbit system has an H-normal form. In the extreme case $H = SO(3)$ the subgroup H is so large that every spin-orbit system (j, A) has an H-normal form but H gives no evidence if the dynamics of (j, A) is "simple", i.e., if (j, A) can be transformed to a "simple" spin-orbit system. In the other extreme case, considered in great detail in Chapter 7, H is the group consisting only of $I_{3\times 3}$ and is thus so small that the spin-orbit systems of a (j, A), where A is H-valued, have static spin-motions. Also, in the latter extreme case, it is very unlikely that a given spin-orbit system has an H-normal form (for example if $j = \mathcal{P}[\omega]$ this only happens if (j, A) is on spin-orbit resonance, see Section 7.2).

This ordering aspect of $\trianglelefteq$ brings into play the notion of "algebraic hull" from Dynamical Systems Theory. In fact the algebraic hull of (j, A) is, roughly speaking, the smallest (w.r.t. $\trianglelefteq$) subgroup H of $SO(3)$ for which (j, A) has an H-normal form. The notion of the algebraic hull is of great interest for the existence problem of the ISF, see Remark 1 in Chapter 6. However the use of the algebraic hull is beyond the scope of this work (for details, see, e.g., [Fe, Section 6],[HK1, Section 9]).

It is also a simple exercise to show that if H and H' are conjugate subgroups of $SO(3)$ then $H' \trianglelefteq H$ and $H \trianglelefteq H'$ whence, by (5.3),

$$\mathcal{CB}_{H'}(d, j) = \mathcal{CB}_H(d, j) . \tag{5.4}$$

The relation $\trianglelefteq$ is well-known in Mathematics even beyond Dynamical Systems Theory (see, e.g., [Ka]) and will also be an important tool in our follow-up work.

6 $SO(2)$-normal forms and the IFF Theorem

In this chapter we consider H-normal forms in the special case $H = SO(2)$ where $SO(2) \subset SO(3)$ is defined by

$$SO(2) := \{\exp(x\mathcal{J}) : x \in \mathbb{R}\} = \{\exp(x\mathcal{J}) : x \in [0, 2\pi)\} , \tag{6.5}$$

and where the matrix $\mathcal{J}$ is defined by

$$\mathcal{J} := \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} , \tag{6.6}$$

whence

$$\exp(x\mathcal{J}) = \begin{pmatrix} \cos(x) & -\sin(x) & 0 \\ \sin(x) & \cos(x) & 0 \\ 0 & 0 & 1 \end{pmatrix} . \tag{6.7}$$

The second equality in (6.5) follows from (6.7) and it is also easy to check, by (6.5), that $SO(2)$ is a group w.r.t. matrix multiplication whence is a subgroup of $SO(3)$.

We will see that the notion of $SO(2)$-normal form is not new and is connected with the notion of the ISF via the IFF Theorem, Theorem 6.2c. For reasons that will become clear below, we first define:

Definition 6.1 *(Invariant frame field)*
Let $(j, A) \in \mathcal{SOS}(d, j)$. We call every element of $\mathcal{TF}_{SO(2)}(j, A)$ an "Invariant Frame Field (IFF) of (j, A)". Clearly, by Definition 5.1, $\mathcal{TF}_{SO(2)}(j, A)$ is nonempty iff $(j, A) \in \mathcal{CB}_{SO(2)}(d, j)$. $\square$

Moreover, for any subgroup $H \neq SO(2)$ of $SO(3)$, we will view the elements of $\mathcal{TF}_H(j, A)$ as generalized IFF's of (j, A). Definition 6.1 sets the stage for

Theorem 6.2 *a) A matrix r in $SO(3)$ belongs to $SO(2)$ iff $r(0, 0, 1)^t = (0, 0, 1)^t$. Moreover the set $SO(2)$ can be written as follows:*

$$SO(2) = \left\{ \begin{pmatrix} a & b & 0 \\ -b & a & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \mathbb{R}^{3\times3} : |a|^2 + |b|^2 = 1 \right\}$$
$$= \left\{ \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{pmatrix} : a, b, c, d \in \mathbb{R} \right\} \cap SO(3) \ . \tag{6.8}$$

b) Let $(j, A) \in \mathcal{SOS}(d, j)$ and let A be $SO(2)$-valued. Then the constant function on $\mathbb{T}^d$ with value $(0, 0, 1)^t$ is an ISF of (j, A).
c) (IFF Theorem) Let $(j, A) \in \mathcal{SOS}(d, j)$. Then T is an IFF of (j, A) iff $T \in \mathcal{C}(\mathbb{T}^d, SO(3))$ and the third column of T is an ISF of (j, A). In other words, a $T \in \mathcal{C}(\mathbb{T}^d, SO(3))$ belongs to $\mathcal{TF}_{SO(2)}(j, A)$ iff $f(z) := T(z)(0, 0, 1)^t$ satisfies (3.9).

Proof of Theorem 6.2a: To prove (6.8) we note that if $r \in SO(2)$ then, by (6.5),(6.7), r belongs to the set on the rhs of the first equality in (6.8). If conversely r belongs to the set on the rhs of the first equality in (6.8) then $r = \begin{pmatrix} a & b & 0 \\ -b & a & 0 \\ 0 & 0 & 1 \end{pmatrix}$ where a, b are real numbers with $a^2 + b^2 = 1$ whence we abbreviate $z_1 := a - ib$ and observe that $|z_1|^2 = 1$. Thus there exists an $x \in \mathbb{R}$ such that $z_1 = \exp(ix)$ whence $a = \cos(x), b = -\sin(x)$ so that $r \in SO(2)$. To prove the second equality in (6.8) let r belong to the set on the lhs of this equality. Then, by the first equality, $r \in SO(2) \subset SO(3)$ whence r belongs to the set on the rhs of the second equality in (6.8). Let conversely r belong to the set on the rhs of the second equality in (6.8), i.e., $r \in SO(3)$ and

$$r = \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{pmatrix} . \tag{6.9}$$

Since $r \in SO(3)$ it follows from (6.9) that $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}^t = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = 1$ whence $1 = ad - bc$ and $a^2 + b^2 = 1 = c^2 + d^2$ so that, by defining $z_2 := d + ic$ and recalling that $z_1 = a - ib$, we get $|z_1|^2 = 1 = |z_2|^2$ and $|z_1 - z_2|^2 = (a-d)^2 + (b+c)^2 = 0$. Thus $z_1 = z_2$ whence $c = -b$ and $d = a$ so that r belongs to the set on the lhs of the equality. Thus the second equality in (6.8) is valid, too, which completes the proof of (6.8).

To prove the first claim let $r \in SO(2)$ whence, by (6.5),(6.7), $r(0,0,1)^t = (0,0,1)^t$. Conversely, let r be in $SO(3)$ and $r(0,0,1)^t = (0,0,1)^t$. Then $r^t(0,0,1)^t = (0,0,1)^t$ whence the third column and third row of r are equal to $(0,0,1)^t$ so that r belongs to the set on the rhs of the second equality in (6.8). Thus, by (6.8), $r \in SO(2)$. □

Proof of Theorem 6.2b: The claim readily follows from Definition 3.2 and Theorem 6.2a. □

Proof of Theorem 6.2c: "$\Rightarrow$": Let $T \in \mathcal{TF}_{SO(2)}(j, A)$. Then, by Definition 5.1, T is a transfer field from (j, A) to (j, A') where A' is $SO(2)$-valued. Also, by Theorem 6.2b, $(0,0,1)^t$ is an ISF of (j, A'). Of course, by Remark 0 in Chapter 4, T^t is a transfer field from (j, A') to (j, A) whence, by the transformation formula (4.9) of invariant polarization fields and since $(0,0,1)^t$ is an ISF of (j, A'), we conclude that $T(0,0,1)^t$ is an ISF of (j, A).

"$\Leftarrow$": Let $T \in \mathcal{C}(\mathbb{T}^d, SO(3))$ and let $T(0,0,1)^t$ be an ISF of (j, A) whence, by Definition 3.2, $A(z)T(z)(0,0,1)^t = T(j(z))(0,0,1)^t$ so that $T^t(j(z))A(z)T(z)(0,0,1)^t = (0,0,1)^t$. Thus, by Theorem 6.2a, $T^t(j(z))A(z)T(z) \in SO(2)$. It now follows from Definition 5.1 that $T \in \mathcal{TF}_{SO(2)}(j, A)$. □

Theorem 6.2c connects the concepts of normal form and invariant field since, by Theorem 6.2c, IFF's are those continuous T's whose third columns are ISF's. In fact this is to be expected given the definition of the IFF in the continuous-time formalism in [BEH]. There, we begin with the ISF at each point in phase space, and then construct the IFF by attaching two unit vectors to the ISF at each point so as to form a local orthonormal coordinate system for spin at each point in phase space. Spin vector motion within the IFF is then a simple precession around the ISF. Here, in constrast, we come from the opposite direction by noting that by definition spin vector motion w.r.t. an element of $T \in \mathcal{TF}_{SO(2)}(j, A)$ as obtained by a transformation of the kind in (4.1) (say), is a simple precession around the third axis. We then discover that the third axis must be an ISF. We will apply Theorem 6.2c in Chapter 7. Moreover, since the spin vector motion within the IFF is a simple precession around the ISF, Theorem 6.2c is of practical importance as will be explained in Section 7.2 when we discuss the computer code SPRINT.

Remark:

(1) By the IFF Theorem and Definition 6.1, every (j, A) in $\mathcal{CB}_{SO(2)}(d, j)$ has an ISF. This fact makes the notion of the algebraic hull (recall Chapter 5) an interesting tool for addressing the existence problem of the ISF. In fact it implies that if the algebraic hull of (j, A) is less or equal to $SO(2)$ w.r.t. $\trianglelefteq$ then (j, A) has an ISF. □

We now demonstrate that, in terms of our philosophy mentioned at the beginning of Chapter 5, the subgroup $SO(2)$ of $SO(3)$ is "small" because the dynamics of the spin-orbit systems in $\mathcal{CB}_{SO(2)}(d, j)$ are "simple". We will accomplish this by showing that the dynamics of (j, A) is manifestly "simple" when A is $SO(2)$-valued. For that matter we need part b) of the following theorem which is an implication of the Baby Lift Theorem, Theorem 2.5c, above.

Theorem 6.3 *Let $A \in \mathcal{C}(\mathbb{T}^d, SO(3))$ be $SO(2)$-valued. Let us define the function $f : \mathbb{T}^d \to \mathbb{T}$ by $f(z) := (A_{11}(z), A_{21}(z))^t$ where $A_{11}(z)$ and $A_{21}(z)$ denote the (11)- and (21)-matrix elements of $A(z)$ (note that f is $\mathbb{T}$-valued because of Definition 2.1 and (6.5),(6.7)). Then the following hold.*

a) The functions f and $f \circ \pi_d$ are continuous, i.e., $f \in \mathcal{C}(\mathbb{T}^d, \mathbb{T})$ and $(f \circ \pi_d) \in \mathcal{C}(\mathbb{R}^d, \mathbb{T})$ and $f \circ \pi_d$ is 2π-periodic in its arguments. Moreover let

$$f \circ \pi_d = \pi_1 \circ \beta \,, \tag{6.10}$$

where $\beta \in \mathcal{C}(\mathbb{R}^d, \mathbb{R})$ (note that such an β exists due to the Baby Lift Theorem, Theorem 2.5c). Then

$$A(z)(1,0,0)^t = \begin{pmatrix} \pi_1(\beta(\phi)) \\ 0 \end{pmatrix} \,, \tag{6.11}$$

where $z = \pi_d(\phi)$. Also $A(z) = \exp(\mathcal{J}\beta(\mathrm{Arg}(z))$ and

$$A(z) = \exp(\mathcal{J}\beta(\phi)) \,, \tag{6.12}$$

where $z = \pi_d(\phi)$.
b) A function $\alpha \in \mathcal{C}(\mathbb{T}^d, \mathbb{R})$ and an $N \in \mathbb{Z}^d$ exist such that

$$A(z) = \exp(\mathcal{J}[N \cdot \phi + 2\pi\alpha(z)]) \,, \tag{6.13}$$

where $z = \pi_d(\phi)$. Furthermore N is unique and α is unique up to a constant in the following sense: if $\tilde{\alpha} \in \mathcal{C}(\mathbb{T}^d, \mathbb{R})$ and $\tilde{N} \in \mathbb{Z}^d$ satisfy $A(z) = \exp(\mathcal{J}[\tilde{N} \cdot \phi + 2\pi\tilde{\alpha}(z)])$ then $\tilde{N} = N$ and an $n \in \mathbb{Z}$ exists such that $\tilde{\alpha}(z) = \alpha(z) + n$.

Proof of Theorem 6.3a: We first note that f is continuous since A is continuous. It thus follows from Theorem 2.5b that $f \circ \pi_d$ belongs to $\mathcal{C}(\mathbb{R}^d, \mathbb{T})$ and is 2π-periodic in its arguments. Let $\beta \in \mathcal{C}(\mathbb{R}^d, \mathbb{R})$ satisfy (6.10) whence

$$(A_{11}(\pi_d(\phi)), A_{21}(\pi_d(\phi)))^t = f(\pi_d(\phi)) = \pi_1(\beta(\phi)) = (\cos(\beta(\phi)), \sin(\beta(\phi)))^t \,, \tag{6.14}$$

where in the third equality we used Definition 2.4. Since A is $SO(2)$-valued it follows from (6.5),(6.7) and (6.14) that

$$A(\pi_d(\phi)) = \begin{pmatrix} A_{11}(\pi_d(\phi)) & -A_{21}(\pi_d(\phi)) & 0 \\ A_{21}(\pi_d(\phi)) & A_{11}(\pi_d(\phi)) & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \cos(\beta(\phi)) & -\sin(\beta(\phi)) & 0 \\ \sin(\beta(\phi)) & \cos(\beta(\phi)) & 0 \\ 0 & 0 & 1 \end{pmatrix} \,, \tag{6.15}$$

whence, and by again using (6.14), $A(\pi_d(\phi)) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos(\beta(\phi)) \\ \sin(\beta(\phi)) \\ 0 \end{pmatrix} = \begin{pmatrix} \pi_1(\beta(\phi)) \\ 0 \end{pmatrix}$ which proves (6.11). Moreover (6.12) follows from (6.7) and (6.15). Defining $\phi := \mathrm{Arg}(z)$ we conclude from (6.12) and Definition 2.4 that $A(z) = A(\pi_d(\mathrm{Arg}(z))) = A(\pi_d(\phi)) = \exp(\mathcal{J}\beta(\phi)) = \exp(\mathcal{J}\beta(\mathrm{Arg}(z)))$ which completes the proof of Theorem 6.3a. □

Proof of Theorem 6.3b: To show (6.13) we first note, by Theorem 6.3a, that a $\beta \in \mathcal{C}(\mathbb{R}^d, \mathbb{R})$ exists which satisfies (6.10),(6.11) and (6.12). For fixed but arbitrary $M \in \mathbb{Z}^d$ we define $g \in \mathcal{C}(\mathbb{R}^d, \mathbb{R})$ by $g(\phi) := \beta(\phi + 2\pi M)$ whence, by (6.11) and the periodicity of π_d, $\begin{pmatrix} \pi_1(\beta(\phi)) \\ 0 \end{pmatrix} = A(\pi_d(\phi))(1,0,0)^t = A(\pi_d(\phi + 2\pi M))(1,0,0)^t = \begin{pmatrix} \pi_1(\beta(\phi + 2\pi M)) \\ 0 \end{pmatrix} = \begin{pmatrix} \pi_1(g(\phi)) \\ 0 \end{pmatrix}$ so that $\pi_1 \circ \beta = \pi_1 \circ g$ which implies, by the Baby Lift Theorem, Theorem 2.5c, that an $m \in \mathbb{Z}$ exists such that $g(\phi) = \beta(\phi) + 2\pi m$, i.e.,

$$\beta(\phi + 2\pi M) = \beta(\phi) + 2\pi m \; . \tag{6.16}$$

To show how m in (6.16) depends on M we note that if $M_1, M_2 \in \mathbb{Z}^d, m_1, m_2 \in \mathbb{Z}$ and $\beta(\phi + 2\pi M_1) = \beta(\phi) + 2\pi m_1, \beta(\phi + 2\pi M_2) = \beta(\phi) + 2\pi m_2$ then $\beta(\phi + 2\pi(M_1 + M_2)) = \beta(\phi + 2\pi M_1) + 2\pi m_2 = \beta(\phi) + 2\pi(m_1 + m_2)$ whence the dependence of m on M is linear so that an $N \in \mathbb{Z}^d$ exists such that, for $M \in \mathbb{Z}^d$,

$$\beta(\phi + 2\pi M) = \beta(\phi) + 2\pi(M \cdot N) \; . \tag{6.17}$$

To identify α we define the function $h \in \mathcal{C}(\mathbb{R}^d, \mathbb{R})$ by $h(\phi) := \beta(\phi) - N \cdot \phi$ whence

$$\beta(\phi) = N \cdot \phi + h(\phi) \; , \tag{6.18}$$

and we compute, for fixed but arbitrary $M \in \mathbb{Z}^d$ and by (6.17), $h(\phi + 2\pi M) = \beta(\phi + 2\pi M) - N \cdot (\phi + 2\pi M) = \beta(\phi) + 2\pi(M \cdot N) - N \cdot \phi - 2\pi(M \cdot N) = \beta(\phi) - N \cdot \phi = h(\phi)$ whence h is 2π-periodic in its arguments. Thus following Theorem 2.5b we can define $\alpha : \mathbb{T}^d \to \mathbb{R}$ by $\alpha(z) := (1/2\pi) h(\mathrm{Arg}(z))$ and obtain that $\alpha \in \mathcal{C}(\mathbb{T}^d, \mathbb{R})$ and we compute $2\pi\alpha(\pi_d(\phi)) = h(\mathrm{Arg}(\pi_d(\phi))) = h(\phi)$ where in the second equality we used Theorem 2.5b and the periodicity of h whence, by (6.18),

$$\beta(\phi) = N \cdot \phi + 2\pi\alpha(\pi_d(\phi)) \; . \tag{6.19}$$

It follows from (6.12) and (6.19) that $A(\pi_d(\phi)) = \exp(\mathcal{J}\beta(\phi)) = \exp(\mathcal{J}[N \cdot \phi + 2\pi\alpha(\pi_d(\phi))])$ so that indeed (6.13) holds. To prove the second claim let $\tilde{\alpha} \in \mathcal{C}(\mathbb{T}^d, \mathbb{R})$ and $\tilde{N} \in \mathbb{Z}^d$ satisfy $A(z) = \exp(\mathcal{J}[\tilde{N} \cdot \phi + 2\pi\tilde{\alpha}(z)])$. Defining $\tilde{\beta} \in \mathcal{C}(\mathbb{R}^d, \mathbb{R})$ by $\tilde{\beta}(\phi) := \tilde{N} \cdot \phi + 2\pi\tilde{\alpha}(\pi_d(\phi))$ we get $A(z) = \exp(\mathcal{J}\tilde{\beta}(\phi))$ so we compute, by (6.11) and Definition 2.4,

$$\begin{pmatrix} \pi_1(\beta(\phi)) \\ 0 \end{pmatrix} = A(\pi_d(\phi))(1,0,0)^t = \exp(\mathcal{J}\tilde{\beta}(\phi))(1,0,0)^t$$
$$= \begin{pmatrix} \cos(\tilde{\beta}(\phi)) & -\sin(\tilde{\beta}(\phi)) & 0 \\ \sin(\tilde{\beta}(\phi)) & \cos(\tilde{\beta}(\phi)) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos(\tilde{\beta}(\phi)) \\ \sin(\tilde{\beta}(\phi)) \\ 0 \end{pmatrix} = \begin{pmatrix} \pi_1(\tilde{\beta}(\phi)) \\ 0 \end{pmatrix} ,$$

whence $\pi_1 \circ \beta = \pi_1 \circ \tilde{\beta}$ so that, by the Baby Lift Theorem, Theorem 2.5c, an $m \in \mathbb{Z}$ exists such that $\tilde{\beta}(\phi) = \beta(\phi) + 2\pi m$ which implies that $\tilde{N} \cdot \phi + 2\pi\tilde{\alpha}(\pi_d(\phi)) = \tilde{\beta}(\phi) = \beta(\phi) + 2\pi m = N \cdot \phi + 2\pi\alpha(\pi_d(\phi)) + 2\pi m$, i.e.,

$$(\tilde{N} - N) \cdot \phi = 2\pi\alpha(\pi_d(\phi)) - 2\pi\tilde{\alpha}(\pi_d(\phi)) + 2\pi m \; . \tag{6.20}$$

Since $\alpha \circ \pi_d$ and $\tilde{\alpha} \circ \pi_d$ are 2π-periodic in their arguments, the rhs of (6.20) is 2π-periodic in all components of ϕ whence the lhs of (6.20) is 2π-periodic in all components of ϕ so that $\tilde{N} - N = 0$ which implies, by (6.20), that $\tilde{\alpha}(\pi_d(\phi)) = \alpha(\pi_d(\phi)) + m$ completing the proof of the second claim. □

We will now apply Theorem 6.3b which will also be used in Chapter 7 below and whose practical importance will be explained in Section 7.2 when we discuss the computer code SPRINT. Thus let $A \in \mathcal{C}(\mathbb{T}^d, SO(3))$ be $SO(2)$-valued whence A reads as in (6.13). If (Z, S) is a particle-spin-vector trajectory of (j, A) then, by (2.31) and (6.13), S evolves simply as:

$$S(n+1) = A(Z(n))S(n) = \exp\Big(\mathcal{J}[N \cdot \phi(n) + 2\pi\alpha(Z(n))\Big)S(n) \ , \tag{6.21}$$

where $\pi_d(\phi(n)) = Z(n)$. Note that the spin vector motion in (6.21) is planar, i.e., the points $S(n)$ lie in a plane parallel to the 1-2-plane. This simple planar motion suggests to consider the spin-orbit system (j, A) as "simple" and thus the subgroup $SO(2)$ of $SO(3)$ as "small". Therefore according to our philosophy, mentioned at the beginning of this chapter, the dynamics of all spin-orbit systems in $\mathcal{CB}_{SO(2)}(d, j)$ is "simple" and the dynamics for (j, A) with A in (6.13) is not only "simple" but manifestly "simple".

Remark:

(2) If (j, A) belongs to $\mathcal{CB}_{SO(2)}(d, j)$ then $(j, A) \sim (j, A')$ where A' is $SO(2)$-valued whence, by Theorem 6.3b, a constant $N' \in \mathbb{Z}^d$ and an $\alpha' \in \mathcal{C}(\mathbb{T}^d, \mathbb{R})$ exist such that

$$A'(z) = \exp(\mathcal{J}[N' \cdot \phi + 2\pi\alpha'(z)]) \ , \tag{6.22}$$

where $\pi_d(\phi) = z$. Of course if (Z, S') is a particle-spin-vector trajectory of (j, A') then, by (6.21), S' is the planar spin vector motion, determined by

$$S'(n+1) = \exp\Big(\mathcal{J}[N' \cdot \phi(n) + 2\pi a'(Z(n))\Big)S'(n) \ , \tag{6.23}$$

where $\pi_d(\phi(n)) = Z(n)$. If $T \in \mathcal{TF}(A, A'; d, j)$ and if (Z, S) is a particle-spin-vector trajectory of (j, A) then, by the transformation formula (4.1), (Z, S) transforms into the particle-spin-vector trajectory (Z, S') of (j, A') where $S'(n) := T^t(Z(n))S(n)$. Thus S' obeys (6.23). □

The following remark gives us further insight into $\mathcal{CB}_{SO(2)}(d, j)$ and into the relation $\trianglelefteq$.

Remark:

(3) The subgroup $SO(2)$ of $SO(3)$ allows us to show that the relation $\trianglelefteq$ is finer than $\subset$ and it also demonstrates that $\trianglelefteq$ contains more information than $\subset$. We thus define the subgroup $\widehat{SO}(2)$ of $SO(3)$

$$\widehat{SO}(2) := \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(x) & -\sin(x) \\ 0 & \sin(x) & \cos(x) \end{pmatrix} : x \in \mathbb{R} \right\} . \tag{6.24}$$

It is easy to show that $\widehat{SO}(2)$ and $SO(2)$ are conjugate whence $\widehat{SO}(2) \trianglelefteq SO(2)$ and $SO(2) \trianglelefteq \widehat{SO}(2)$ so that, by (5.4),

$$\mathcal{CB}_{\widehat{SO}(2)}(d,j) = \mathcal{CB}_{SO(2)}(d,j) \; . \tag{6.25}$$

In fact (6.25) captures the intuition that, in terms of our philosophy of "smallness" of subgroups of $SO(3)$, both $\widehat{SO}(2)$ and $SO(2)$ are of the same "size". While this is nicely captured by $\trianglelefteq$, it is not captured by $\subset$ since neither $\widehat{SO}(2) \subset SO(2)$ nor $SO(2) \subset \widehat{SO}(2)$. The latter fact shows that $\trianglelefteq$ is finer than $\subset$ and that $\trianglelefteq$ contains more information than $\subset$. □

A question closely related to Theorem 6.2c is: if $f \in \mathcal{C}(\mathbb{T}^d, \mathbb{R}^3)$ with $|f| = 1$, is there a $T \in \mathcal{C}(\mathbb{T}^d, SO(3))$ such that f is the third column of T? In fact it is shown in [He2] by artificially constructed f that in general such a T does not exist. The above question will be generalized in our follow-up work.

7 G_ν-normal forms and the notions of spin tune and spin-orbit resonance

In this chapter we continue our study of normal forms by considering H-normal forms in the special case $H = G_\nu$ where the subgroup G_ν of $SO(3)$ is defined by (7.1), i.e., $G_\nu := \{\exp(2\pi n\nu\mathcal{J}) : n \in \mathbb{Z}\}$. Note that the G_ν-normal forms are closely related to the $SO(2)$-normal forms from the previous chapter since $G_\nu \subset SO(2)$. We will see that the notion of G_ν-normal form is not new and is intimately connected with the notion of spin tune and spin-orbit resonance. Thus in Section 7.1 this approach will enable us to associate tunes in addition to ω, namely spin tunes, with our spin-orbit systems. As in other dynamical systems, tunes can lead to the recognition of resonances and consequent instabilities. Here, spin tunes will lead to recognition of spin-orbit resonances, see Section 7.2. In the case of real spin vector motion, where spins are subject to the electric and magnetic fields on synchro-betatron trajectories, the definition of spin-orbit resonance allows us to predict at which orbital tunes spin vector motion might be particularly unstable.

7.1 G_ν-normal forms and the subset $\mathcal{ACB}(d,j)$ of $\mathcal{SOS}(d,j)$. Spin tunes

We first define, for every $\nu \in [0,1)$,

$$G_\nu := \{\exp(2\pi n\nu\mathcal{J}) : n \in \mathbb{Z}\} = \{\exp(2\pi(n\nu + m)\mathcal{J}) : m,n \in \mathbb{Z}\} \; , \tag{7.1}$$

where the, trivial, second equality highlights the fact that G_ν consists of matrices $\exp(2\pi\mu\mathcal{J})$ where $\mu \in [0,1)$. It is clear by (6.5) and (7.1) that G_ν is a subgroup of $SO(2)$. We will see by Theorem 7.3c below that if $(j,A) \in \mathcal{SOS}(d,j)$ has a G_ν-normal form, say (j,A'), then A' is constant, i.e., $A'(z)$ is independent of z (of course A' is G_ν-valued, too). This leads us to the following definition:

Definition 7.1 *($\mathcal{ACB}(d,j)$)*
We denote by $\mathcal{ACB}(d,j)$ the set of those $(j,A) \in \mathcal{SOS}(d,j)$ for which $\overline{(j,A)}$ contains a (j,A') such that A' is constant, i.e., such that $A'(z)$ is independent of z. □

From the remarks before Definition 7.1 it is clear that $\mathcal{ACB}(d,j) \supset \bigcup_{\nu\in[0,1)} \mathcal{CB}_{G_\nu}(d,j)$ and in Theorem 7.3d below we will see that even the reverse inclusion holds. The problem of deciding whether a given spin-orbit system is in $\mathcal{ACB}(d,j)$, i.e., has a G_ν-normal form is fruitful both theoretically and practically. The set $\mathcal{ACB}(d,j)$ contains the most important spin-orbit systems in $\mathcal{SOS}(d,j)$ when it comes to applications. See the remarks after Definition 7.7 too. However it is easy to artificially construct $(j,A) \in \mathcal{SOS}(d,j)$ which are not in $\mathcal{ACB}(d,j)$. For examples, see Section 3.3 and Theorem 7.6 below.

The following remarks reveal some simple properties of $\mathcal{ACB}(d,j)$.

Remarks:

(1) Definition 7.1 gives us another property shared by equivalent spin-orbit systems since it implies that if (j,A) belongs to $\mathcal{ACB}(d,j)$ then every spin-orbit system in $\overline{(j,A)}$ belongs to $\mathcal{ACB}(d,j)$.

(2) If $(j,A) \in \mathcal{ACB}(d,j)$ then, by Definition 7.1, a $T \in \mathcal{C}(\mathbb{T}^d, SO(3))$ exists such that $(T^t \circ j)AT = A'$ where A' is constant whence, by (2.36),

$$\Psi[j,A'](n;z) = (A')^n \ , \tag{7.2}$$

so that every n-turn spin transfer matrix function of (j,A') is a constant function which implies that (j,A) is equivalent to a spin-orbit system for which every n-turn spin transfer matrix function is a constant function. On the other hand and recalling Section 4.1, in the terminology of Dynamical Systems Theory $\Psi[j,A]$ and $\Psi[j,A']$ are cohomologous cocycles. Moreover since every $\Psi[j,A'](n;\cdot)$ is a constant function, it is common in this terminology (see, e.g., [KR]) to call $\Psi[j,A]$ an "almost coboundary". This motivates our acronym $\mathcal{ACB}$ in Definition 7.1.

(3) We now reconsider Remark 2 above. If $(\mathcal{P}[\omega],A) \in \mathcal{SOS}(d,\mathcal{P}[\omega])$ such that A is constant, then $(\mathcal{P}[\omega],A) \in \mathcal{SOS}_{CT}(d,\omega)$ since one easily shows that a function $\mathcal{A} : \mathbb{R}^{d+1} \to \mathbb{R}^{3\times3}$ exists which is constant and whose constant value is a skew-symmetric matrix and such that $A = \exp(2\pi\mathcal{A})$. Then we see, by Section 2.2, that

$$\Phi(2\pi; \mathrm{Arg}(z)) = A \ , \tag{7.3}$$

whence indeed $(\mathcal{P}[\omega],A) \in \mathcal{SOS}_{CT}(d,\omega)$. □

For Theorem 7.3 below, we need some more notation. We begin by defining, for every $\nu \in [0,1)$ and every positive integer d, the constant function $A_{d,\nu} \in \mathcal{C}(\mathbb{T}^d, SO(3))$ as

$$A_{d,\nu}(z) := \exp(2\pi\nu\mathcal{J}) = \begin{pmatrix} \cos(2\pi\nu) & -\sin(2\pi\nu) & 0 \\ \sin(2\pi\nu) & \cos(2\pi\nu) & 0 \\ 0 & 0 & 1 \end{pmatrix} . \tag{7.4}$$

Clearly, for every $j \in \mathrm{Homeo}(\mathbb{T}^d)$, the spin-orbit system $(j, A_{d,\nu})$ belongs to $\mathcal{ACB}(d, j)$ since $A_{d,\nu}$ is a constant function. Also, since $A_{d,\nu}$ is constant and by Remark 2 above,

$$\Psi[j, A_{d,\nu}](n; z) = \exp(2\pi \mathcal{J} n\nu) \ . \tag{7.5}$$

Due to (7.4) every $A_{d,\nu}$ is G_ν-valued whence $(j, A_{d,\nu}) \in \mathcal{CB}_{G_\nu}(d, j)$. Since $A_{d,\nu}$ is G_ν-valued and because $G_\nu \subset SO(2)$, it follows that $A_{d,\nu}$ is $SO(2)$-valued whence, by Theorem 6.2b, the constant function on $\mathbb{T}^d$ with value $(0, 0, 1)^t$ is an ISF of $(j, A_{d,\nu})$. Also, by (7.4), $G_0 = \{I_{3\times 3}\}$ is the trivial subgroup of $SO(3)$.

While Remark 2 above explains the acronym $\mathcal{ACB}$, the following remark explains the acronym $\mathcal{CB}$.

Remark:

(4) Let $(j, A) \in \mathcal{CB}_{G_0}(d, j)$. Thus, by Definition 5.1, a $T \in \mathcal{C}(\mathbb{T}^d, SO(3))$ exists such that $(T^t \circ j)AT = A_{d,0}$ whence, by (4.5),(7.4),

$$\Psi[j, A](n; z) = T(j^n(z))T^t(z) \ . \tag{7.6}$$

On the other hand and recalling Section 4.1, in the terminology of Dynamical Systems Theory $\Psi[j, A]$ is a cocycle whence, and by (7.6), it is common in this terminology (see, e.g., [HK2] and Chapter 1 in [HK1]) to call $\Psi[j, A]$ a "coboundary". This motivates our acronym $\mathcal{CB}$ in Definition 5.1. We will see in Remark 8 below that the spin-orbit systems in $\mathcal{CB}_{G_0}(d, \mathcal{P}[\omega])$ are on a so-called spin-orbit resonance. Moreover (7.6) will play a role in Section 7.2 below where, in the case $n = 1$, it becomes (7.29). $\square$

Eq. (7.4) and Definition 7.1 lead us naturally to the notion of spin tune. A $\nu \in [0, 1)$ is said to be a spin tune for $(j, A) \in \mathcal{SOS}(d, j)$ if (j, A) is equivalent to (j, A') with $A'(z) = \exp(2\pi\nu\mathcal{J})$, i.e., if $(j, A_{d,\nu})$ belongs to $\overline{(j, A)}$. We thus arrive at the following definition:

Definition 7.2 *(Spin tune)*
For every $(j, A) \in \mathcal{SOS}(d, j)$ we define the set

$$\Xi(j, A) := \{\nu \in [0, 1) : (j, A_{d,\nu}) \in \overline{(j, A)}\} = \{\nu \in [0, 1) : \mathcal{TF}(A, A_{d,\nu}, d, j) \neq \emptyset\} \ , \tag{7.7}$$

where in the second equality of (7.7) we used Definitions 4.1 and 4.2. We call each element of $\Xi(j, A)$ a "spin tune" of (j, A). Eq. (7.7) gives us another property shared by equivalent spin-orbit systems since it implies that if $(j, A') \in \overline{(j, A)}$ then $\Xi(j, A) = \Xi(j, A')$. In other words, equivalent spin-orbit systems have the same spin tunes. In particular in the absence of spin tunes, i.e., when $\Xi(j, A) = \emptyset$ we have $\Xi(j, A') = \emptyset$ for all $(j, A') \in \overline{(j, A)}$. $\square$

In the present section we focus on the mathematical properties of the spin tunes whereas in the following section we discuss physical aspects. The following is our main theorem about G_ν-normal forms, spin tunes and $\mathcal{ACB}(d, j)$. The meaning of Theorem 7.3 is discussed in great detail below.

Theorem 7.3 *a) Let $(j, A) \in \mathcal{SOS}(d, j)$. Then $(j, A) \in \mathcal{ACB}(d, j)$ iff a $\nu \in [0, 1)$ exists such that $(j, A_{d,\nu})$ belongs to $\overline{(j, A)}$.*

b) Let $(j,A)\in\mathcal{SOS}(d,j)$. Then $(j,A)\in\mathcal{ACB}(d,j)$ iff (j,A) has spin tunes.

c) Let $\nu\in[0,1)$ and $A\in\mathcal{C}(\mathbb{T}^d,SO(3))$ be G_ν-valued. Then A is a constant function.

d) Let $j\in\mathrm{Homeo}(\mathbb{T}^d)$. Then

$$\mathcal{ACB}(d,j)=\bigcup_{\nu\in[0,1)}\mathcal{CB}_{G_\nu}(d,j)\ . \tag{7.8}$$

e) Let $T\in\mathcal{C}(\mathbb{T}^d,SO(3))$ and let $(j,A')\in\mathcal{SOS}(d,j)$ be the transform of $(j,A)\in\mathcal{SOS}(d,j)$ under T, i.e., $T\in\mathcal{TF}(A,A';d,j)$. Then T belongs to $\bigcup_{\nu\in[0,1)}\mathcal{TF}_{G_\nu}(j,A)$ iff T is an IFF of (j,A) and A' is a constant function.

f) Let $(\mathcal{P}[\omega],A)\in\mathcal{SOS}(d,\mathcal{P}[\omega])$. If ν is a spin tune of $(\mathcal{P}[\omega],A)$ then

$$\Xi(\mathcal{P}[\omega],A)=[0,1)\cap\Big\{\varepsilon\nu+m\cdot\omega+n:\varepsilon\in\{1,-1\},m\in\mathbb{Z}^d,n\in\mathbb{Z}\Big\}\ . \tag{7.9}$$

Proof of Theorem 7.3a: If $\nu\in[0,1)$ exists such that $(j,A_{d,\nu})$ belongs to $\overline{(j,A)}$ then, by Definition 7.1, $(j,A)\in\mathcal{ACB}(d,j)$ since $A_{d,\nu}$ is constant. To prove the converse, let $(j,A)\in\mathcal{ACB}(d,j)$. Then, by Definition 7.1, $\overline{(j,A)}$ contains a (j,A') such that A' is constant with constant value, say r. By some simple Linear Algebra, a $\nu\in[0,1)$ and a $W\in SO(3)$ can be found such that

$$r=W\exp(2\pi\nu\mathcal{J})W^t\ . \tag{7.10}$$

See, e.g., Lemma 2.1 of [BEH]. Thus, defining the constant function $T\in\mathcal{C}(\mathbb{T}^d,SO(3))$ by $T(z):=W$ we observe by (7.10) and Definition 4.1 that $T\in\mathcal{TF}(A',A_{d,\nu},d,j)$ whence $(j,A')\sim(j,A_{d,\nu})$ so that $(j,A)\sim(j,A_{d,\nu})$ which implies that $(j,A_{d,\nu})$ belongs to $\overline{(j,A)}$. □
Proof of Theorem 7.3b: The claim is a simple consequence of Definition 7.2 and Theorem 7.3a. □
Proof of Theorem 7.3c: Since A is G_ν-valued it follows from (7.1) that a function $\tilde{n}:\mathbb{T}^d\to\mathbb{Z}$ exists such that $A(z)=\exp(\mathcal{J}2\pi\nu\tilde{n}(z))$ whence

$$A(z)=\exp(\mathcal{J}2\pi\nu\tilde{n}(z))\ . \tag{7.11}$$

Clearly A is $SO(2)$-valued whence, by Theorem 6.3b, a constant $N\in\mathbb{Z}^d$ and an $\alpha\in\mathcal{C}(\mathbb{T}^d,\mathbb{R})$ exist such that

$$\exp(\mathcal{J}2\pi\nu\tilde{n}(z))=A(z)=\exp(\mathcal{J}[N\cdot\phi+2\pi\alpha(z)])\ , \tag{7.12}$$

where $\pi_d(\phi)=z$ and where in the first equality of (7.12) we used (7.11). It follows from (6.7) and (7.12) that a function $k:\mathbb{T}^d\to\mathbb{Z}$ exists such that

$$2\pi\nu\tilde{n}(z)+2\pi k(z)=N\cdot\phi+2\pi\alpha(z)\ , \tag{7.13}$$

where $\pi_d(\phi) = z$. Since π_d is 2π-periodic in its d arguments, so are $\tilde{n}(\pi_d(\phi))$, $k(\pi_d(\phi))$ and $\alpha(\pi_d(\phi))$ whence it follows from (7.13) that $N \cdot \phi$ is 2π-periodic in the d components of ϕ so that $N = 0$ which implies, by (7.13) and for all $z \in \mathbb{T}^d$,

$$\nu\tilde{n}(z) + k(z) = \alpha(z) \, . \tag{7.14}$$

Since $\tilde{n}$ and k are $\mathbb{Z}$-valued, the function $\nu\tilde{n}+k$ can take only countably many values whence, by (7.14), the function α can take only countably many values. "Countably many" values means no more values than elements of $\mathbb{Z}$. On the other hand since α is continuous and since its domain, $\mathbb{T}^d$, is a path-connected topological space, the range of α is a path-connected subset of $\mathbb{R}$, i.e., a nonempty interval, say I (for the notion of range see also Appendix A.1). However since α takes only countably many values, I is a nonempty interval which contains only countably many points whence I contains just one point which implies that α is constant. Since α is constant and $N = 0$ it follows from (7.12) that A is constant. □

Proof of Theorem 7.3d: "$\subset$": Let $\underline{(j, A)} \in \mathcal{ACB}(d, j)$. Then, by Theorem 7.3a, a $\nu \in [0, 1)$ exists such that $(j, A_{d,\nu})$ belongs to $\overline{(j, A)}$. By a remark after (7.1), $A_{d,\nu}$ is G_ν-valued whence, by Definition 5.1, $(j, A) \in \mathcal{CB}_{G_\nu}(d, j)$.

"$\supset$": Let $\nu \in [0, 1)$ and $(j, A) \in \mathcal{CB}_{G_\nu}(d, j)$ whence, by Definition 5.1, $\mathcal{TF}_{G_\nu}(j, A)$ is nonempty. So pick a $T \in \mathcal{TF}_{G_\nu}(j, A)$. Then, by Definitions 4.1 and 5.1, $T \in \mathcal{TF}(A, A'; d, j)$ where A' is G_ν-valued. Since A' is G_ν-valued it follows from Theorem 7.3c that A' is constant which implies, by Definition 7.1, that $(j, A) \in \mathcal{ACB}(d, j)$. □

Proof of Theorem 7.3e: "$\Rightarrow$": Let $T \in \mathcal{TF}_{G_\nu}(j, A)$. Since G_ν is a subgroup of $SO(2)$ we conclude from Remark 2 in Chapter 5 that $T \in \mathcal{TF}_{SO(2)}(j, A)$ whence T is an IFF of (j, A). Also, A' is G_ν-valued whence, by Theorem 7.3c, A' is constant.

"$\Leftarrow$": Let T be an IFF of (j, A) and let A' be constant. Clearly, by Definition 6.1, A' is $SO(2)$-valued whence $\nu \in [0, 1)$ exists such that $A' = A_{d,\nu}$ which implies that A' is G_ν-valued so that $T \in \mathcal{TF}_{G_\nu}(j, A)$. □ □

Proof of Theorem 7.3f: The claim is proved in Appendix B.3. Note that our proof of the inclusion $\Xi(\mathcal{P}[\omega], A) \supset [0, 1) \cap \left\{ \varepsilon\nu + m \cdot \omega + n : \varepsilon \in \{1, -1\}, m \in \mathbb{Z}^d, n \in \mathbb{Z} \right\}$ is a simple consequence of (7.7). In contrast our proof of the converse inclusion needs the technique of quasiperiodic functions. □

Theorems 7.3a and 7.3b are elementary statements about $\mathcal{ACB}(d, j)$ which are needed in the proof of Theorem 7.3d and in Section 7.2 below. Theorem 7.3c played a key role in the motivation of Definition 7.1 (see the remarks before that definition). Theorem 7.3c is also used in the proof of Theorem 7.3d. Theorem 7.3d is the insight that every $\mathcal{ACB}(d, j)$ can be understood in terms of normal forms, a fact that is not obvious by Definition 7.1. Theorem 7.3e will lead us to the definition of the uniform IFF below and it will allow us to prove the Uniform IFF Theorem, Theorem 7.5 below. Theorem 7.3f gives a key insight into the notion of spin tune and, as we will see in Section 7.2, into the notion of spin-orbit resonance. Theorem 7.3f also shows that every $\Xi(\mathcal{P}[\omega], A)$ has only countably many elements which will allow us to show, after Remark 6 below, that every $\mathcal{SOS}(d, \mathcal{P}[\omega])$ is partitioned w.r.t. to the equivalence relation $\sim$ into infinitely many equivalence classes.

The following remark relates the notions of G_ν-normal form and $SO(2)$-normal form.

Remark:

(5) Let $j \in \mathrm{Homeo}(\mathbb{T}^d)$ and $\nu \in [0,1)$. By the remarks after (7.1) we have $G_\nu \subset SO(2)$ whence, by (5.3),

$$\mathcal{CB}_{G_\nu}(d,j) \subset \mathcal{CB}_{SO(2)}(d,j)\ . \tag{7.15}$$

In Chapter 6 we found that the dynamics of all spin-orbit systems in $\mathcal{CB}_{SO(2)}(d,j)$ is "simple" whence, by (7.15), the dynamics of all spin-orbit systems in $\mathcal{CB}_{G_\nu}(d,j)$ is "simple". Moreover, in the language of Chapter 6 and by the fact that $A_{d,\nu}$ is G_ν-valued (whence $SO(2)$-valued), the dynamics of $(j, A_{d,\nu})$ is even manifestly "simple". It also follows from (7.15) and Theorem 7.3d that

$$\mathcal{ACB}(d,j) \subset \mathcal{CB}_{SO(2)}(d,j)\ . \tag{7.16}$$

Then, by Definition 6.1, every $(j,A) \in \mathcal{ACB}(d,j)$ has an IFF whence, by Remark 1 in Chapter 6, every $(j,A) \in \mathcal{ACB}(d,j)$ has an ISF. □

We saw in Chapter 6 that the IFF is important for understanding the concept of the $SO(2)$-normal form. Analogously we will now see that the uniform IFF, defined below, is important for understanding the concept of the G_ν-normal form.

Definition 7.4 *(Uniform invariant frame field)*
Let $(j,A) \in \mathcal{SOS}(d,j)$. We call every element of every $\mathcal{TF}_{G_\nu}(j,A)$ a "Uniform Frame Field (uniform IFF) of (j,A)". Note, by (7.1), that $\bigcup_{\nu\in[0,1)} \mathcal{TF}_{G_\nu}(j,A)$ is the set of uniform IFF's of (j,A). □

In analogy to the IFF Theorem, Theorem 6.2c, we now get:

Theorem 7.5 *(Uniform IFF Theorem) Let $(j,A) \in \mathcal{SOS}(d,j)$. Then T is a uniform IFF of (j,A) iff the following hold: T is an IFF of (j,A) and it is a transfer field from (j,A) to some (j,A') such that A' is a constant function.*
Remark: Thus, and by Theorem 6.2c, T is a uniform IFF of (j,A) iff the following hold: the third column of T is an ISF of (j,A) and T is a transfer field from (j,A) to some (j,A') such that A' is a constant function. We will also see in (7.21) that A' is of the form $A_{d,\nu}$.

Proof of Theorem 7.5 "⇒": Let T be a uniform IFF of (j,A), i.e., let $T \in \mathcal{TF}_{G_\nu}(j,A)$. We recall from the remarks after (7.1) that $G_\nu \subset SO(2)$ whence, by Remark 2 in Chapter 5, $T \in \mathcal{TF}_{SO(2)}(j,A)$ so that, by Definition 6.1, T is an IFF of (j,A). Of course T is a transfer field from (j,A) to some spin-orbit system, say (j,A') whence $T \in \mathcal{TF}(A,A';d,j)$ so that, and since $T \in \mathcal{TF}_{G_\nu}(j,A)$, we conclude from Theorem 7.3e that A' is a constant function.
"⇐": Let $T \in \mathcal{TF}(A,A';d,j)$ where A' is a constant function. Then, by Theorem 7.3e, $T \in \bigcup_{\nu\in[0,1)} \mathcal{TF}_{G_\nu}(j,A)$ whence T is a uniform IFF of (j,A). □

Because of Theorem 7.5, the uniform IFF is the discrete-time analogue of the so-called uniform invariant frame field introduced in the continuous-time formalism of [BEH]. The concept of the uniform IFF is of great importance and we use it in this work at several places, for example in the proof of Theorem 7.3f (see Appendix B.3) and in the proof of (7.28) (see Appendix B.5). The concept of the uniform IFF is also of practical importance as will be explained in Section 7.2 when we discuss the computer code SPRINT.

It is useful to characterize the set of uniform IFF's of (j, A) in terms of the $A_{d,\nu}$ leading us to (7.21) as follows. Thus let $(j, A) \in \mathcal{SOS}(d, j)$ and T be a uniform IFF of (j, A). Therefore, by Definition 7.4, there exists a $\nu \in [0, 1)$ such that $T \in \mathcal{TF}_{G_\nu}(j, A)$ whence, by Definition 5.1, A', defined by $A'(z) := T^t(j(z))A(z)T(z)$ is G_ν-valued so that, by Theorem 7.3c, the function A' is constant valued taking the value, say r. Of course $r \in G_\nu$ whence, by (7.1), $r = \exp(2\pi\mu\mathcal{J})$ where $\mu \in [0, 1)$ is the fractional part of $N\nu$ where N is a constant integer. Note, by Definition 7.2 and Theorem 7.3f, that μ is a spin tune of (j, A) but ν in general is not. Clearly the function A' is constant valued taking the value $\exp(2\pi\mu\mathcal{J})$ whence, by (7.4), $A' = A_{d,\mu}$. We thus have shown that if T is a uniform IFF of (j, A) then there exists a $\mu \in [0, 1)$ so that $T^t(j(z))A(z)T(z) = A_{d,\mu}(z)$, i.e., T is a transfer field from (j, A) to $(j, A_{d,\mu})$ whence

$$\bigcup_{\nu\in[0,1)} \mathcal{TF}_{G_\nu}(j, A) \subset \bigcup_{\mu\in[0,1)} \mathcal{TF}(A, A_{d,\mu}; d, j) = \bigcup_{\nu\in[0,1)} \mathcal{TF}(A, A_{d,\nu}; d, j) \; , \tag{7.17}$$

where the equality in (7.17) is a trivial reparametrization. To show the reverse inclusion let $T \in \mathcal{TF}(A, A_{d,\nu}; d, j)$ for some $\nu \in [0, 1)$. Clearly $A_{d,\nu}$ is G_ν-valued whence, by Definition 5.1, $T \in \mathcal{TF}_{G_\nu}(j, A)$ so that $\bigcup_{\nu\in[0,1)} \mathcal{TF}_{G_\nu}(j, A) \supset \bigcup_{\nu\in[0,1)} \mathcal{TF}(A, A_{d,\nu}; d, j)$ which implies, by (7.17),

$$\bigcup_{\nu\in[0,1)} \mathcal{TF}_{G_\nu}(j, A) = \bigcup_{\nu\in[0,1)} \mathcal{TF}(A, A_{d,\nu}; d, j) \; . \tag{7.18}$$

To show the second equality of (7.21) we first note, by (7.7), that $\Xi(j, A) \subset [0, 1)$ whence

$$\bigcup_{\nu\in[0,1)} \mathcal{TF}(A, A_{d,\nu}; d, j) \supset \bigcup_{\nu\in\Xi(j,A)} \mathcal{TF}(A, A_{d,\nu}; d, j) \; . \tag{7.19}$$

To show the reverse inclusion let $T \in \mathcal{TF}(A, A_{d,\nu}; d, j)$ where $\nu \in [0, 1)$ whence $\mathcal{TF}(A, A_{d,\nu}; d, j)$ is nonempty so that, by (7.7), $\nu \in \Xi(j, A)$ which implies that $\bigcup_{\nu\in[0,1)} \mathcal{TF}(A, A_{d,\nu}; d, j) \subset \bigcup_{\nu\in\Xi(j,A)} \mathcal{TF}(A, A_{d,\nu}; d, j)$. Thus, by (7.19),

$$\bigcup_{\nu\in[0,1)} \mathcal{TF}(A, A_{d,\nu}; d, j) = \bigcup_{\nu\in\Xi(j,A)} \mathcal{TF}(A, A_{d,\nu}; d, j) \; . \tag{7.20}$$

It follows from (7.17),(7.20) and Definition 7.4 that indeed the set of uniform IFF's of (j, A) satisfies

$$\bigcup_{\nu\in[0,1)} \mathcal{TF}_{G_\nu}(j, A) = \bigcup_{\nu\in[0,1)} \mathcal{TF}(A, A_{d,\nu}; d, j) = \bigcup_{\nu\in\Xi(j,A)} \mathcal{TF}(A, A_{d,\nu}; d, j) \; . \tag{7.21}$$

It follows from the first equality in (7.21) that the set of uniform IFF's of (j, A) is equal to the set of all transfer fields from (j, A) to the spin-orbit systems $(j, A_{d,\nu})$.

The following remark is an important application of (7.21).

Remark:

(6) Let $(j, A) \in \mathcal{SOS}(d, j)$. We first consider the case that (j, A) has a uniform IFF, say T. Then, by (7.21), a $\nu \in \Xi(j, A)$ exists such that T is a transfer field from (j, A) to $(j, A_{d,\nu})$. Also, since $\Xi(j, A)$ is nonempty and by Theorem 7.3b, $(j, A) \in \mathcal{ACB}(d, j)$. We now consider the case where $\Xi(j, A)$ is nonempty so we pick a $\nu \in \Xi(j, A)$. Then, by (7.7), (7.21), a transfer field from (j, A) to $(j, A_{d,\nu})$ exists and it is a uniform IFF of (j, A). Thus, and by Theorem 7.3b, if $(j, A) \in \mathcal{ACB}(d, j)$ then (j, A) has a uniform IFF. Summarizing both cases we observe that (j, A) has a uniform IFF iff $(j, A) \in \mathcal{ACB}(d, j)$. Thus, by Theorem 7.3b, (j, A) has a uniform IFF iff it has spin tunes. □

As mentioned in Section 4.3, in this work we do not fully address how the $\mathcal{SOS}(d, j)$ are partitioned w.r.t. to the equivalence relation $\sim$. Nevertheless Theorem 7.3f sheds light on this issue. In fact if j is of the form $\mathcal{P}[\omega]$ then it is easy to show that $\mathcal{SOS}(d, j)$ is partitioned into uncountably many equivalence classes as follows.

To prove this claim we first of all note that $\mathcal{SOS}(d, \mathcal{P}[\omega])$ has uncountably many elements since ν is a continuous parameter whence there are uncountably many $A_{d,\nu}$, i.e., the spin-orbit systems $(\mathcal{P}[\omega], A_{d,\nu})$ form an uncountable subset, say B, of $\mathcal{SOS}(d, \mathcal{P}[\omega])$ (note that ω is fixed but ν varies over $[0, 1)$). Note also that both B and $\overline{(\mathcal{P}[\omega], A_{d,\nu})}$ have uncountably many elements but, as will be shown below, $B \cap \overline{(\mathcal{P}[\omega], A_{d,\nu})}$ has only countably many elements. In fact in our proof the sets $B \cap \overline{(\mathcal{P}[\omega], A_{d,\nu})}$ for each ν will play a key role and we already note here that they form a partition of B since the $\overline{(\mathcal{P}[\omega], A_{d,\nu})}$, being equivalence classes, are mutually disjoint. In particular, if $\overline{(\mathcal{P}[\omega], A_{d,\nu})}$ and $\overline{(\mathcal{P}[\omega], A_{d,\mu})}$ are different then they are disjoint and belong to different equivalence classes of the equivalence relation $\sim$. The crucial question now is: how many of the sets $B \cap \overline{(\mathcal{P}[\omega], A_{d,\nu})}$ are different? In other words how common is it that two spin-orbit systems in B are equivalent? This is where Theorem 7.3 engages. In fact, by Theorem 7.3f, each set $\Xi(\mathcal{P}[\omega], A_{d,\nu})$ contains only countably many elements. On the other hand if $\nu, \mu \in [0, 1)$ then, by (7.7), $(\mathcal{P}[\omega], A_{d,\mu}) \in \overline{(\mathcal{P}[\omega], A_{d,\nu})}$ iff $\mu \in \Xi(\mathcal{P}[\omega], A_{d,\nu})$. Thus every set of the form $B \cap \overline{(\mathcal{P}[\omega], A_{d,\nu})}$ contains only countably many elements of B. Thus we need uncountably many of the sets $B \cap \overline{(\mathcal{P}[\omega], A_{d,\nu})}$ to overlap B whence the $B \cap \overline{(\mathcal{P}[\omega], A_{d,\nu})}$ form an uncountable partition of B. Since different $B \cap \overline{(\mathcal{P}[\omega], A_{d,\nu})}$ are contained in different equivalence classes we thus have shown that there are uncountably many equivalence classes of the form $\overline{(\mathcal{P}[\omega], A_{d,\nu})}$. Thus, as was to be shown, $\mathcal{SOS}(d, \mathcal{P}[\omega])$ is partitioned into uncountably many equivalence classes w.r.t. to the equivalence relation $\sim$.

To put that into context we note, by Definition 5.1, that every $\mathcal{CB}_{G_\nu}(d, \mathcal{P}[\omega])$ is a union of equivalence classes w.r.t. $\sim$ whence, by Theorem 7.3d, $\mathcal{ACB}(d, \mathcal{P}[\omega])$ is a union of equivalence classes, too. Thus we just have shown that $\mathcal{ACB}(d, \mathcal{P}[\omega])$ is partitioned into uncountably many equivalence classes w.r.t. $\sim$. To get a little further insight into this we now ask the natural question: are there spin-orbit systems in $\mathcal{SOS}(d, \mathcal{P}[\omega])$ which do not belong to $\mathcal{ACB}(d, \mathcal{P}[\omega])$, i.e., is $\mathcal{SOS}(d, \mathcal{P}[\omega]) \neq \mathcal{ACB}(d, \mathcal{P}[\omega])$? In other words do there exists equivalence classes w.r.t. $\sim$ which are disjoint to $\mathcal{ACB}(d, \mathcal{P}[\omega])$? In fact the following theorem shows that this in indeed the case if $(1, \omega)$ is nonresonant.

Theorem 7.6 *Let $(\mathcal{P}[\omega], A) \in \mathcal{SOS}(d, \mathcal{P}[\omega])$ be off orbital resonance, i.e., let $(1, \omega)$ be nonresonant and let $A \in \mathcal{C}(\mathbb{T}^d, SO(3))$ be defined by*

$$A(z) = \exp(\mathcal{J}(N \cdot \phi)) \,, \tag{7.22}$$

where $N \in \mathbb{Z}^d$ and $\pi_d(\phi) = z$. Then $(\mathcal{P}[\omega], A)$ has an ISF and an IFF. Moreover $(\mathcal{P}[\omega], A)$ has spin tunes iff $N = 0$.
Remark: Thus $(\mathcal{P}[\omega], A) \in \mathcal{ACB}(d, \mathcal{P}[\omega])$ iff $N = 0$. Also $(\mathcal{P}[\omega], A)$ has a uniform IFF iff $N = 0$.

Proof of Theorem 7.6: The claims are proved in Appendix C.3. □

Theorem 7.6 shows that if $(1, \omega)$ is nonresonant then $\mathcal{SOS}(d, \mathcal{P}[\omega]) \neq \mathcal{ACB}(d, \mathcal{P}[\omega])$, i.e., that there is at least one equivalence class in $\mathcal{SOS}(d, \mathcal{P}[\omega])$ which is disjoint to $\mathcal{ACB}(d, \mathcal{P}[\omega])$. In fact one can even show there are infinitely many equivalence classes in $\mathcal{SOS}(d, \mathcal{P}[\omega])$ which are disjoint to $\mathcal{ACB}(d, \mathcal{P}[\omega])$ (see [He2, Chapter 8] where Theorem 7.6 is substantially generalized). Theorem 7.6 also shows, if $(1, \omega)$ is nonresonant and since A is $SO(2)$-valued, that $(\mathcal{P}[\omega], A) \in \mathcal{CB}_{SO(2)}(d, \mathcal{P}[\omega])$ whence $\mathcal{CB}_{SO(2)}(d, \mathcal{P}[\omega]) \neq \mathcal{ACB}(d, \mathcal{P}[\omega])$. For more details on the special case where $d = 1$, see Appendix B.6 below.

7.2 Physical aspects of spin tunes. Spin-orbit resonances

Definitions 7.1 and 7.2 and Theorem 7.3 lead us naturally to the notion of spin-orbit resonance.

Definition 7.7 *(Spin-orbit resonance)*
We say that $(\mathcal{P}[\omega], A)$ is "on spin-orbit resonance" if it has a spin tune and if for every spin tune ν of $(\mathcal{P}[\omega], A)$ one can find $m \in \mathbb{Z}^d, n \in \mathbb{Z}$ such that

$$\nu = m \cdot \omega + n \, . \tag{7.23}$$

We say that $(\mathcal{P}[\omega], A)$ is "off spin-orbit resonance" iff $(\mathcal{P}[\omega], A) \in \mathcal{ACB}(d, \mathcal{P}[\omega])$ and if $(\mathcal{P}[\omega], A)$ is not on spin-orbit resonance. Note that a spin-orbit system (j, A) which has no spin tunes is neither on nor off spin-orbit resonance. Moreover (j, A) is neither on nor off spin-orbit resonance when j is not a torus translation, i.e., not of the form $\mathcal{P}[\omega]$. □

In [BEH] spin-orbit systems with spin tunes belong to the class of "well tuned" systems and most of the systems with no spin tunes are said to be "ill-tuned".

In [He2] the spin tune and spin-orbit resonances defined here are called spin tune of the first kind and spin-orbit resonances of the first kind respectively since [He2] finds it convenient to distinguish between two kind of spin tune. That distinction is not needed here.

Remarks:

(7) By Theorem 7.3f and Definition 7.7 a spin-orbit system of the form $(\mathcal{P}[\omega], A)$ is on spin-orbit resonance iff (7.23) holds for just one choice of $m \in \mathbb{Z}^d, n \in \mathbb{Z}, \nu \in \Xi(\mathcal{P}[\omega], A)$. Thus a single spin tune ν of $(\mathcal{P}[\omega], A)$ is sufficient to determine if this spin-orbit system is on spin-orbit resonance. Note also, by Theorem 7.3f and Definition 7.7, that a spin-orbit system $(\mathcal{P}[\omega], A)$ is on spin-orbit resonance iff $0 \in \Xi(\mathcal{P}[\omega], A)$.

(8) Let $(\mathcal{P}[\omega], A) \in \mathcal{SOS}(d, \mathcal{P}[\omega])$. It can be easily shown, by using (7.7) and Remark 6 above, that $(\mathcal{P}[\omega], A) \in \mathcal{CB}_{G_0}(d, \mathcal{P}[\omega])$ iff $0 \in \Xi(\mathcal{P}[\omega], A)$. Thus, by Remark 7, $(\mathcal{P}[\omega], A)$ is on spin-orbit resonance iff $(\mathcal{P}[\omega], A) \in \mathcal{CB}_{G_0}(d, \mathcal{P}[\omega])$. We will use this fact in the proof of Theorem 7.8b below. □

If one considers a family $(j_J, A_J)_{J\in\Lambda}$ of spin-orbit systems (see the Introduction and Chapter 8) and if every (j_J, A_J) has a spin tune, say ν_J, then ν_J is called an amplitude dependent spin tune (ADST). Recall from Remark 6 above that if T_J is a uniform IFF of (j_J, A_J) then $T_J^t(j(z))A_J(z)T_J(z) = A_{d,\nu_J}(z) = \exp(2\pi\nu_J\mathcal{J})$.

As stated at the beginning of this chapter spin-orbit resonance can lead to a large angular spread of the ISF and that can lead to unacceptably low equilibrium polarization as explained in Chapter 8. The large angular spread also means that if a particle beam occupies a large volume of phase space at injection while the spins all point in roughly the same direction, the polarization of the beam can be very unstable while the spin precess around their individual ISF's. See [Ho] for an example of this. See [Ho],[Ma],[Vo],[Yo1] for formalisms and calculations which have demonstrated the potential for a large spread of the ISF near spin-orbit resonances. For detailed further comments see Section X in [BEH].

Moreover, since the ADST can vary with orbital amplitude J, particles at one amplitude can be close to spin-orbit resonance while particles as nearby amplitudes need not be. Manifestations of this are beautifully demonstrated in [Ho, Vo, BHV00, HV] where the value of a rigorous definition of spin tune is made crystal clear. Note that as shown in those works, spin-orbit resonances tend to be rather repelling than attractive. The rigorous definition of spin tune and of spin-orbit resonance also will lead us in Chapter 8 to the Uniqueness Theorem for the ISF [Yo1, DK73]. In summary, a rigorous definition, as in Definitions 7.2 and 7.7, is very important for a detailed understanding of real spin vector motion.

As explained in Section X of [BEH] and in [BV1], as well as in other literature, a real spin-orbit system $(\mathcal{P}[\omega], A)$ on orbital resonance normally has no spin tune. One exception is the so-called single resonance model underlying the model with two Siberian snakes in Section 3.3. Nevertheless, such a system can, but need not, have an ISF of the continuous kind defined here. An example of a spin-orbit system on orbital resonance which has no ISF, and thus no spin tune, is studied in Section 3.3. We recall from Remark 3 in Chapter 3 that, if the d components of ω are rational numbers, then it is easy to calculate an ISF f by finding the real eigenvector $f(z)$ of the matrix $\Psi[\mathcal{P}[\omega], A](n; z)$ for the number of turns n for which the particle returns to its starting position z. The discontinuous "ISF" of [BV1] can also be calculated in this way (and this is also done in our example in Section 3.3). Recall also from the ISF conjecture in Chapter 3 that we expect an ISF to exist off orbital resonance.

The ISF and the ADST for real spin vector motion off orbital resonance in storage rings can be computed in a number of ways [Be],[Fo],[HH],[Ho],[Ma],[Vo],[Yo3]. Here we describe two of them and we start with a method of computing the ADST, implemented in the computer code SPRINT [He2, Ho, Vo] (as an alternative method, SPRINT offers an implementation of the SODOM-2 algorithm). The calculations proceed in two steps [BEH00, BHV98, Ho, Vo]. For simplicity we consider a fixed but arbitrary action value J and assume that the spin-orbit system belongs to $\mathcal{ACB}(d, \mathcal{P}[\omega])$ and is off orbital resonance and off spin-orbit resonance. As we will see in Chapter 8, by the Uniqueness Theorem, Theorem 8.1b, the given spin-orbit system $(\mathcal{P}[\omega], A)$ has only two ISF's, say f and $-f$. Of course f and $-f$ in general are unknown and in fact one only attempts to compute a discretization of them. In the first step, f is computed at some point z on the torus at some point θ on a ring using stroboscopic averaging [EH, HH] giving us $f(z)$. Since $(\mathcal{P}[\omega], A) \in \mathcal{ACB}(d, \mathcal{P}[\omega])$ it follows from Remark 5 above that an IFF, say T, exists and, due to Theorem 6.2c, the third column of T is either f or $-f$ and here $T(z)$ is constructed by a simple orthonormalization procedure in which $f(z)$ is the third column is $T(z)$. The axis represented by the second column of $T(z)$ could, for example, be chosen so as to have no component along the direction of the beam. In the next step the spin value $f(z)$ is tracked forwards turn by turn, according to (3.9), resulting in the discretization $f(z), f(\mathcal{P}[\omega](z)), f(\mathcal{P}[\omega]^2(z)), ..., f(\mathcal{P}[\omega]^N(z))$ of f for some large integer N. Accordingly $T(z), T(\mathcal{P}[\omega](z)), T(\mathcal{P}[\omega]^2(z)), ..., T(\mathcal{P}[\omega]^N(z))$ are constructed at the end of each turn according to the chosen prescription. Then, the average spin precession angle around the ISF w.r.t. this IFF is computed for a very large number of turns N. In fact since T is an IFF, by Remark 2 in Chapter 6, an $N \in \mathbb{Z}^d$ and an $\alpha \in \mathcal{C}(\mathbb{T}^d, \mathbb{R})$ exist such that

$$T^t(\mathcal{P}[\omega]^{n+1}(z))A(\mathcal{P}[\omega]^n(z))T(\mathcal{P}[\omega]^n(z)) = \exp\bigg(\mathcal{J}[N \cdot (\phi + 2\pi n\omega) + 2\pi\alpha(\mathcal{P}[\omega]^n(z))]\bigg) , \tag{7.24}$$

where $\pi_d(\phi) = z$ and $n = 0, ..., N$. Note that, for every $\phi \in \mathbb{R}^d$ with $\pi_d(\phi) = z$, the rhs of (7.24) is the same (recall Section 2.3) whence (7.24) is independent of the choice of ϕ and thus well-defined. One can show [He2, Vo], by using Theorem 3.3a above, that the average $< \alpha >$ of α, given by

$$< \alpha > := \frac{1}{(2\pi)^d} \int_{[0,2\pi]^d} \alpha(\pi_d(\phi)) d\phi , \tag{7.25}$$

is a spin tune of $(\mathcal{P}[\omega], A)$. On the other hand, (7.24), gives us $\alpha(z), \alpha(\mathcal{P}[\omega](z))$, $\alpha(\mathcal{P}[\omega]^2(z)), ..., \alpha(\mathcal{P}[\omega]^N(z))$ which allows one to approximate the average of α. This delivers an ADST for the given J but the member of the set $\Xi(\mathcal{P}[\omega], A)$ that emerges will depend on the convention used to choose the first and second axes of T.

Another practical way to compute spin tunes is by using the spectrum of the spin vector motion as follows. For simplicity we consider a fixed but arbitrary action value J and assume that the spin-orbit system belongs to $\mathcal{ACB}(d, \mathcal{P}[\omega])$. Then let $(\mathcal{P}[\omega], A)$ have a particle-spin-vector trajectory (Z, S) and let $S_j(n)$ denotes the j-th component of $S(n)$. The discrete Fourier transform (DFT) of $S_j(0), ..., S_j(N)$ is defined by $\hat{S}_j$ where

$$\hat{S}_j(n) := \frac{1}{N+1} \sum_{k=0}^{N} S_j(k) \exp(-\frac{2\pi ink}{N+1}) , \tag{7.26}$$

and where $n = 0, ..., N$. We define, for $\lambda \in [0, 1)$ and nonnegative integer N,

$$a_N(S_j, \lambda) := (N+1)^{-1} \sum_{n=0}^{N} S_j(n) \exp(-2\pi in\lambda) . \tag{7.27}$$

Since, by Remark 6 above, $(\mathcal{P}[\omega], A)$ has a uniform IFF it can be easily shown (see Appendix B.5) that $a_N(S_j, \lambda)$ converges as $N \to \infty$ and we denote the limit of $a_N(S_j, \lambda)$ by $a(S_j, \lambda)$ and we define the " "Cesàro spectrum" $\Lambda(S_j)$ of S_j by $\Lambda(S_j) := \{\lambda \in [0, 1) : a(S_j, \lambda) \neq 0\}$. From (7.26) and (7.27) it is clear that $a(S_j, \lambda)$ can be approximated by using standard DFT software. Then spin tunes are contained in the Cesàro spectrum since, as shown in Appendix B.5 (by using the fact that $(\mathcal{P}[\omega], A)$ has a uniform IFF),

$$\Lambda(S_j) \subset \Xi(\mathcal{P}[\omega], A) \cup \{l \cdot \omega + n : l \in \mathbb{Z}^d, n \in \mathbb{Z}\} . \tag{7.28}$$

Moreover, the Cesàro spectrum can contain many of the spin tunes in $\Xi(\mathcal{P}[\omega], A)$. Theorem 9.1c in the continuous-time formalism of [BEH] reaches the same conclusions. With this we have a direct relationship between the set $\Xi(\mathcal{P}[\omega], A)$ appearing in Theorem 7.3 and a "measurable" quantity, namely the Cesàro spectrum. This way of getting ADST's has been essential for interpreting spin vector motion near to resonance with oscillating external magnetic fields [Ba]. For more details on the notion of Cesàro spectrum, see Appendix B.

In analogy with Theorem 6.2c we now state:

Theorem 7.8 *a) Let $(j, A) \in \mathcal{SOS}(d, j)$ and $T \in \mathcal{C}(\mathbb{T}^d, SO(3))$. Then T satisfies*

$$T \circ j = AT \,, \tag{7.29}$$

iff it belongs to $\mathcal{TF}_{G_0}(j, A)$.
Remark: If $T \in \mathcal{C}(\mathbb{T}^d, SO(3))$ satisfies (7.29) then, by Definition 7.4, T is a uniform IFF of (j, A).
b) (Spin-Orbit Resonance Theorem) Let $(\mathcal{P}[\omega], A) \in \mathcal{SOS}(d, \mathcal{P}[\omega])$. Then $(\mathcal{P}[\omega], A)$ is on spin-orbit resonance iff $\mathcal{TF}_{G_0}(\mathcal{P}[\omega], A)$ is nonempty.

Proof of Theorem 7.8a: By Definition 5.1, $T \in \mathcal{TF}_{G_0}(j, A)$ iff $T^t(j(z))A(z)T(z) \in G_0$ whence, by (7.1), $T \in \mathcal{TF}_{G_0}(j, A)$ iff $T^t(j(z))A(z)T(z) = I_{3\times 3}$ which proves the claim. □
Proof of Theorem 7.8b: By Remark 8, $(\mathcal{P}[\omega], A)$ is on spin-orbit resonance iff $(\mathcal{P}[\omega], A) \in \mathcal{CB}_{G_0}(d, \mathcal{P}[\omega])$. The claim now follows from Definition 5.1. □

We will use Theorem 7.8 in the proof of the Uniqueness Theorem, Theorem 8.1b. Moreover, Theorem 7.8a and Theorem 6.2c are the special cases $H = G_0$ resp. $H = SO(2)$ of a theorem which is valid for every subgroup H of $SO(3)$. This will be addressed in our follow-up work.

8 Polarization

In this chapter we tie together the concepts of polarization field and polarization.

8.1 Estimating the polarization

Consider a family $(j_J, A_J)_{J\in\Lambda}$ of spin-orbit systems where $(j_J, A_J) \in \mathcal{SOS}(d, j_J)$ and $\Lambda \subset \mathbb{R}^d$ is the set of action values.

We note (see also [BH, BV1]) that, for every $J \in \Lambda$, we have a so-called "local polarization", say $\mathcal{S}_{loc,J}$, which by definition is a polarization-field trajectory of (j_J, A_J) satisfying

$$|\mathcal{S}_{loc,J}| \leq 1 \,. \tag{8.1}$$

The associated polarization on the torus J at time n is then given by

$$P_J(n) := (\frac{1}{2\pi})^d \left| \int_{[0,2\pi]^d} d\phi \mathcal{S}_{loc,J}(n, \pi_d(\phi)) \right| \,. \tag{8.2}$$

We will see below how P_J can be estimated by (8.5) which makes P_J a convenient tool for analyzing the bunch polarization. In the so-called "spin equilibrium" the polarization-field trajectory $\mathcal{S}_{loc,J}$ is, by the definition of the spin equilibrium, time-independent for every J whence its initial value, $\mathcal{S}_{loc,J}(0,\cdot)$ is an invariant polarization field of (j_J, A_J). Thus for the spin equilibrium we get

$$P_J(n) = P_J(0) = (\frac{1}{2\pi})^d \Big| \int_{[0,2\pi]^d} d\phi \mathcal{S}_{loc,J}(0,\pi_d(\phi)) \Big| . \tag{8.3}$$

Let j_J be topologically transitive. Then, by Theorem 3.3b, $|\mathcal{S}_{loc,J}(0,z)|$ is independent of z and, if $\mathcal{S}_{loc,J}(0,\cdot)$ is not the zero function, then $|\mathcal{S}_{loc,J}(0,z)| > 0$ and $\mathcal{S}_{loc,J}(0,\cdot)/|\mathcal{S}_{loc,J}(0,\cdot)|$ is an ISF of (j_J, A_J) whence, by (8.1),(8.3),

$$\begin{aligned} P_J(n) = P_J(0) &= (\frac{1}{2\pi})^d \Big| \int_{[0,2\pi]^d} d\phi |\mathcal{S}_{loc,J}(0,\pi_d(\phi))| \frac{\mathcal{S}_{loc,J}(0,\pi_d(\phi))}{|\mathcal{S}_{loc,J}(0,\pi_d(\phi))|} \Big| \\ &\le (\frac{1}{2\pi})^d \Big| \int_{[0,2\pi]^d} d\phi \frac{\mathcal{S}_{loc,J}(0,\pi_d(\phi))}{|\mathcal{S}_{loc,J}(0,\pi_d(\phi))|} \Big| , \end{aligned} \tag{8.4}$$

so that

$$P_J(n) = P_J(0) \le P_{J,max} , \tag{8.5}$$

where

$$P_{J,max} := (\frac{1}{2\pi})^d \sup \Big\{ \Big| \int_{[0,2\pi]^d} d\phi f(\pi_d(\phi)) \Big| : f \in \mathcal{ISF}(j_J, A_J) \Big\} . \tag{8.6}$$

Of course (8.5) also holds if $\mathcal{S}_{loc,J}(0,\cdot)$ is the zero function because in that case $P_J(n) = P_J(0) = 0$. Thus (8.5) holds for the spin equilibrium if j_J is topologically transitive and (j_J, A_J) has an ISF. We conclude from (8.5) that the ISF's provide an upper bound for P_J and this is one reason why they are so important in practice. One can simplify (8.6) in the important case where the spin-orbit system (j_J, A_J) in (8.6) has exactly two ISF's, say $f_J, -f_J$. Then (8.6) simplifies to

$$P_{J,max} = (\frac{1}{2\pi})^d \Big| \int_{[0,2\pi]^d} d\phi f_J(\pi_d(\phi)) \Big| . \tag{8.7}$$

Clearly $P_{J,max}$ is small if the range of f_J is spread out. In Section 8.2 we will see how the Uniqueness Theorem leads to the situation underlying (8.7).

Of course P_J can also be used for an estimation of the bunch polarization which is given by

$$P(n) = (\frac{1}{2\pi})^d \Big| \int_\Lambda dJ \rho_{eq}(J) \int_{[0,2\pi]^d} d\phi \mathcal{S}_{loc,J}(n,\pi_d(\phi)) \Big| , \tag{8.8}$$

where $(\frac{1}{2\pi})^d \rho_{eq}$ is the equilibrium particle phase-space density. We will take a closer look at (8.8) in our follow-up work. With (8.8) the bunch polarization for the combined beam equilibrium and spin equilibrium reads as

$$P(n) = P(0) = (\frac{1}{2\pi})^d \Big| \int_\Lambda dJ \rho_{eq}(J) \int_{[0,2\pi]^d} d\phi \mathcal{S}_{loc,J}(0,\pi_d(\phi)) \Big| . \tag{8.9}$$

Let the conditions underlying (8.5) hold for almost all J, i.e., let a set $M \subset \Lambda$ exist which has Lebesgue measure zero and such that, for every $J \in (\Lambda \setminus M)$, the spin-orbit system (j_J, A_J) has an ISF and j_J is topologically transitive. Then, by (8.3),(8.5),(8.9), we have for the spin equilibrium

$$P(n) = P(0) \leq (\frac{1}{2\pi})^d \int_\Lambda dJ \rho_{eq}(J) \Big| \int_{[0,2\pi]^d} d\phi \mathcal{S}_{loc,J}(0, \pi_d(\phi)) \Big|$$
$$= \int_\Lambda dJ \rho_{eq}(J) P_J(0) \leq \int_\Lambda dJ \rho_{eq}(J) P_{J,max} \ . \tag{8.10}$$

Note that we assume that $\rho_{eq}(J)$ and $P_{J,max}$ depend on J regularly enough to ensure that the integrals in (8.8), (8.9) and (8.10) are meaningful. Using (8.7) one can simplify (8.10) in the case where, for every $J \in (\Lambda \setminus M)$, the spin-orbit system (j_J, A_J) has two ISF's $f_J, -f_J$ and no others. Then (8.10) simplifies, thanks to (8.7), to

$$P(n) = P(0) \leq (\frac{1}{2\pi})^d \int_\Lambda dJ \rho_{eq}(J) \Big| \int_{[0,2\pi]^d} d\phi f_J(\pi_d(\phi)) \Big| \ , \tag{8.11}$$

where we also assume that the functional dependences of $\rho_{eq}(J)$ and f_J on J are regular enough to ensure that the integrals in (8.11) are meaningful. For more details on estimating the bunch polarization, also for non-equilibrium spin fields, see [Ho, Vo].

8.2 The Uniqueness Theorem of invariant spin fields

We saw in (8.5) and (8.7), how in a situation where only two ISF's exist, the invariant spin fields govern the estimation of P_J. In this section we will see that this situation is very common off spin-orbit resonance.

Let $(j, A) \in \mathcal{ACB}(d, j)$. Then, by Remark 5 in Chapter 7, (j, A) has an ISF and so it natural to ask about the impact of the set $\Xi(j, A)$ on $\mathcal{ISF}(j, A)$. In fact, if $j = \mathcal{P}[\omega]$ and $(\mathcal{P}[\omega], A)$ is off orbital resonance, this question is partially answered by part b) of the following theorem.

Theorem 8.1 *a) Let $(j, A) \in \mathcal{SOS}(d, j)$ and let f and g be invariant polarization fields of (j, A). Then $h \in \mathcal{C}(\mathbb{T}^d, \mathbb{R}^3)$, defined by $h(z) := f(z) \times g(z)$, is an invariant polarization field of (j, A) where $\times$ denotes the vector product.*

b) (The Uniqueness Theorem) Let $(\mathcal{P}[\omega], A) \in \mathcal{ACB}(d, \mathcal{P}[\omega])$ be off orbital resonance, i.e., let $(1, \omega)$ be nonresonant. Also, let $(\mathcal{P}[\omega], A)$ be off spin-orbit resonance. Then $(\mathcal{P}[\omega], A)$ has an ISF, say F, and F and $-F$ are the only ISF's of $(\mathcal{P}[\omega], A)$.

Proof of Theorem 8.1a: Since f and g are invariant polarization fields of (j, A) it follows from Definition 3.2 that $f \circ j = Af$ and $g \circ j = Ag$ whence

$$h(j(z)) = (f(j(z)) \times g(j(z))) = (A(z)f(z) \times A(z)g(z)) = A(z)(f(z) \times g(z)) = A(z)h(z) \ ,$$

so that, by Definition 3.2, h is an invariant polarization field of (j, A). □

Proof of Theorem 8.1b: Let $(\mathcal{P}[\omega], A) \in \mathcal{ACB}(d, \mathcal{P}[\omega])$ be off orbital resonance. The claim to be proved is equivalent to its contrapositive which is the following claim: If the total number of ISF's of $(\mathcal{P}[\omega], A)$ is not 2, then $(\mathcal{P}[\omega], A)$ is not off spin-orbit resonance. Now, we know from Remark 5 in Chapter 7 that $(\mathcal{P}[\omega], A)$ has an ISF whence, by Section 3.2, it has at least two ISF's so that if the number of ISF's differs from 2, there are more than two ISF's. Also, since $(\mathcal{P}[\omega], A) \in \mathcal{ACB}(d, \mathcal{P}[\omega])$ we know from Theorem 7.3b that $(\mathcal{P}[\omega], A)$ has spin tunes. Then if the system is not off spin-orbit resonance, it must be on spin-orbit resonance. Thus the above claim we have to prove is equivalent to the following claim: If the total number of ISF's of $(\mathcal{P}[\omega], A)$ is larger than two, then $(\mathcal{P}[\omega], A)$ is on spin-orbit resonance.

In fact we will now prove the latter claim. So let $(\mathcal{P}[\omega], A)$ have more than two ISF's. Recalling Section 3.2, we then conclude that $(\mathcal{P}[\omega], A)$ has ISF's, say f and g, such that $g \neq f$ and $g \neq -f$. Note that $f, -f$ and g are three different ISF's of $(\mathcal{P}[\omega], A)$. We define $h \in \mathcal{C}(\mathbb{T}^d, \mathbb{R}^3)$ by $h(z) := f(z) \times g(z)$ and observe, by Theorem 8.1a, that h is an invariant polarization field of $(\mathcal{P}[\omega], A)$. On the other hand, since $(\mathcal{P}[\omega], A)$ is off orbital resonance, $\mathcal{P}[\omega]$ is topologically transitive whence, by Theorem 3.3b, $|h|$ is constant, i.e., $|h(z)| =: \lambda$ is independent of z. We first consider the case where $\lambda = 0$, i.e., where $f \times g$ is the zero function. Then a function $\hat{h} : \mathbb{T}^d \to \mathbb{R}$ exists such that $g = \hat{h} f$ whence $g \cdot f = \hat{h}|f|^2 = \hat{h}$ which implies that $\hat{h}$ is continuous. On the other hand $1 = |g| = |\hat{h} f| = |\hat{h}|$ whence $\hat{h}$ can take values only in $\{1, -1\}$ whence, since $\hat{h}$ is continuous and $\mathbb{T}^d$ is path-connected, $\hat{h}$ is constant. Thus either $g = f$ or $g = -f$ which is a contradiction. So the case where $\lambda = 0$ cannot occur. Thus $\lambda > 0$. Since h is an invariant polarization field of $(\mathcal{P}[\omega], A)$ and since the real number λ is positive we define $k \in \mathcal{C}(\mathbb{T}^d, \mathbb{R}^3)$ by $k(z) := h(z)/\lambda = h(z)/|h(z)|$ and observe, by using Definition 3.2, that k is an invariant polarization field of $(\mathcal{P}[\omega], A)$. Of course $|k(z)| = |h(z)|/|h(z)| = 1$ whence k is an ISF of $(\mathcal{P}[\omega], A)$. We also define $l \in \mathcal{C}(\mathbb{T}^d, \mathbb{R}^3)$ by $l(z) := k(z) \times f(z)$ and observe, by Theorem 8.1a, that l is an invariant polarization field of $(\mathcal{P}[\omega], A)$. Of course $f(z) \cdot k(z) = (f(z) \cdot h(z))/|h(z)| = f(z) \cdot (f(z) \times g(z))/\lambda = 0$ whence, for every $z \in \mathbb{T}^d$,

$$0 = l(z) \cdot k(z) = l(z) \cdot f(z) = f(z) \cdot k(z) \ . \tag{8.12}$$

Clearly $|l(z)| = |k(z) \times f(z)| = \sqrt{|k(z)|^2\,|f(z)|^2 - (k(z) \cdot f(z))^2} = \sqrt{1 - (k(z) \cdot f(z))^2} = 1$ which implies that l is an ISF of $(\mathcal{P}[\omega], A)$ and that

$$1 = |l(z)| = |k(z)| = |f(z)| \ . \tag{8.13}$$

It follows from (8.12) and (8.13) that

$$[l(z), k(z), f(z)]^t [l(z), k(z), f(z)] = I_{3\times 3} \ . \tag{8.14}$$

Moreover, by (8.13), $\det([l(z), k(z), f(z)]) = l(z) \cdot (k(z) \times f(z)) = |l(z)|^2 = 1$ whence, by (8.14), for every $z \in \mathbb{T}^d$, the 3×3-matrix $[l(z), k(z), f(z)]$ belongs to $SO(3)$. We thus define $T \in \mathcal{C}(\mathbb{T}^d, SO(3))$ by $T(z) := [l(z), k(z), f(z)]$. Since all three columns of T are invariant polarization fields of $(\mathcal{P}[\omega], A)$ we have, by Definition 3.2,

$$\begin{aligned} A(z)T(z) &= A(z)[l(z), k(z), f(z)] = [A(z)l(z), A(z)k(z), A(z)f(z)] \\ &= [l(\mathcal{P}[\omega](z)), k(\mathcal{P}[\omega](z)), f(\mathcal{P}[\omega](z))] = T(\mathcal{P}[\omega](z)) \ , \end{aligned}$$

whence $T \circ \mathcal{P}[\omega] = AT$ so that, by Theorem 7.8a, T belongs to $\mathcal{TF}_{G_0}(j, A)$. Thus, by Theorem 7.8b, $(\mathcal{P}[\omega], A)$ is on spin-orbit resonance as was to be shown. $\square$

The claim of Theorem 8.1b that $(\mathcal{P}[\omega], A)$ has an ISF is trivial because of Remark 5 in Chapter 7. Thus the essence of the claim of Theorem 8.1b is that $(\mathcal{P}[\omega], A)$ has only two ISF's. Recall also from Chapter 3 that the set of ISF's of a spin-orbit system is either infinite or contains an even number of elements. Note that in this work the term "finite number" includes the case of zero. Indeed if a spin-orbit system has no ISF then its number of ISF's is zero, an even number!

9 Summary and Outlook

In this work we have studied the discrete-time spin-vector motion in storage rings in mathematically rigorous terms as follows. A spin-orbit system is a pair (j, A) where $j \in \mathrm{Homeo}(\mathbb{T}^d)$ is the particle 1-turn map and $A \in \mathcal{C}(\mathbb{T}^d, SO(3))$ with the torus $\mathbb{T}^d$ introduced in Section 2.3. In the special case $j = \mathcal{P}[\omega]$, ω is the orbital tune and $\mathcal{P}[\omega]$ is the corresponding translation on the torus after one turn. For every spin-orbit system (j, A) in $\mathcal{SOS}(d, j)$ a 1-turn particle-spin-vector map $\mathcal{P}[j, A] \in \mathrm{Homeo}(\mathbb{T}^d \times \mathbb{R}^3)$ is defined by (2.22), i.e., $\mathcal{P}[j, A](z, S) := (j(z), A(z)S)$. Also a 1-turn field map $\tilde{\mathcal{P}}[j, A]$ is a bijection on $\mathcal{C}(\mathbb{T}^d, \mathbb{R}^3)$ defined by (3.2), i.e.,

$$\tilde{\mathcal{P}}[j, A](f) := (Af) \circ j^{-1} \, .$$

We note also that the particle-spin-vector maps are just characteristic maps of the field maps. If $f \in \mathcal{C}(\mathbb{T}^d, \mathbb{R}^3)$ satisfies $\tilde{\mathcal{P}}[j, A](f) = f$ then f is called an invariant polarization field of (j, A) and in the subcase $|f| = 1$ it is called an invariant spin field. A $j \in \mathrm{Homeo}(\mathbb{T}^d)$ is called topologically transitive if a $z_0 \in \mathbb{T}^d$ exists such that the topological closure $\overline{\{j^n(z_0) : n \in \mathbb{Z}\}}$ of $\{j^n(z_0) : n \in \mathbb{Z}\}$ equals $\mathbb{T}^d$. The ISF-conjecture states that a spin-orbit system (j, A) has an ISF if j is topologically transitive. Note that a special case of this conjecture is: If a spin-orbit system $(\mathcal{P}[\omega], A)$ is off orbital resonance, then it has an ISF. If $(j, A) \in \mathcal{SOS}(d, j)$ and $T \in \mathcal{C}(\mathbb{T}^d, SO(3))$ then $(j, A') \in \mathcal{SOS}(d, j)$ is called the transform of (j, A) under T where A' is defined by (4.2), i.e., $A'(z) := T^t(j(z))A(z)T(z)$. If H is a subgroup of $SO(3)$ and $(j, A) \in \mathcal{SOS}(d, j)$ then (j, A') in $\mathcal{SOS}(d, j)$ is an H-normal form of (j, A) if A' is H-valued and (j, A') is a transform of (j, A). A spin-orbit system has an $SO(2)$-normal form iff it has an invariant frame field. Following Chapter 7, a spin-orbit system (j, A) has a spin tune $\nu \in [0, 1)$ if (j, A') with $A'(z) = \exp(2\pi\nu\mathcal{J})$ is a transform of (j, A). We say that $(\mathcal{P}[\omega], A)$ is on spin-orbit resonance if it has spin tunes and if for every spin tune ν we can find $m \in \mathbb{Z}^d, n \in \mathbb{Z}$ such that $\nu = m \cdot \omega + n$. A spin-orbit system has spin tunes iff it has a G_ν-normal form. Moreover a spin-orbit system has an G_ν-normal form iff it has an uniform invariant frame field. A spin-orbit system $(\mathcal{P}[\omega], A)$ is on spin-orbit resonance iff it has a G_0-normal form. Also we used the notions of quasiperiodicity and Cesàro to study the impact of the spin tune on the spin-vector motion. The Uniqueness Theorem, Theorem 8.1b, states that, if $(\mathcal{P}[\omega], A)$ has spin tunes and is off orbital resonance and off spin-orbit resonance, then it has only two ISF's and they differ only by a sign. The polarization of a bunch is defined and its size is estimated in Section 8.1. In the follow-up work we will reconsider and generalize the present work by using using concepts from fibre bundle theory introduced in [He2].

Table of Notation

$\mathcal{A}, A_{d,\nu}$	(2.3), (7.4)
$\mathcal{ACB}(d,j)$	Definition 7.1
Arg	(2.18)
$a_N(F,\lambda,a(F,\lambda)$	Definition $B.2c$
$\mathcal{BMT}$	Section 2.1
$\mathcal{C}(X,Y)$ set of continuous functions from X to Y	Appendix A.4
$\mathcal{CB}_H$	Definition 5.1
Equivalent spin – orbit systems, $\sim$, $\overline{(j,A)}$	Definition 4.2
Fun_d	Section 2.2
$GL(d,\mathbb{Z})$	Appendix B.4
Group, Abelian group, conjugate subgroups	Appendix A.6
H – normal Form	Definition 5.1
$\mathrm{Homeo}(X,Y), \mathrm{Homeo}(X)$	Appendix A.4
Invariant frame field (IFF)	Definition 6.1
Invariant spin field (ISF)	Definition 3.2
$\mathcal{J}, \mathcal{J}'$	Section 2.4, (B.32)
Map_d	Section 2.2
$\mathcal{P}[\omega], \hat{\mathcal{P}}[\omega]$	(2.24), Section 2.2
$\mathcal{P}[j,A], \tilde{\mathcal{P}}[j,A]$	(2.22), (3.2)
$\hat{\mathcal{P}}[\hat{j},\hat{A}], \tilde{\hat{\mathcal{P}}}[\hat{j},\hat{A}]$	(2.12), (3.8)
Resonant, nonresonant, orbital resonance	Section 2.5
$SO(3), SO(2)$	Section 2.1, (6.5)
$\mathcal{SOS}(d,j), \mathcal{SOS}_{CT}(d,\omega)$	(2.27), Section 2.4
Spin tune, spin – orbit resonance	Definitions 7.2, 7.7
Topologically transitive	Section 2.5
$\mathbb{T}^d$ (d – torus)	(2.15)
$\mathcal{TF}(A,A';d,j), \mathcal{TF}_H(j,A)$	Definition 4.1, Definition 5.1
Uniform IFF	Definition 7.4
Y_χ	(B.6)
$\mathbb{Z}, \mathbb{Z}_+$	Set of integers, Set of nonnegative integers
$\Lambda(F), \Lambda_{tot}(F)$	Definition $B.2c$
$\Xi(j,A)$	(7.7)
π_d	(2.19)
τ_d, τ_d^{fin}	Section 2.3
$\Psi[j,A]$ (spin transfer matrix function)	(2.36)
ω (orbital tune)	Section 2.1
$\lfloor c \rfloor$	Appendix B.1
$\trianglelefteq$	Chapter 5

Appendix

A Conventions and terminology

In this appendix we introduce some terminology and notions of the main text.

A.1 Function, image, inverse image

A "function" $f : X \to Y$ is determined by its graph and its codomain. The "graph" of f is the set $\{(x, f(x)) : x \in X\}$ and the "codomain" of f is Y. The "domain" of f is X and the "range" of f is the set $f(X) := \{f(x) : x \in X\}$. More generally, if M is a subset of X then the "image" of M under f is the set $f(M) := \{f(x) : x \in M\}$. If M is a subset of Y then the "inverse image" of M under f is the set $f^{-1}(M) := \{x \in X : f(x) \in M\}$.

One calls f a "surjection" or "onto" if its range and codomain are equal. One calls f "one-one" or an "injection" if $f(x) = f(x')$ implies that $x = x'$. One calls f a "bijection" if it is one-one and a surjection.

If $f : X \to Y$ and $g : Y \to Z$ are functions then $g \circ f$ is the function $g \circ f : X \to Z$ defined by $(g \circ f)(x) := g(f(x))$. One calls the operation $\circ$ the "composition" of functions. If X is a set then the function $id_X : X \to X$ is defined by $id_X(x) := x$ and is called the "identity function" on X. If $f : X \to Y$ is a bijection then a unique function $f^{-1} : Y \to X$ exists such that $f^{-1} \circ f = id_X, f \circ f^{-1} = id_Y$ and it is called the "inverse" of f. Clearly f is a bijection iff it has an inverse. Note that if $f : X \to Y$ is a bijection then f^{-1} can either mean the inverse function or the inverse image operation. However it will always be clear from the context what the meaning is.

Note finally that according to our, very common, definition of a function two functions with the same graph are different iff they have different codomains. Thus the alternative, and equally common, way to define a function in terms of its graph (i.e., without invoking the codomain) is different from our definition.

A.2 Partition, equivalence relation

If X is a set and if P is a set whose elements are disjoint nonempty subsets of X whose union is X then one calls P a "partition of X".

If X is a set and B a subset of $X \times X$ then B is called a "relation" on X. The relation B is called "symmetric" if $(x, y) \in B$ implies that $(y, x) \in B$. The relation B is called "reflexive" if $(x, x) \in B$ for all $x \in X$. The relation B is called "transitive" if $(x, y) \in B$ and $(y, z) \in B$ implies that $(x, z) \in B$.

A relation on X is called an "equivalence relation on X" if it is symmetric,reflexive, and transitive. If B is an equivalence relation on X and $x \in X$ then the set $\{y \in Y : (x, y) \in B\}$ is called the "equivalence class of x under the equivalence relation B".

The equivalence classes of B form a partition of X as follows. Clearly the equivalence classes of B are nonempty sets and overlap X. Moreover by, transitivity, if two equivalence classes of B have a common element then they are equal. Conversely every partition of X defines an equivalence relation on X as can be easily checked.

A.3 Topology, topological space, open set

A collection, τ, of subsets of a set X is called a "topology on X" if τ is closed under arbitrary unions and finite intersections and if $X, \emptyset \in \tau$. Any pair (X, τ) is called a "topological space (over X)". The elements of τ are called the "open" sets of (X, τ).

The "closed" sets of (X, τ) are the complements of the open sets (X, τ). If M is a subset of X then its "closure" $\bar{M}$ is defined as the intersection of all closed supsets of M.

If (X, τ) is a topological space and if X' is a subset of X then the "subspace topology" τ' of X' from X is the collection $\{X' \cap M : M \in \tau\}$ and the topological space (X', τ') is called a "topological subspace" of (X, τ).

If the topology τ is clear from the context then we write X instead of (X, τ). For example the topology of $\mathbb{R}^d$ is obtained from the Euclidean norm and the topology of $\mathbb{Z}^d$ is discrete, i.e., every subset of $\mathbb{Z}^d$ is open.

If (X, τ) is a topological space then $\sigma \subset \tau$ is called a "base" of (X, τ) (and of τ) if every $M \in \tau$ is a union of elements of σ.

A.4 Continuous function, homeomorphism

Let (X, τ) and (X', τ') be topological spaces. Then a function $f : X \to X'$ is called "continuous w.r.t. (X, τ) and (X', τ')" if for every $M \in \tau'$ the inverse image of M under f belongs to τ, i.e., $f^{-1}(M) \in \tau$. We denote the collection of continuous functions by $\mathcal{C}(X, X')$. A function $f \in \mathcal{C}(X, X')$ is called a "homeomorphism" and X, X' are called a "homeomorphic" if f is a bijection and if its inverse is continuous. We denote the collection of those homeomorphisms by $\mathrm{Homeo}(X, X')$ and we also define $\mathrm{Homeo}(X) := \mathrm{Homeo}(X, X)$. The topological spaces X and X' are called "homeomorphic" if $\mathrm{Homeo}(X, X')$ is nonempty.

Let $F : X \to \mathbb{C}$ and $F^N : X \to \mathbb{C}$ be functions where $N \in \mathbb{Z}_+$. Then the sequence F^N is said to converge uniformly on X to F as $N \to \infty$ if, for all $\varepsilon > 0$ there exists an $M \in \mathbb{Z}_+$ such that, for all $N \in \mathbb{Z}_+$ with $N \geq M$, $\sup_{x \in X} |F^N(x) - F(x)| < \varepsilon$. Even in the case of uniform convergence, in general neither F nor the F^N are bounded functions. However in the important situation where all F^N are bounded, and since $|F(x)| \leq |F^N(x) - F(x) + F^N(x)|$, F is bounded too.

A.5 Product topology, Hausdorff space, compact space, path-connected space, final topology

If (X, τ_X) and (Y, τ_Y) are topological spaces then the product topology τ on $X \times Y$ is defined such that the sets $M \times N$ with $M \in \tau_X, Y \in \tau_Y$ form a base of τ. topology from $\mathbb{R}^{3\times 3}$.

A topological space X is called "Hausdorff" if for every pair of distinct elements x, x' of X open sets M, M' exists such that $x \in M, x' \in M'$ and $M \cap M' = \emptyset$. A topological space X is called "compact" if for any union of X by open sets of X already the union of finitely many of those open sets equals X. If X is a topological space and then a subset A of X is called "compact" if A is, as a topological subspace of X, compact.

A topological space X is called "path-connected" if for elements $x, x' \in X$ a continuous function $f : [0, 1] \to X$ exists such that $f(0) = x$ and $f(1) = x'$. A subset A of X is called "path-connected" if A is, as a topological subspace of X, path-connected. One has the following intermediate-value theorem: If X, Y are topological spaces such that X is path-connected and if $g : X \to Y$ is a continuous function then the range of g is a path-connected subset of Y.

Let X be a topological space and let $p : X \to Y$ be a surjection where Y is a set. A natural topology on Y is defined such that a subset $B \subset Y$ is open iff the inverse image $p^{-1}(B) \subset X$ is open. One calls the topology on Y the "final topology" w.r.t. p [wiki1].

A.6 Groups

A "group" is a pair $(G, *)$ where G is a set and $*$ is a binary operation such that

$$\begin{array}{ll} \text{(G0) (Binary operation)} & \forall_{g_1,g_2 \in G}\ (g_1 * g_2) \in G\ , \\ \text{(G1) (Associativity)} & \forall_{g_1,g_2,g_3 \in G}\ (g_1 * g_2) * g_3 = g_1 * (g_2 * g_3)\ , \\ \text{(G2) (Identity element } e_G\text{)} & \exists_{e_G \in G}\ \forall_{g \in G}\ e_G = e_G * g = g * e_G\ , \\ \text{(G3) (Inverse elements)} & \forall_{g_1 \in G}\ \exists_{g_2 \in G}\ e_G = g_1 * g_2 = g_2 * g_1\ . \end{array}$$

We will abbreviate $(G, *)$ as G when the operation $*$ is clear from the context and we often write $g_1 * g_2$ as $g_1 g_2$ when the operation $*$ is clear from the context. The inverse element of a $g \in G$ is denoted by g^{-1}. If H is a subset of G and if $g, g' \in G$ then we define $gHg' := \{ghg' : h \in H\}$. A group G is called "Abelian" if, in addition to G1-G3,

$$\text{(G4) (Commutativity)} \qquad \forall_{g_1,g_2 \in G}\ g_1 * g_2 = g_2 * g_1\ ,$$

in which case $*$ is often replaced by $+$.

A subset G' of G is called a "subgroup of G" if it is a group w.r.t. to the restriction of $*$ to G'. Two elements g', g'' of a group G are called "conjugate" if $g \in G$ exists such that $g'' = gg'g^{-1}$. Two subgroups G', G'' of a group G are called "conjugate" if $g \in G$ exists such that $G'' = gG'g^{-1}$.

The most important groups in our work are $SO(3)$ and its subgroups and, in Appendix B.4, the Abelian group $\mathbb{Z}^d$ and its subgroups.

B Quasiperiodic functions and the Cesàro spectrum of a function

In this appendix we define and apply the notions of quasiperiodic function and Cesàro spectrum with the main aim of proving two major results of the main text: Theorem 7.3f and Eq. (7.28). In fact in Appendix B.3 we prove Theorem 7.3f by using, from Appendix B.2, the Exponential Theorem. The Exponential Theorem is proved by using, from Appendix B.2, the First Spectral Theorem. The First Spectral Theorem is proved by using, from Appendix B.2, Fejér's multivariate theorem. Moreover in Appendix B.5 we prove Eq. (7.28) by using the First Spectral Theorem and, from Appendix B.4, the Second, Third, and Fourth Spectral Theorems. In Appendix B.6 we show how pathologies in the Cesàro spectra of the spin motions can lead to the absence of spin tunes.

B.1 Defining Quasiperiodic functions and the Cesàro spectrum

Definition B.1 *Let $f \in \mathcal{C}(\mathbb{R}^d, \mathbb{C})$ be 2π-periodic in its arguments. If $\chi \in \mathbb{R}^d$ then f is called a "χ-generator" of the function $F : \mathbb{Z} \to \mathbb{C}$ defined by $F(n) = f(2\pi n \chi)$. A function $F : \mathbb{Z} \to \mathbb{C}$ is called "χ–quasiperiodic" if it has a χ-generator and it is called "quasiperiodic" if it has a χ-generator for some χ. Note that, as sets, $\mathbb{C} = \mathbb{R}^2$ so we equip $\mathbb{C}$ with the natural topology of $\mathbb{R}^2$. Note also that we choose the sets $\mathbb{R}$ and $\mathbb{C}$ such that $\mathbb{R} \subset \mathbb{C}$.*

Note that if f is a χ-generator of the χ–quasiperiodic function $F : \mathbb{Z} \to \mathbb{C}$ then, by Theorem 2.5b, there exists a unique function $h \in \mathcal{C}(\mathbb{T}^d, \mathbb{C})$ such that $f = h \circ \pi_d$. Thus we could have defined quasiperiodicity in terms of h instead of f, however the latter is more convenient. Again we see that ϕ and z have the same expressive power. □

Remark:

(1) The following facts immediately follow from Definition B.1. Let F and G be χ–quasiperiodic functions where $\chi \in \mathbb{R}^d$ and let $c_1, c_2 \in \mathbb{C}$. Then the functions $c_1 F + c_2 G$ and FG are χ–quasiperiodic. Moreover if $\tilde{\chi} \in \mathbb{R}^k$ then F is $(\chi, \tilde{\chi})$-quasiperiodic and $(\tilde{\chi}, \chi)$-quasiperiodic. Furthermore if H is $\tilde{\chi}$-quasiperiodic then FH is $(\chi, \tilde{\chi})$-quasiperiodic and $(\tilde{\chi}, \chi)$-quasiperiodic. □

Let Z be the particle trajectory of a spin-orbit system $\mathcal{P}[\omega]$ with $Z(0) = \pi_d(\phi_0)$ whence, by (2.25),(2.26) and the remarks after (2.31), $Z(n) = \mathcal{P}[\omega]^n(\pi_d(\phi_0)) = \mathcal{P}[n\omega](\pi_d(\phi_0)) = \pi_d(\phi_0 + 2\pi n\omega)$ so that, by (2.31), the equations of motion for the spin-vector trajectory S read as $S(n+1) = A(\pi_d(\phi_0 + 2\pi n\omega))S(n)$. Thus, and by Definition B.1, the matrix elements of $A(\pi_d(\phi_0 + 2\pi n\omega))$ are ω–quasiperiodic functions of n so that the equation of motion $S(n+1) = A(\pi_d(\phi_0 + 2\pi n\omega))S(n)$ is ω–quasiperiodic (see also Remark 2 in Chapter 4). This circumstance explains why the concept of quasiperiodicity is relevant for the present work.

Part c) of the following definition gives us the notion of Cesàro spectrum.

Definition B.2 *a) Let $f : \mathbb{R}^d \to \mathbb{C}$ be continuous and 2π–periodic in its arguments. Then for $m \in \mathbb{R}^d$ the "m-th Fourier coefficient" of f is defined by*

$$f_m := \frac{1}{(2\pi)^d} \int_0^{2\pi} \cdots \int_0^{2\pi} f(\phi) \exp(-i(m \cdot \phi)) d\phi_1 \cdots d\phi_d \, . \tag{B.1}$$

b) Let $m \in \mathbb{R}^d, N \in \mathbb{Z}_+$ where where $\mathbb{Z}_+$ denotes the set of nonnegative integers. We define

$$A^d_{N,m} := \prod_{n=1}^{d} \frac{N+1-|m_n|}{N+1} \, , \qquad ||m|| := max(|m_1|, ..., ||m_d|) \, . \tag{B.2}$$

Let $f : \mathbb{R}^d \to \mathbb{C}$ be continuous and 2π–periodic in its arguments. We define, for $N \in \mathbb{Z}_+$, function $f^N : \mathbb{R}^d \to \mathbb{C}$ by

$$f^N(\phi) := \sum_{\substack{m \in \mathbb{Z}^d \\ ||m|| \le N}} A^d_{N,m} f_m \exp(i(m \cdot \phi)) \, . \tag{B.3}$$

c) If $F:\mathbb{Z}\to\mathbb{C}$ is a function and $\lambda\in[0,1), N\in\mathbb{Z}_+$, we define

$$a_N(F,\lambda) := \frac{1}{N+1}\sum_{n=0}^{N} F(n)\exp(-2\pi i n\lambda)\ . \tag{B.4}$$

*We denote by $\Lambda_{tot}(F)$ the set of those $\lambda\in[0,1)$ for which $a_N(F,\lambda)$ converges as $N\to\infty$. If $\lambda\in\Lambda_{tot}(F)$ we denote the limit of $a_N(F,\lambda)$ by $a(F,\lambda)$ and we define the "*Cesàro *spectrum" $\Lambda(F)$ of F by $\Lambda(F):=\{\lambda\in\Lambda_{tot}(F): a(F,\lambda)\neq 0\}$.* □

The terminology "Cesàro spectrum" in Definition B.2c is justified by the fact that the limit $a(F,\lambda)$ is often called the "Cesàro sum" of the sequence $F(n)\exp(-2\pi i n\lambda)$ where $n=0,1,\cdots$ [wiki6]. We now define the simple but important function $E_c:\mathbb{Z}\to\mathbb{C}$ by

$$E_c(n) := \exp(i2\pi nc) = \exp(i2\pi n\lfloor c\rfloor)\ , \tag{B.5}$$

where c is an arbitrary real number and where $\lfloor c\rfloor$ denotes the fractional part of c. Clearly, by (B.5) and Definition B.1, the function $E_{1/2\pi}$ is a c-generator of E_c whence E_c is c–quasiperiodic. We will strengthen this result in Theorem B.4 below. We will use E_c also in Appendix B.3 where we prove Theorem 7.3f.

B.2 Fejér's multivariate theorem and the First Spectral Theorem. The Exponential Theorem

In this section we first present (see Theorem B.3b) Fejér's multivariate theorem and then derive from that the First Spectral Theorem (Theorem B.3d) and some other properties. We then use the First Spectral Theorem to prove the Exponential Theorem, Theorem B.4b, which allow us, in Appendix B.3, to prove Theorem 7.3f. To study the Cesàro spectrum of quasiperiodic functions, we define, for $\chi\in\mathbb{R}^d$,

$$Y_\chi := \{m\cdot\chi + n : m\in\mathbb{Z}^d, n\in\mathbb{Z}\}\ . \tag{B.6}$$

Theorem B.3 *a) Let $c\in\mathbb{R}$ and $\lambda\in[0,1)$. Then*

$$\Lambda_{tot}(E_c) = [0,1)\ , \tag{B.7}$$

$$a(E_c,\lambda) = \begin{cases} 1 & \text{if } \lambda=\lfloor c\rfloor \\ 0 & \text{if } \lambda\neq\lfloor c\rfloor\ , \end{cases} \tag{B.8}$$

$$\Lambda(E_c) = \{\lfloor c\rfloor\}\ . \tag{B.9}$$

Let $F:\mathbb{Z}\to\mathbb{C}$ be a function. Then, for $N\in\mathbb{Z}_+$,

$$a_N(FE_c,\lambda) = a_N(F,\lfloor\lambda-c\rfloor)\ . \tag{B.10}$$

If, in addition, $\Lambda_{tot}(F)=[0,1)$ then

$$\Lambda_{tot}(FE_c) = [0,1)\ , \tag{B.11}$$

$$a(FE_c,\lambda) = a(F,\lfloor\lambda-c\rfloor)\ , \tag{B.12}$$

$$\Lambda(FE_c) = \{\lambda\in[0,1): \lfloor\lambda-c\rfloor\in\Lambda(F)\}\ . \tag{B.13}$$

b) (Fejér's multivariate theorem) Let $f : \mathbb{R}^d \to \mathbb{C}$ *be* 2π*–periodic in its arguments and continuous. Let also, for* $N \in \mathbb{Z}_+$*, the function* $f^N : \mathbb{R}^d \to \mathbb{C}$ *be defined by (B.3), i.e.,* $f^N(\phi) := \sum_{\substack{m\in\mathbb{Z}^d \\ ||m||\leq N}} A^d_{N,m} f_m \exp(i(m\cdot\phi))$*. Then* f^N *is continuous and* 2π*–periodic in its arguments. Moreover the sequence* f^N *converges uniformly on* $\mathbb{R}^d$ *to* f *as* $N \to \infty$*.*
c) Let $F : \mathbb{Z} \to \mathbb{C}$ *be a* χ*–quasiperiodic function where* $\chi \in \mathbb{R}^d$ *and let* f *be a* χ*-generator of* F*, i.e.,* $F(n) = f(2\pi n\chi)$*. Defining for* $N \in \mathbb{Z}_+$ *the function* $F^N : \mathbb{Z} \to \mathbb{C}$ *by*

$$F^N(n) := \sum_{\substack{m\in\mathbb{Z}^d \\ ||m||\leq N}} A^d_{N,m} h_m \exp(i2\pi n(m\cdot\chi)) , \tag{B.14}$$

where f_m *is the* m*-th Fourier coefficient of* f*, then the sequence* F^N *converges uniformly on* $\mathbb{Z}$ *to* F *as* $N \to \infty$*. Furthermore* F^N *is* χ*-quasiperiodic and* $\Lambda_{tot}(F^N) = [0,1)$*. Moreover, for* $\lambda \in [0,1)$*,*

$$a(F^N,\lambda) = \sum_{\substack{m\in\mathbb{Z}^d \\ ||m||\leq N}} A^d_{N,m} f_m a(E_{m\cdot\chi},\lambda) , \tag{B.15}$$

as well as $\Lambda(F^N) \subset Y_\chi$*.*
d) (First Spectral Theorem) Let $F : \mathbb{Z} \to \mathbb{C}$ *be a* χ*–quasiperiodic function and let* $\Lambda_{tot}(F) = [0,1)$*. Then* $\Lambda(F) \subset Y_\chi$*.*

Proof of Theorem B.3a: Clearly, for $N \in \mathbb{Z}_+, \lambda \in [0,1)$ and by (B.4),(B.5),

$$\begin{aligned} a_N(E_c,\lambda) &= \frac{1}{N+1}\sum_{n=0}^{N} E_c(n)\exp(-2\pi i n\lambda) = (N+1)^{-1}\sum_{n=0}^{N}\exp(2\pi i n(c-\lambda)) \\ &= \begin{cases} 1 & \text{if } \lambda = \lfloor c \rfloor \\ \frac{1}{N+1}\frac{1-\exp(2\pi i(N+1)(c-\lambda))}{1-\exp(2\pi i(c-\lambda))} & \text{if } \lambda \neq \lfloor c \rfloor , \end{cases} \end{aligned} \tag{B.16}$$

whence

$$\lim_{N\to\infty} a_N(E_c,\lambda) = \begin{cases} 1 & \text{if } \lambda = \lfloor c \rfloor \\ 0 & \text{if } \lambda \neq \lfloor c \rfloor . \end{cases} \tag{B.17}$$

It follows from (B.17) and Definition B.2c and for $c \in \mathbb{R}, \lambda \in [0,1)$, that (B.7),(B.8),(B.9) hold.

Let $F : \mathbb{Z} \to \mathbb{C}$ whence, by (B.5) and Definition B.2c,

$$\begin{aligned} a_N(FE_c,\lambda) &= \frac{1}{N+1}\sum_{n=0}^{N} F(n)\exp(2\pi i n c)\exp(-2\pi i n\lambda) \\ &= \frac{1}{N+1}\sum_{n=0}^{N} F(n)\exp(-2\pi i n\lfloor \lambda - c\rfloor) = a_N(F,\lfloor \lambda - c\rfloor) , \end{aligned}$$

so that (B.10) holds. Let, in addition, $\Lambda_{tot}(F) = [0,1)$. It then follows from Definition B.2c and (B.10) that (B.11),(B.13) and (B.13) hold. $\square$

Proof of Theorem B.3b: That the sequence f^N converges uniformly on $\mathbb{R}^d$ to f is the generalization of Fejér's univariate theorem from $d = 1$ to arbitrary positive integer d (see for example [Maa, Sec. III.22],[Ko, Sec. 79]). $\square$

Proof of Theorem B.3c: Defining for $N \in \mathbb{Z}_+$ the function $f^N : \mathbb{R}^d \to \mathbb{C}$ by (B.3), it follows from Theorem B.3b that the sequence f^N converges uniformly on $\mathbb{R}^d$ to f as $N \to \infty$. It is clear, by (B.3),(B.14), that $F^N(n) = f^N(2\pi n\chi)$. By the uniform convergence of f^N we have $\lim_{N\to\infty} \sup_{\phi\in\mathbb{R}^d} |f^N(\phi) - f(\phi)| = 0$ whence $0 = \lim_{N\to\infty} \sup_{n\in\mathbb{Z}} |f^N(2\pi n\chi\phi) - f(2\pi n\chi\phi)| = \lim_{N\to\infty} \sup_{n\in\mathbb{Z}} |F^N(n) - F(n)|$ so that the sequence F^N converges uniformly on $\mathbb{Z}$ to F as $N \to \infty$. Let $\lambda \in [0, 1)$. It follows from (B.4),(B.5),(B.14) and for $M, N \in \mathbb{Z}_+$ that

$$a_M(F^N, \lambda) = \frac{1}{M+1} \sum_{n=0}^{M} \sum_{\substack{m\in\mathbb{Z}^d \\ ||m||\le N}} A^d_{N,m} f_m \exp(i2\pi n(m\cdot\chi)) \exp(-2\pi i n\lambda)$$

$$= \sum_{\substack{m\in\mathbb{Z}^d \\ ||m||\le N}} \frac{A^d_{N,m} f_m}{M+1} \sum_{n=0}^{M} E_{m\cdot\chi}(n) \exp(-2\pi i n\lambda) = \sum_{\substack{m\in\mathbb{Z}^d \\ ||m||\le N}} A^d_{N,m} f_m a_M(E_{m\cdot\chi}, \lambda) \; . \quad \text{(B.18)}$$

It follows from (B.7) and Definition B.2c that

$$\lim_{M\to\infty} a_M(E_{m\cdot\chi}, \lambda) = a(E_{m\cdot\chi}, \lambda) \; , \quad \text{(B.19)}$$

whence, by (B.18),

$$\lim_{M\to\infty} a_M(F^N, \lambda) = \sum_{\substack{m\in\mathbb{Z}^d \\ ||m||\le N}} A^d_{N,m} f_m a(E_{m\cdot\chi}, \lambda) \; , \quad \text{(B.20)}$$

which implies, by Definition B.2c, that $\Lambda_{tot}(F^N) = [0, 1)$ and that $a(F^N, \lambda)$ satisfies (B.15). It is also clear, by (B.14), that F^N is a finite sum of χ-quasiperiodic functions whence F^N is χ-quasiperiodic. To prove the last claim let $\mu \in \Lambda(F^N)$, i.e., $a(F^N, \mu) \neq 0$ whence, by (B.15), there exists an $m \in \mathbb{Z}^d$ such that $a(E_{m\cdot\chi}, \mu) \neq 0$ so that, by Definition B.2c, $\mu \in \Lambda(E_{m\cdot\chi})$. Thus, by (B.9), $\mu = \lfloor m \cdot \chi \rfloor$ whence $\mu = m \cdot \chi + k$ with $k \in \mathbb{Z}$ so that, by (B.6), $\mu \in Y_\chi$. We thus have shown that indeed $\Lambda(F^N) \subset Y_\chi$. $\square$

Proof of Theorem B.3d: Let λ be in $[0, 1)$. It follows from Theorem B.3c that a sequence of functions $F^N : \mathbb{Z} \to \mathbb{C}$ exists which converges uniformly on $\mathbb{Z}$ to F as $N \to \infty$ and such that $\Lambda_{tot}(F^N) = [0, 1)$, $\Lambda(F^N) \subset Y_\chi$. Thus and since $a(F^N, \lambda)$ and $a(F, \lambda)$ exist, we have

$$|a(F^N, \lambda) - a(F, \lambda)| = |a(F^N - F, \lambda)|$$

$$= |\lim_{M\to\infty} \frac{1}{M+1} \sum_{n=0}^{M} (F^N(n) - F(n)) \exp(-2\pi i\lambda n)| \le \sup_n |F^N(n) - F(n)| \; , \quad \text{(B.21)}$$

where we also used the fact that F^N and F are bounded functions. Since F^N converges uniformly on $\mathbb{Z}$ to F as $N \to \infty$ we have $0 = \lim_{N\to\infty} \sup_{n\in\mathbb{Z}} |F^N(n) - F(n)|$ whence, by (B.21),

$$\lim_{N\to\infty} a(F^N, \lambda) = a(F, \lambda) \; . \quad \text{(B.22)}$$

If $\mu \in [0,1) \setminus Y_\chi$, then, since $\Lambda(F^N) \subset Y_\chi$ and $\Lambda_{tot}(F^N) = [0,1)$, we have that $\mu \in \Lambda_{tot}(F^N) \setminus \Lambda(F^N)$. Thus $a(F^N, \mu) = 0$ and (B.22) gives $a(F, \mu) = 0$ whence $\mu \in [0,1) \setminus \Lambda(F)$. Thus $[0,1) \setminus Y_\chi \subset [0,1) \setminus \Lambda(F)$ whence $\Lambda(F) \subset Y_\chi$. □

We will show in Section B.4 that every quasiperiodic function $F : \mathbb{Z} \to \mathbb{C}$ has the property $\Lambda_{tot}(F) = [0,1)$ and we will apply this in Appendix B.5. We don't prove this property in the present section since we don't want to clutter the present section with the machinery of Section B.4. Of course the First Spectral Theorem is powerful enough in a situation when applied to a quasiperiodic functions F for which we know that $\Lambda_{tot}(F) = [0,1)$. In fact this is the case when $F = E_c$ and so we will use the First Spectral Theorem in the proof of the Exponential Theorem below.

Remark:

(2) Let $(\mathcal{P}[\omega], A) \in \mathcal{ACB}(d, \mathcal{P}[\omega])$ and $\nu \in \Xi(\mathcal{P}[\omega], A)$ (note, by Theorem 7.3b, that such a ν exists). Let also (Z, S) be a particle-spin-vector trajectory of $(\mathcal{P}[\omega], A)$ and let S_j denote the j-th component of S. We here show that S_j is (ω, ν)-quasiperiodic. First of all, since $\nu \in \Xi(\mathcal{P}[\omega], A)$ and by (7.7), a transfer field T from $(\mathcal{P}[\omega], A)$ to $(\mathcal{P}[\omega], A_{d,\nu})$ exists (note, by (7.21), that T is a uniform IFF of $(\mathcal{P}[\omega], A)$). Thus, and by Remark 0 in Chapter 4, T^t is a transfer field from $(\mathcal{P}[\omega], A_{d,\nu})$ to $(\mathcal{P}[\omega], A)$ so that, by (4.5),

$$\Psi[\mathcal{P}[\omega], A](n; Z(0)) = T\Big(\mathcal{P}[\omega]^n(Z(0))\Big)\Psi[\mathcal{P}[\omega], A_{d,\nu}](n; Z(0))T^t(Z(0)) \ . \quad \text{(B.23)}$$

We now pick a $\phi_0 \in \mathbb{R}^d$ such that $\pi_d(\phi_0) = Z(0)$ and we define the function $t : \mathbb{Z} \to SO(3)$ by

$$t(n) := T(Z(n)) = T(\mathcal{P}[\omega]^n(Z(0)) = T(\mathcal{P}[n\omega](\pi_d(\phi_0)) = T(\pi_d(\phi_0 + 2\pi n\omega)) \ , \quad \text{(B.24)}$$

where we used (2.25),(2.26), (2.32). It follows from (B.23),(B.24) that

$$\Psi[\mathcal{P}[\omega], A](n; Z(0)) = t(n)\Psi[\mathcal{P}[\omega], A_{d,\nu}](n; Z(0))t^t(0) \ , \quad \text{(B.25)}$$

whence, by (2.35),

$$S(n) = t(n)\Psi[\mathcal{P}[\omega], A_{d,\nu}](n; Z(0))t^t(0)S(0) \ . \quad \text{(B.26)}$$

We define the function $t_{jk} : \mathbb{Z} \to \mathbb{C}$ as the jk-matrix element of t and the function $T_{jk} \in \mathcal{C}(\mathbb{T}^d, \mathbb{C})$ as the jk-matrix element of T. Defining the function $\tilde{T}_{jk} \in \mathcal{C}(\mathbb{T}^d, \mathbb{C})$ by $\tilde{T}_{jk} := (T_{jk} \circ \pi_d)(\phi_0 + \cdot)$, i.e., $\tilde{T}_{jk}(\phi) = T_{jk}(\pi_d(\phi_0 + \phi)$, we obtain from (B.24) that $t_{jk}(n) = \tilde{T}_{jk}(2\pi n\omega)$ whence, by Definition B.1, $\tilde{T}_{jk}$ is an ω-generator of t_{jk} and so t_{jk} is ω-quasiperiodic. We define the function $\psi_{jk} : \mathbb{Z} \to \mathbb{C}$ as the jk-matrix element of $\Psi[\mathcal{P}[\omega], A_{d,\nu}](\cdot; Z(0))$ whence, by (7.5), $\psi_{jk}(n)$ is the jk-matrix element of $\exp(2\pi\mathcal{J}n\nu)$ so that ψ_{jk} is ν-quasiperiodic. Since ψ_{jk} is ν-quasiperiodic and t_{jk} is ω-quasiperiodic it follows from (B.26) and Remark 1 above that each S_j is (ω, ν)-quasiperiodic. We will return to this result in Remark 3 below. □

In the remarks after (B.9) we have seen that E_c is c–quasiperiodic. We now strengthen this result to part a) of the following theorem. The proof of part a) is a simple application of quasiperiodicity. In constrast, our proof of part b) needs the First Spectral Theorem, Theorem B.3d, i.e., involves the notion of Cesàro spectrum. As a matter of fact, part b) is the part which matters here since it will be used in the proof of Theorem 7.3f (see Appendix B.3 below).

Theorem B.4 *Let c be a real number and let $E_c : \mathbb{Z} \to \mathbb{C}$ be defined by (B.5), i.e., $E_c(n) := \exp(i2\pi nc)$. Let also $\chi \in \mathbb{R}^d$. Then the following hold.*
a) If $c \in Y_\chi$ then E_c is χ–quasiperiodic.
Remark: This confirms that E_c is c–quasiperiodic (choose the special case $c = \chi$!).
b) (Exponential Theorem) If E_c is χ–quasiperiodic then $c \in Y_\chi$.

Proof of Theorem B.4a: If $\chi \in \mathbb{R}^d$ and $c \in Y_\chi$ then, by (B.6), $m \in \mathbb{Z}^d, k \in \mathbb{Z}$ exist such that $c = m \cdot \chi + k$ whence, by (B.5), $E_c(n) = \exp(i2\pi nc) = \exp(i2\pi n(m \cdot \chi))$ so that, by Definition B.1, E_c is χ–quasiperiodic. □
Proof of Theorem B.4b: Let E_c be χ–quasiperiodic. Recalling from (B.7) that $\Lambda_{tot}(E_c) = [0, 1)$, we can apply Theorem B.3d and thus obtain $\Lambda(E_c) \subset Y_\chi$. It thus follows from (B.9) that $\{\lfloor c \rfloor\} \subset Y_\chi$, i.e., that $\lfloor c \rfloor \in Y_\chi$ whence (recall (B.6)) there exist $m \in \mathbb{Z}^d, n \in \mathbb{Z}$ such that $\lfloor c \rfloor = m \cdot \chi + n$. It follows that $c \in Y_\chi$. □

B.3 Proof of Theorem 7.3f

Let $(\mathcal{P}[\omega], A) \in \mathcal{SOS}(d, \mathcal{P}[\omega])$ and $\nu \in \Xi(\mathcal{P}[\omega], A)$. The claim of Theorem 7.3f is (7.9), i.e., $\Xi(\mathcal{P}[\omega], A) = [0, 1) \cap \Big\{\varepsilon\nu + m \cdot \omega + n : \varepsilon \in \{1, -1\}, m \in \mathbb{Z}^d, n \in \mathbb{Z}\Big\}$. We first define, for $\mu \in [0, 1)$,

$$\hat{\Xi}[\omega, \mu, \pm] := [0, 1) \cap \Big\{\pm\mu + m \cdot \omega + n : m \in \mathbb{Z}^d, n \in \mathbb{Z}\Big\} , \tag{B.27}$$

whence

$$\hat{\Xi}[\omega, \mu, +] \cup \hat{\Xi}[\omega, \mu, -] = [0, 1) \cap \Big\{\varepsilon\mu + m \cdot \omega + n : \varepsilon \in \{1, -1\}, m \in \mathbb{Z}^d, n \in \mathbb{Z}\Big\} , \tag{B.28}$$

so that our task of this section boils down to prove that

$$\Xi(\mathcal{P}[\omega], A) = \hat{\Xi}[\omega, \nu, +] \cup \hat{\Xi}[\omega, \nu, -] . \tag{B.29}$$

We will prove (B.29) by showing $\supset$ and $\subset$ separately. Our proof of $\supset$ is a simple application of (7.7) while our proof of $\subset$ needs the Exponential Theorem of Appendix B.3.

"$\supset$": We first show that $\Xi(\mathcal{P}[\omega], A) \supset \hat{\Xi}[\omega, \nu, +]$ so let $\mu \in \hat{\Xi}[\omega, \nu, +]$, i.e., by (B.27) $\mu = \nu + m \cdot \omega + n$ where $m \in \mathbb{Z}^d, n \in \mathbb{Z}$. Since $\nu \in \Xi(\mathcal{P}[\omega], A)$ it follows from (7.7) that a transfer field T from $(\mathcal{P}[\omega], A)$ to $(\mathcal{P}[\omega], A_{d,\nu})$ exists whence, by Definition 4.1,

$$(T^t \circ \mathcal{P}[\omega])AT = A_{d,\nu} . \tag{B.30}$$

We define $R_+ \in \mathcal{C}(\mathbb{T}^d, SO(3))$ by $R_+(z) := T(z)\exp(-\mathcal{J}(m\cdot\phi))$ where $z = \pi_d(\phi)$ whence, by (2.25),

$$R_+(\mathcal{P}[\omega](z)) = R_+(\pi_d(\phi+2\pi\omega)) = T(\pi_d(\phi+2\pi\omega))\exp(-\mathcal{J}[m\cdot(\phi+2\pi\omega)])$$
$$= T(\mathcal{P}[\omega](z))\exp(-\mathcal{J}[m\cdot\phi])\exp(-\mathcal{J}[2\pi(m\cdot\omega)]) \; ,$$

so that

$$R_+^t(\mathcal{P}[\omega](z)) = \exp(\mathcal{J}[2\pi(m\cdot\omega)])\exp(\mathcal{J}[m\cdot\phi])T^t(\mathcal{P}[\omega](z)) \; . \tag{B.31}$$

We now show that R_+ is a transfer field from $(\mathcal{P}[\omega], A)$ to $(\mathcal{P}[\omega], A_{d,\mu})$ whence we compute, by (7.4),(B.30),(B.31),

$$R_+^t(\mathcal{P}[\omega](z))A(z)R_+(z)$$
$$= \exp(\mathcal{J}[2\pi(m\cdot\omega)])\exp(\mathcal{J}[m\cdot\phi])T^t(\mathcal{P}[\omega](z))A(z)T(z)\exp(-\mathcal{J}(m\cdot\phi))$$
$$= \exp(\mathcal{J}[2\pi(m\cdot\omega)])\exp(\mathcal{J}[m\cdot\phi])A_{d,\nu}\exp(-\mathcal{J}(m\cdot\phi))$$
$$= \exp(\mathcal{J}[2\pi(m\cdot\omega)])\exp(\mathcal{J}[m\cdot\phi])\exp(\mathcal{J}2\pi\nu)\exp(-\mathcal{J}(m\cdot\phi))$$
$$= \exp(2\pi\mathcal{J}[\nu+m\cdot\omega]) = \exp(2\pi\mathcal{J}[\nu+m\cdot\omega+n]) = \exp(2\pi\mathcal{J}\mu) = A_{d,\mu} \; ,$$

so that, by Definition 4.1, R_+ is a transfer field from $(\mathcal{P}[\omega], A)$ to $(\mathcal{P}[\omega], A_{d,\mu})$ whence, by (7.7), $\mu \in \Xi(\mathcal{P}[\omega], A)$ which implies that indeed $\Xi(\mathcal{P}[\omega], A) \supset \hat{\Xi}[\omega,\nu,+]$. To show that $\Xi(\mathcal{P}[\omega], A) \supset \hat{\Xi}[\omega,\nu,-]$, let $\mu \in \hat{\Xi}[\omega,\nu,-]$, i.e., by (B.27), $\mu = -\nu + m\cdot\omega + n$ where $m \in \mathbb{Z}^d, n \in \mathbb{Z}$. We define $R_- \in \mathcal{C}(\mathbb{T}^d, SO(3))$ by $R_-(z) := T(z)\exp(\mathcal{J}(m\cdot\phi))\mathcal{J}'$ where $z = \pi_d(\phi)$ and

$$\mathcal{J}' := \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \; , \tag{B.32}$$

whence, by (6.6),(6.7),

$$\mathcal{J}'\mathcal{J}\mathcal{J}' = -\mathcal{J} \; , \quad \mathcal{J}'\exp(x\mathcal{J})\mathcal{J}' = \exp(x\mathcal{J}'\mathcal{J}\mathcal{J}') = \exp(-x\mathcal{J}) \; , \tag{B.33}$$

and, by (2.25),

$$R_-(\mathcal{P}[\omega](z)) = R_-(\pi_d(\phi+2\pi\omega)) = T(\pi_d(\phi+2\pi\omega))\exp(\mathcal{J}[m\cdot(\phi+2\pi\omega)])\mathcal{J}'$$
$$= T(\mathcal{P}[\omega](z))\exp(\mathcal{J}[m\cdot\phi])\exp(\mathcal{J}[2\pi(m\cdot\omega)])\mathcal{J}' \; ,$$

so that, by (B.32),

$$R_-^t(\mathcal{P}[\omega](z)) = \mathcal{J}'\exp(-\mathcal{J}[2\pi(m\cdot\omega)])\exp(-\mathcal{J}[m\cdot\phi])T^t(\mathcal{P}[\omega](z)) \; . \tag{B.34}$$

We now show that R_- is a transfer field from $(\mathcal{P}[\omega], A)$ to $(\mathcal{P}[\omega], A_{d,\mu})$ whence we compute, by (7.4),(B.30),(B.33), (B.34),

$$R_-^t(\mathcal{P}[\omega](z))A(z)R_-(z)$$
$$= \mathcal{J}'\exp(-\mathcal{J}[2\pi(m\cdot\omega)])\exp(-\mathcal{J}[m\cdot\phi])T^t(\mathcal{P}[\omega](z))A(z)T(z)\exp(\mathcal{J}(m\cdot\phi))\mathcal{J}'$$
$$= \mathcal{J}'\exp(-\mathcal{J}[2\pi(m\cdot\omega)])\exp(-\mathcal{J}[m\cdot\phi])A_{d,\nu}\exp(\mathcal{J}(m\cdot\phi))\mathcal{J}'$$
$$= \mathcal{J}'\exp(-\mathcal{J}[2\pi(m\cdot\omega)])\exp(-\mathcal{J}[m\cdot\phi])\exp(\mathcal{J}2\pi\nu)\exp(\mathcal{J}(m\cdot\phi))\mathcal{J}'$$
$$= \mathcal{J}'\exp(2\pi\mathcal{J}[\nu-m\cdot\omega])\mathcal{J}' = \exp(2\pi\mathcal{J}[-\nu+m\cdot\omega])$$
$$= \exp(2\pi\mathcal{J}[-\nu+m\cdot\omega+n]) = \exp(2\pi\mathcal{J}\mu) = A_{d,\mu} \; ,$$

so that, by Definition 4.1, R_- is a transfer field from $(\mathcal{P}[\omega], A)$ to $(\mathcal{P}[\omega], A_{d,\mu})$ whence, by (7.7), $\mu \in \Xi(\mathcal{P}[\omega], A)$ which implies that indeed $\Xi(\mathcal{P}[\omega], A) \supset \hat{\Xi}[\omega, \nu, -]$.

"$\subset$": Let $\tilde{\nu} \in \Xi(\mathcal{P}[\omega], A)$ so we have to show that either $\tilde{\nu} \in \hat{\Xi}[\omega, \nu, +]$ or $\tilde{\nu} \in \hat{\Xi}[\omega, \nu, -]$. Since $\nu, \tilde{\nu} \in \Xi(\mathcal{P}[\omega], A)$ it follows from (7.7) that transfer fields $T, \tilde{T}$ from $(\mathcal{P}[\omega], A)$ to $(\mathcal{P}[\omega], A_{d,\nu}), (\mathcal{P}[\omega], A_{d,\tilde{\nu}})$ exist whence, by Definition 4.1, we have (B.30) and

$$(\tilde{T}^t \circ \mathcal{P}[\omega]) A \tilde{T} = A_{d,\tilde{\nu}} \ . \tag{B.35}$$

We define the functions $t : \mathbb{Z} \to SO(3)$ and $\tilde{t} : \mathbb{Z} \to SO(3)$ by $t(n) := T(\pi_d(2\pi n\omega)) = (T \circ \pi_d)(2\pi n\omega)$ and $\tilde{t}(n) := \tilde{T}(\pi_d(2\pi n\omega)) = (\tilde{T} \circ \pi_d)(2\pi n\omega)$. We also define the functions $t_{jk} : \mathbb{Z} \to \mathbb{C}$ and $\tilde{t}_{jk} : \mathbb{Z} \to \mathbb{C}$ as the jk-matrix elements of t and $\tilde{t}$ whence, by Remark 2 above, t_{jk} and $\tilde{t}_{jk}$ are ω-quasiperiodic. We also define the two functions $g_\pm : \mathbb{Z} \to \mathbb{C}$ by

$$g_\pm(n) := \Big(t(n)(1, \pm i, 0)^t\Big) \cdot \Big(\tilde{t}(n)(1, i, 0)^t\Big) = \Big((1, \pm i, 0)^t\Big) \cdot \Big(t^t(n)\tilde{t}(n)(1, i, 0)^t\Big) \ . \tag{B.36}$$

Because the matrix elements of t and $\tilde{t}$ are ω-quasiperiodic it follows from (B.36) and Remark 1 above that g_+ and g_- are ω-quasiperiodic. We now show that the functions $g_\pm$ satisfy (B.45), i.e., $g_\pm(n) = \exp(i2\pi n(\pm\nu + \tilde{\nu}))g_\pm(0)$. Once we have shown (B.45) the claim easily follows. We first take a closer look at t. By (7.4),(B.30) we have

$$T(\mathcal{P}[\omega](z)) = A(z)T(z)A^t_{d,\nu} = A(z)T(z)\exp(-2\pi\nu\mathcal{J}) \ , \tag{B.37}$$

whence, by (2.25) and for $n \in \mathbb{Z}$,

$$\begin{aligned} t(n+1) &= T(\pi_d(2\pi(n+1)\omega)) = T(\mathcal{P}[\omega](\pi_d(2\pi n\omega))) \\ &= A(\pi_d(2\pi n\omega))T(\pi_d(2\pi n\omega))\exp(-2\pi\nu\mathcal{J}) = A(\pi_d(2\pi n\omega))t(n)\exp(-2\pi\nu\mathcal{J}) \ , \end{aligned}$$

so that

$$t(n+1)(1, \pm i, 0)^t = A(\pi_d(2\pi n\omega))t(n)\exp(-2\pi\nu\mathcal{J})(1, \pm i, 0)^t \ . \tag{B.38}$$

On the other hand, by (6.7),

$$\exp(-2\pi\nu\mathcal{J})(1, \pm i, 0)^t = \exp(\pm i2\pi\nu)(1, \pm i, 0)^t \ , \tag{B.39}$$

whence, by (B.38), $t(n+1)(1, \pm i, 0)^t = \exp(\pm i2\pi\nu)A(\pi_d(\phi_0 + 2\pi n\omega))t(n)(1, \pm i, 0)^t$ so that

$$\exp(\mp i2\pi\nu)\tilde{t}(n+1)(1, \pm i, 0)^t = A(\pi_d(\phi_0 + 2\pi n\omega))\tilde{t}(n)(1, \pm i, 0)^t \ . \tag{B.40}$$

We now take a closer look at $\tilde{t}$. By (7.4),(B.35) we have

$$\tilde{T}(\mathcal{P}[\omega](z)) = A(z)\tilde{T}(z)A^t_{d,\tilde{\nu}} = A(z)\tilde{T}(z)\exp(-2\pi\tilde{\nu}\mathcal{J}) \ , \tag{B.41}$$

whence, by (2.25) and for $n \in \mathbb{Z}$,

$$\begin{aligned} \tilde{t}(n+1) &= \tilde{T}(\pi_d(2\pi(n+1)\omega)) = \tilde{T}(\mathcal{P}[\omega](\pi_d(2\pi n\omega))) \\ &= A(\pi_d(2\pi n\omega))\tilde{T}(\pi_d(2\pi n\omega))\exp(-2\pi\tilde{\nu}\mathcal{J}) = A(\pi_d(2\pi n\omega))\tilde{t}(n)\exp(-2\pi\tilde{\nu}\mathcal{J}) \ , \end{aligned}$$

so that

$$\tilde{t}(n+1)(1,i,0)^t = A(\pi_d(2\pi n\omega))\tilde{t}(n)\exp(-2\pi\tilde{\nu}\mathcal{J})(1,i,0)^t \ . \tag{B.42}$$

On the other hand, by (6.7),

$$\exp(-2\pi\tilde{\nu}\mathcal{J})(1,i,0)^t = \exp(i2\pi\tilde{\nu})(1,i,0)^t \ , \tag{B.43}$$

whence, by (B.42), $\tilde{t}(n+1)(1,i,0)^t = \exp(i2\pi\tilde{\nu})A(\pi_d(\phi_0+2\pi n\omega))\tilde{t}(n)(1,i,0)^t$ so that

$$\exp(-i2\pi\tilde{\nu})\tilde{t}(n+1)(1,i,0)^t = A(\pi_d(\phi_0+2\pi n\omega))\tilde{t}(n)(1,i,0)^t \ . \tag{B.44}$$

We conclude from (B.36),(B.40),(B.44) that

$$\begin{aligned}
&\exp(\mp i2\pi\nu)\exp(-i2\pi\tilde{\nu})g_\pm(n+1) \\
&\quad = \exp(\mp i2\pi\nu)\exp(-i2\pi\tilde{\nu})\Big(t(n+1)(1,\pm i,0)^t\Big)\cdot\Big(\tilde{t}(n+1)(1,i,0)^t\Big) \\
&= \Big(A(\pi_d(\phi_0+2\pi n\omega))t(n)(1,\pm i,0)^t\Big)\cdot\Big(A(\pi_d(\phi_0+2\pi n\omega))\tilde{t}(n)(1,i,0)^t\Big) \\
&= \Big(t(n)(1,\pm i,0)^t\Big)\cdot\Big(A^t(\pi_d(\phi_0+2\pi n\omega))A(\pi_d(\phi_0+2\pi n\omega))\tilde{t}(n)(1,i,0)^t\Big) \\
&= \Big(t(n)(1,\pm i,0)^t\Big)\cdot\Big(\tilde{t}(n)(1,i,0)^t\Big) = g_\pm(n) \ ,
\end{aligned}$$

so that $g_\pm(n+1) = \exp(i2\pi(\pm\nu+\tilde{\nu}))g_\pm(n)$ which implies that indeed

$$g_\pm(n) = \exp(i2\pi n(\pm\nu+\tilde{\nu}))g_\pm(0) \ . \tag{B.45}$$

We will now finish the proof by showing, via (B.45), that $\tilde{\nu}\in\hat{\Xi}[\omega,\nu,-]$ if $g_+(0)\neq 0$ and that $\tilde{\nu}\in\hat{\Xi}[\omega,\nu,+]$ if $g_-(0)\neq 0$. We first have to show that either $g_+(0)\neq 0$ or $g_-(0)\neq 0$. In fact, by (B.36),

$$\begin{aligned}
\frac{1}{2}(g_+(n)+g_-(n)) &= \Big((1,0,0)^t\Big)\cdot\Big(t^t(n)\tilde{t}(n)(1,i,0)^t\Big) \ , \\
\frac{i}{2}(g_-(n)-g_+(n)) &= \Big((0,1,0)^t\Big)\cdot\Big(t^t(n)\tilde{t}(n)(1,i,0)^t\Big) \ ,
\end{aligned}$$

whence, and since the matrices $t^t(n)\tilde{t}(n)$ are real,

$$\Re e\{\frac{1}{2}(g_+(n)+g_-(n))\} = \Big((1,0,0)^t\Big)\cdot\Big(t^t(n)\tilde{t}(n)(1,0,0)^t\Big) \ , \tag{B.46}$$

$$\Im m\{\frac{1}{2}(g_+(n)+g_-(n))\} = \Big((1,0,0)^t\Big)\cdot\Big(t^t(n)\tilde{t}(n)(0,1,0)^t\Big) \ , \tag{B.47}$$

$$\Re e\{\frac{i}{2}(g_-(n)-g_+(n))\} = \Big((0,1,0)^t\Big)\cdot\Big(t^t(n)\tilde{t}(n)(1,0,0)^t\Big) \ , \tag{B.48}$$

$$\Im m\{\frac{i}{2}(g_-(n)-g_+(n))\} = \Big((0,1,0)^t\Big)\cdot\Big(t^t(n)\tilde{t}(n)(0,1,0)^t\Big) \ . \tag{B.49}$$

It follows from (B.46)-(B.49) that if $g_+(0) = g_-(0) = 0$ then the (11)-,(12)-,(21)-,(22)-matrix elements of $t^t(0)\tilde{t}(0)$ vanish whence $t^t(0)\tilde{t}(0)$ has zero determinant which contradicts the fact that $\det(t^t(0)\tilde{t}(0)) = 1$ (note that $t^t(0)\tilde{t}(0) \in SO(3)$!). We conclude that either $g_+(0) \neq 0$ or $g_-(0) \neq 0$.

We first consider the case when $g_+(0) \neq 0$. Then, by (B.45) and the remarks after (B.36), $g_+(n)/g_+(0) = \exp(i2\pi n(\nu + \tilde{\nu})) = E_{\nu+\tilde{\nu}}(n)$ is an ω-quasiperiodic function of n where in the second equality we used (B.5). Since this function is exponential we can apply the Exponential Theorem, Theorem B.4b, giving us that $(\nu + \tilde{\nu}) \in Y_\omega$ whence, by (B.6), $\nu+\tilde{\nu} = m\cdot\omega+k$ with $k \in \mathbb{Z}$ so that, by (B.29), $\tilde{\nu} \in \hat{\Xi}[\omega, \nu, -]$. We now consider the case when $g_-(0) \neq 0$. Then, by (B.45) and the remarks after (B.36), $g_-(n)/g_-(0) = \exp(i2\pi n(-\nu + \tilde{\nu})) = E_{-\nu+\tilde{\nu}}(n)$ is an ω-quasiperiodic function of n where in the second equality we used (B.5). Since this function is exponential we can apply once again the Exponential Theorem B.4b giving us that $(-\nu + \tilde{\nu}) \in Y_\omega$ whence, by (B.6), $-\nu + \tilde{\nu} = m \cdot \omega + k$ with $k \in \mathbb{Z}$ so that, by (B.29), $\tilde{\nu} \in \hat{\Xi}[\omega, \nu, +]$. This completes the proof that either $\tilde{\nu} \in \hat{\Xi}[\omega, \nu, +]$ or $\tilde{\nu} \in \hat{\Xi}[\omega, \nu, -]$. Thus the theorem is proven. Note, by (7.21), that $T, \tilde{T}, R_+, R_-$ are uniform IFF's of $(\mathcal{P}[\omega], A)$. $\square$

B.4 The Second, Third and Fourth Spectral Theorems

We here state and prove the Second, Third and Fourth Spectral Theorems which show for every quasiperiodic function F two facts: $\Lambda_{tot}(F) = [0, 1)$ and, in the case $\Lambda(F) = \emptyset$, $F = 0$. We will apply the first fact in Appendix B.5 and the second fact in Appendix B.6. These three theorems correspond to three different cases of χ. In fact the Second Spectral Theorem considers χ-quasiperiodic functions in the case where the components of χ are rational and the Third Spectral Theorem considers the case of nonresonant $(1, \chi)$ whereas the Fourth Spectral Theorem is concerned with the remaining case.

We call a function $F : \mathbb{Z} \to \mathbb{C}$ "p–periodic" if p is a positive integer and if, for every $n \in \mathbb{Z}$, $F(n + p) = F(n)$. Part b) of the following Theorem considers χ–quasiperiodic functions in the case where the components of χ are rational. Part a) is used to prove part b) and to prove Theorem B.7 below.

Theorem B.5 *a) Let p be a positive integer and let the function $F : \mathbb{Z} \to \mathbb{C}$ be p-periodic. Then $\Lambda_{tot}(K) = [0, 1)$. Furthermore, for $n \in \mathbb{Z}$,*

$$F(n) = \sum_{r=0}^{p-1} \exp(2\pi irn/p)a(F, r/p) \; . \tag{B.50}$$

b) (Second Spectral Theorem) Let $\chi \in \mathbb{Q}^d$ and let $F : \mathbb{Z} \to \mathbb{C}$ be χ–quasiperiodic. Let also p be a positive integer such that $p\chi$ belongs to $\mathbb{Z}^d$ (note that such a p exists). Then F is p–periodic and $\Lambda_{tot}(F) = [0, 1)$. If $\Lambda(F) = \emptyset$ then $F = 0$.

Proof of Theorem B.5a: Because F is p–periodic, $F(0), F(1), ..., F(q-1)$ are the only values of F. Let $P : \mathbb{C} \to \mathbb{C}$ be the Lagrange polynomial of F w.r.t. the points $\exp(2\pi in/p)$ in $\mathbb{C}$ [wiki5]. Thus

$$F(n) = P(\exp(2\pi in/p)) \; . \tag{B.51}$$

Because F is p–periodic, we compute, for $n = 0, ..., p-1$ and by (B.51), $F(n+p) = F(n) = P(\exp(2\pi in/p)) = P(\exp(2\pi i(n+p)/p))$ whence (B.51) holds for *all* integers n. Because P is a Lagrange polynomial of F w.r.t. p points in $\mathbb{C}$, it has degree $p-1$ whence $P(z) = \sum_{r=0}^{p-1} P_r z^r$ where $P_r \in \mathbb{C}$ so that, by (B.51), $F(n) = \sum_{r=0}^{p-1} P_r \exp(2\pi irn/p)$ so that, by (B.5),

$$F = \sum_{r=0}^{p-1} P_r E_{r/p} \, . \tag{B.52}$$

Thus F is a finite sum of functions of the form E_c so that, by Theorem B.3a, $\Lambda_{tot}(F) = [0,1)$ and, for every $\lambda \in [0,1)$ and by (B.52),

$$a(F,\lambda) = \sum_{r=0}^{p-1} P_r a(E_{r/p},\lambda) \, . \tag{B.53}$$

Moreover, for $r = 0, ..., p-1$ and by Theorem B.3a, $a(E_{r/p},\lambda) = \begin{cases} 1 & \text{if } \lambda = r/p \\ 0 & \text{if } \lambda \neq r/p \, , \end{cases}$ whence, by (B.53) and for $j = 0, ..., p-1$,

$$a(F,j/p) = \sum_{r=0}^{p-1} P_r a(E_{r/p},j/p) = P_j \, , \tag{B.54}$$

so that, by (B.52), $F = \sum_{r=0}^{p-1} a(K,r/p)E_{r/p}$ which implies (B.50) by using (B.5). □

Proof of Theorem B.5b: Let f be a χ-generator of the χ-quasiperiodic function F whence we compute, by Definition B.1 and using that f is 2π-periodic in its arguments, $F(n+p) = f(2\pi(n+p)\chi) = f(2\pi n\chi) = F(n)$ so that F is p–periodic. It thus follows from Theorem B.5a that $\Lambda_{tot}(F) = [0,1)$. If $\Lambda(F) = \emptyset$ then, by Definition B.2c and for $\lambda \in [0,1)$, we get $a(F,\lambda) = 0$ whence, by (B.50), $F = 0$. □

In part b) of the following theorem, we consider χ–quasiperiodic functions in the case where $(1,\chi)$ is nonresonant. Part a) is used to prove part b) and to prove Theorem B.7 below.

Theorem B.6 *a) For $N \in \mathbb{Z}_+$ let $G^N : \mathbb{Z} \to \mathbb{C}$ be bounded, i.e., let each $\sup_{n\in\mathbb{Z}} |G^N(n)|$ be finite and let the sequence G^N converge uniformly on $\mathbb{Z}$ as $N \to \infty$ to a function $G : \mathbb{Z} \to \mathbb{C}$. Let, for every $N \in \mathbb{Z}_+$, $\lambda \in \Lambda_{tot}(G^N)$ and $a(G^N,\lambda) = 0$. Then G is a bounded function which satisfies $\lambda \in \Lambda_{tot}(G)$ and $a(G,\lambda) = 0$.*
b) (Third Spectral Theorem) Let $\chi \in \mathbb{R}^d$ such that $(1,\chi)$ is nonresonant and let $F : \mathbb{Z} \to \mathbb{C}$ be χ–quasiperiodic and let f be any χ-generator of F. Then $\Lambda_{tot}(F) = [0,1)$ and, for $m \in \mathbb{Z}^d$, $f_m = a(F,\lfloor m\cdot\chi \rfloor)$. Moreover the sequence F^N converges uniformly on $\mathbb{Z}$ to F as $N \to \infty$ where, for $N \in \mathbb{Z}_+$, the function $F^N : \mathbb{Z} \to \mathbb{C}$ is defined by

$$F^N := \sum_{\substack{m\in\mathbb{Z}^d \\ ||m||\leq N}} A^d_{N,m} a(F,\lfloor m\cdot\chi \rfloor) E_{m\cdot\chi} \, . \tag{B.55}$$

If $\Lambda(F) = \emptyset$ then $F = 0$.

Proof of Theorem B.6a: Because the G^N are bounded, we recall from Appendix A.4 that G is bounded. Moreover $a(G^N,\lambda)=0$ whence, by Definition B.2c, $a_M(G^N,\lambda)\to 0$ as $M\to\infty$, i.e.,

$$(\forall\varepsilon>0)(\forall N\in\mathbb{Z}_+)(\exists M[\varepsilon,N]\in\mathbb{Z}_+)(\forall M\in\mathbb{Z}_+)\Big((M\ge M[\varepsilon,N])\Rightarrow |a_M(G^N,\lambda)|<\varepsilon/2\Big)\,. \tag{B.56}$$

To show that $\lambda\in\Lambda_{tot}(G)$ and that $a(G,\lambda)=0$ we use ideas from [MS, p.259]. Because G and G^N are bounded functions and by defining, for $N\in\mathbb{Z}_+$, $\delta^N:=\sup_{n\in\mathbb{Z}}|G(n)-G^N(n)|$ we note that δ^N is finite (and nonnegative). Moreover because $G-G^N$ converges uniformly to zero, $\lim_{N\to\infty}\delta^N=0$, hence

$$(\forall\varepsilon>0)(\exists N[\varepsilon]\in\mathbb{Z}_+)\Big(\delta^{N[\varepsilon]}<\varepsilon/2\Big)\,. \tag{B.57}$$

By Definition B.2c and for all $M,N\in\mathbb{Z}_+$, we compute

$$\begin{aligned}|a_M(G,\lambda)-a_M(G^N,\lambda)|&=|a_M(G-G^N,\lambda)|=\frac{1}{M+1}|\sum_{n=0}^{M}(G(n)-G^N(n))\exp(-2\pi in\lambda)|\\ &\le\frac{1}{M+1}\sum_{n=0}^{M}|G(n)-G^N(n)|\le\frac{1}{M+1}\sum_{n=0}^{M}\sup_{l\in\mathbb{Z}}|G(l)-G^N(l)|=\sup_{l\in\mathbb{Z}}|G(l)-G^N(l)|\\ &=\delta^N\,.\end{aligned} \tag{B.58}$$

Combining (B.57),(B.58) yields, for all $M\in\mathbb{Z}_+$, to

$$(\forall\varepsilon>0)\ \Big(|a_M(G,\lambda)-a_M(G^{N[\varepsilon]},\lambda)|\le\delta^{N[\varepsilon]}<\varepsilon/2\Big)\,. \tag{B.59}$$

From (B.56) and (B.59) it follows that

$$\begin{aligned}(\forall\varepsilon>0)(\forall M\in\mathbb{Z}_+)\Big((M\ge M[\varepsilon,N[\varepsilon]])\Rightarrow\Big(\Big(|a_M(G^{N[\varepsilon]},\lambda)|<\varepsilon/2\Big)\\ \&\Big(|a_M(G,\lambda)-a_M(G^{N(\varepsilon)},\lambda)|<\varepsilon/2\Big)\Big)\Big)\,.\end{aligned} \tag{B.60}$$

Because $|a_M(G,\lambda)|=|a_M(G,\lambda)-a_M(G^{N[\varepsilon]},\lambda)+a_M(G^{N[\varepsilon]},\lambda)|\le|a_M(G,\lambda)-a_M(G^{N[\varepsilon]},\lambda)|+|a_M(G^{N[\varepsilon]},\lambda)|$ it follows from (B.60) that $(\forall\varepsilon>0)(\forall M\ge M[\varepsilon,N[\varepsilon]])|a_M(G,\lambda)|<\varepsilon$. Thus $a_M(G,\lambda)$ converges to zero as $M\to\infty$, i.e. $\lambda\in\Lambda_{tot}(G)$ and $a(G,\lambda)=0$. $\square$

Proof of Theorem B.6b: By using a multivariate 'map' version of Weyl's equidistribution theorem ([CFS, Chapter 3]), we obtain, for $m\in\mathbb{Z}^d$, that $\lfloor m\cdot\chi\rfloor\in\Lambda_{tot}(F)$ and that $f_m=a(F,\lfloor m\cdot\chi\rfloor)$. Note, for every $\lambda\in([0,1)\cap Y_\chi)$ and by (B.6), that $\lambda=\tilde m\cdot\chi+\tilde n$ where $\tilde m\in\mathbb{Z}^d,\tilde n\in\mathbb{Z}$ so that $\lambda=\lfloor\lambda\rfloor=\lfloor\tilde m\cdot\chi+\tilde n\rfloor=\lfloor\tilde m\cdot\chi\rfloor$ whence $([0,1)\cap Y_\chi)\subset\Lambda_{tot}(F)$. Thus to prove that $\Lambda_{tot}(F)=[0,1)$ it remains to be shown that $([0,1)\setminus Y_\chi)\subset\Lambda_{tot}(F)$ so let from now on $\lambda \in ([0,1) \setminus Y_\chi)$. We thus define, as in Theorem B.3c and for $N \in \mathbb{Z}_+$, the function $F^N : \mathbb{Z} \to \mathbb{C}$ by (B.14), i.e.,

$$F^N := \sum_{\substack{m\in\mathbb{Z}^d \\ ||m||\le N}} A^d_{N,m} f_m E_{m\cdot\chi} = \sum_{\substack{m\in\mathbb{Z}^d \\ ||m||\le N}} A^d_{N,m} a(F, \lfloor m\cdot\chi\rfloor) E_{m\cdot\chi} \ , \tag{B.61}$$

whence, by Theorem B.3c, the sequence F^N converges uniformly on $\mathbb{Z}$ to F as $N \to \infty$. Also, by Theorem B.3c, F^N is χ-quasiperiodic and

$$\Lambda_{tot}(F^N) = [0,1) \ , \quad \Lambda(F^N) \subset Y_\chi \ , \tag{B.62}$$

as well as (B.15), i.e., $a(F^N,\lambda) = \sum_{\substack{m\in\mathbb{Z}^d \\ ||m||\le N}} A^d_{N,m} f_m a(E_{m\cdot\chi},\lambda)$. By (B.61) F^N is bounded since it is a finite sum of bounded functions. By (B.62) and Definition B.2c, $a(F^N,\lambda) = 0$. Thus the F^N form a sequence of complex valued and bounded functions for which $\lambda \in \Lambda_{tot}(F^N)$ and $a(F^N,\lambda) = 0$ and which converges uniformly on $\mathbb{Z}$ to F as $N \to \infty$. Thus, by Theorem B.6a, the limit F is a bounded function such that $\lambda \in \Lambda_{tot}(F)$ and $a(F,\lambda) = 0$. Finally we consider the case where $\Lambda(F) = \emptyset$ whence, by Definition B.2c and for $\mu \in [0,1)$, we get $a(F,\mu) = 0$ so that, by (B.55) and for $N \in \mathbb{Z}_+$, $F^N = 0$ which implies, by the convergence of F^N to F, that $F = 0$. $\square$

In part d) of the following theorem, we consider χ–quasiperiodic functions in the remaining case, i.e., the case where neither $(1,\chi)$ is nonresonant nor all components of χ are rational. Parts a)-c) are used to prove part d). We denote by $GL(d,\mathbb{Z})$ the set of those nonsingular matrices in $\mathbb{Z}^{d\times d}$ whose inverse belongs to $\mathbb{Z}^{d\times d}$. Thus $GL(d,\mathbb{Z})$ is the set of those matrices Z in $\mathbb{Z}^{d\times d}$ for which $|\det(Z)| = 1$. We also define, for $\chi \in \mathbb{R}^d$, the set

$$M_\chi := \{m \in \mathbb{Z}^d : m\cdot\chi \in \mathbb{Z}\} \ . \tag{B.63}$$

Clearly $(1,\chi)$ is nonresonant iff $M_\chi = \{0\}$ as can be easily checked. Thus M_χ quantifies how resonant $(1,\chi)$ is.

Theorem B.7 *a) Let F be a χ-quasiperiodic function where $\chi = (\tilde\chi,\hat\chi)$ with $\tilde\chi \in \mathbb{Q}^{d-s}, \hat\chi \in \mathbb{R}^s$ and $0 < s < d$ and where $(1,\hat\chi)$ is nonresonant. Then $\Lambda_{tot}(F) = [0,1)$. If $\Lambda(F) = \emptyset$ then $F = 0$.*
b) Let $\chi \in \mathbb{R}^d$ and $Z \in GL(d,\mathbb{Z})$ and let $F : \mathbb{Z} \to \mathbb{C}$. If F is χ–quasiperiodic then it is also $(Z\chi)$–quasiperiodic.
c) Let χ be in $\mathbb{R}^d \setminus \mathbb{Q}^d$ and let $(1,\chi)$ be resonant. Then a $Z \in GL(d,\mathbb{Z})$ exists such that $Z\chi = (\tilde\chi,\hat\chi)$ with $(1,\hat\chi)$ nonresonant and $\tilde\chi \in \mathbb{Q}^{d-s}$, where $0 < s < d$.
d) (Fourth Spectral Theorem) Let F be a χ–quasiperiodic function and let $(1,\chi)$ be resonant and $\chi \in \mathbb{R}^d \setminus \mathbb{Q}^d$. Let $Z \in GL(d,\mathbb{Z})$ such that $Z\chi = (\tilde\chi,\hat\chi)$ with $(1,\hat\chi)$ nonresonant and $\tilde\chi \in \mathbb{Q}^{d-s}$, where $0 < s < d$ (note, by Theorem B.7c, that such a Z exists). Then F is $(\tilde\chi,\hat\chi)$-quasiperiodic and $\Lambda_{tot}(F) = [0,1)$. If $\Lambda(F) = \emptyset$ then $F = 0$.

Proof of Theorem B.7a: Our proof has three parts. In the first part we use Theorem B.6b to approximate F by a uniformly convergent sequence F^N, in the second part we use this sequence to prove the claim about $\Lambda_{tot}(F)$ by using Theorem B.6a and in the third part we consider the case where $\Lambda(F) = \emptyset$. Let the positive integer p be chosen such that $p\tilde\chi$ is in

$\mathbb{Z}^{d-s}$. The fact that the components of $\tilde{\chi}$ are rational allows us to express the function F in terms of the functions $F_r : \mathbb{Z} \to \mathbb{C}$, defined by $F_r(n) := F(pn + r)$, where $r = 1, ..., p-1$. To express F in terms of the F_r we define the functions $\kappa : \mathbb{Z} \to \mathbb{Z}_+$ and $\kappa' : \mathbb{Z} \to \mathbb{Z}$ by

$$\kappa(n) := p\lfloor n/p \rfloor \ , \qquad \kappa'(n) := (n - \kappa(n))/p \ . \tag{B.64}$$

Note that $\kappa(n+p) = p\lfloor (n+p)/p \rfloor = p\lfloor n/p + 1 \rfloor = p\lfloor n/p \rfloor = \kappa(n)$ whence κ is p–periodic. Also each $\kappa(n)$ is an integer between 0 and $p-1$ whence we compute, for $n \in \mathbb{Z}$ and by (B.64),

$$F(n) = F(n - p\lfloor n/p \rfloor + p\lfloor n/p \rfloor) = F(p\kappa'(n) + \kappa(n)) = F_{\kappa(n)}(\kappa'(n)) \ . \tag{B.65}$$

To apply Theorem B.6b we will use the fact that $(1, \hat{\chi})$ is nonresonant. We first have to establish the quasiperiodicity of the F_r so let $f \in \mathcal{C}(\mathbb{R}^d, \mathbb{C})$ be a χ-generator of F, i.e., $F(n) = f(2\pi n\chi)$. Then, by the periodicity of f,

$$\begin{aligned} F_r(n) &= F(pn+r) = f(2\pi\chi(pn+r)) = f(2\pi(pn+r)(\tilde{\chi}, \hat{\chi})) = f(2\pi r\tilde{\chi}, 2\pi\hat{\chi}(pn+r)) \\ &= g_r(2\pi np\hat{\chi}) \ , \end{aligned} \tag{B.66}$$

where $g_r \in \mathcal{C}(\mathbb{R}^s, \mathbb{C})$ is defined by $g_r(\phi) := f(2\pi r\tilde{\chi}, \phi + 2\pi r\hat{\chi})$. Clearly g_r is 2π–periodic in its arguments whence, by (B.66), g_r is a $p\hat{\chi}$-generator of F_r so that the latter is $p\hat{\chi}$–quasiperiodic. Because $(1, \hat{\chi})$ is nonresonant and the integer p is nonzero, $(1, p\hat{\chi})$ is nonresonant whence we can apply Theorem B.6b to g_r so that the sequence F_r^N, defined by

$$F_r^N(n) := \sum_{\substack{m \in \mathbb{Z}^s \\ ||m|| \le N}} A^s_{N,m} a(F_r, \lfloor p(m \cdot \hat{\chi}) \rfloor) \exp(2\pi i np(m \cdot \hat{\chi})) \ , \tag{B.67}$$

converges uniformly to F_r on $\mathbb{Z}$ as $N \to \infty$. Having got the approximating sequence F_r^N for F_r, (B.65) suggests to approximate F by F^N where, for $N \in \mathbb{Z}_+$, the function $F^N : \mathbb{Z} \to \mathbb{C}$ is defined by

$$\begin{aligned} F^N(n) := F^N_{\kappa(n)}(\kappa'(n)) &= \sum_{\substack{m \in \mathbb{Z}^s \\ ||m|| \le N}} A^s_{N,m} a(F_{\kappa(n)}, \lfloor p(m \cdot \hat{\chi}) \rfloor) \exp(2\pi i \kappa'(n) p(m \cdot \hat{\chi})) \\ &= \sum_{\substack{m \in \mathbb{Z}^s \\ ||m|| \le N}} A^s_{N,m} a(F_{\kappa(n)}, \lfloor p(m \cdot \hat{\chi}) \rfloor) \exp(2\pi i (n - \kappa(n))(m \cdot \hat{\chi})) \ , \end{aligned} \tag{B.68}$$

where in the second equality we used (B.67) and where in the third equality we used (B.64). Because F_r^N converges to F_r it follows that $F^N_{\kappa(n)}(\kappa'(n))$ converges to $F_{\kappa(n)}(\kappa'(n))$ whence, by (B.65), $F^N_{\kappa(n)}(\kappa'(n))$ converges to $F(n)$ so that, by (B.68), F^N converges to F as $N \to \infty$. Note that F^N is bounded since it is a finite sum of bounded functions. To show the uniform convergence of F^N we recall from above that F_r^N converges uniformly to F_r on $\mathbb{Z}$, i.e., for $r = 0, ..., p-1$

$$(\forall \varepsilon > 0)(\exists M[\varepsilon, r] \in \mathbb{Z}_+)(\forall N \in \mathbb{Z}_+)\Big((N \ge M[\varepsilon, r]) \Rightarrow \Big(\sup_{n \in \mathbb{Z}} |F_r^N(n) - F_r(n)| < \varepsilon/p\Big)\Big) \ . \tag{B.69}$$

It trivially follows from (B.69) that, for $r = 0, ..., p-1$,

$$(\forall \varepsilon > 0)(\exists M[\varepsilon, r] \in \mathbb{Z}_+)(\forall N \in \mathbb{Z}_+)\left((N \geq M[\varepsilon, r]) \Rightarrow \left(\sup_{n\in\mathbb{Z}} |F_r^N(\kappa'(n)) - F_r(\kappa'(n))| < \varepsilon/p \right) \right) ,$$

whence, for $r = 0, ..., p-1$,

$$(\forall \varepsilon > 0)(\exists M[\varepsilon] \in \mathbb{Z}_+)(\forall N \in \mathbb{Z}_+)\left((N \geq M[\varepsilon]) \Rightarrow \left(\sup_{n\in\mathbb{Z}} |F_r^N(\kappa'(n)) - F_r(\kappa'(n))| < \varepsilon/p \right) \right) .$$

so that

$$(\forall \varepsilon > 0)(\exists M[\varepsilon] \in \mathbb{Z}_+)(\forall N \in \mathbb{Z}_+)\left((N \geq M[\varepsilon]) \Rightarrow \left(\sum_{r=0}^{p-1} \sup_{n\in\mathbb{Z}} |F_r^N(\kappa'(n)) - F_r(\kappa'(n))| < \varepsilon \right) \right) . \tag{B.70}$$

We also compute, for all $n \in \mathbb{Z}, N \in \mathbb{Z}_+$, $|F_{\kappa(n)}^N(\kappa'(n)) - F_{\kappa(n)}(\kappa'(n))| \leq \sum_{r=0}^{p-1} |F_r^N(\kappa'(n)) - F_r(\kappa'(n))|$ whence, by (B.65),(B.68),

$$\begin{aligned} \sup_{n\in\mathbb{Z}} |F^N(n) - F(n)| &= \sup_{n\in\mathbb{Z}} |F_{\kappa(n)}^N(\kappa'(n)) - F_{\kappa(n)}(\kappa'(n))| \\ &\leq \sup_{n\in\mathbb{Z}} \sum_{r=0}^{p-1} |F_r^N(\kappa'(n)) - F_r(\kappa'(n))| \leq \sum_{r=0}^{p-1} \sup_{n\in\mathbb{Z}} |F_r^N(\kappa'(n)) - F_r(\kappa'(n))| , \end{aligned} \tag{B.71}$$

so that (B.70) yields to

$$(\forall \varepsilon > 0)(\exists M[\varepsilon] \in \mathbb{Z}_+)(\forall N \in \mathbb{Z}_+)\left((N \geq M[\varepsilon]) \Rightarrow \left(\sup_{n\in\mathbb{Z}} |F^N(n) - F(n)| < \varepsilon \right) \right) ,$$

i.e., F^N converges uniformly to F. With the uniform convergence of F^N we have completed the first part in our proof.

In the second part we now apply Theorem B.6a and we first compute $a(F^N, \lambda)$. To simplify the expression (B.68) for F^N we define, for $m \in \mathbb{Z}^s$, the function $\alpha_m : \mathbb{Z} \to \mathbb{C}$ by $\alpha_m(n) := a(F_{\kappa(n)}, \lfloor p(m \cdot \hat{\chi}) \rfloor) \exp(-2\pi i \kappa(n)(m \cdot \hat{\chi}))$ and obtain from (B.68) that

$$F^N(n) = \sum_{\substack{m\in\mathbb{Z}^s \\ ||m||\leq N}} A_{N,m}^s \alpha_m(n) \exp(2\pi i n(m \cdot \hat{\chi})) . \tag{B.72}$$

Note that α_m is p–periodic because χ is p–periodic whence, by Theorem B.5a,

$$\alpha_m(n) = \sum_{r=0}^{p-1} \exp(2\pi i r n/p) a(\alpha_m, r/p) . \tag{B.73}$$

It follows from (B.72),(B.73) that
$F^N(n) = \sum_{\substack{m\in\mathbb{Z}^s \\ ||m||\leq N}} A_{N,m}^s \sum_{r=0}^{p-1} a(\alpha_m, r/p) \exp(2\pi i r n/p) \exp(2\pi i n(m \cdot \hat{\chi}))$, i.e., by (B.5),

$$F^N = \sum_{\substack{m\in\mathbb{Z}^s \\ ||m||\leq N}} A_{N,m}^s \sum_{r=0}^{p-1} a(\alpha_m, r/p) E_{r/p+m\cdot\hat{\chi}} . \tag{B.74}$$

To compute $a(F^N,\lambda)$ we note, by (B.74), that if $a(F^N,\lambda)\neq 0$ then there exist $r=0,...,p-1$ and $m\in\mathbb{Z}^s$ such that $a(E_{r/p+m\cdot\hat\chi},\lambda)\neq 0$ whence, by (B.8), $\lambda=\lfloor r/p+m\cdot\hat\chi\rfloor$ so that, by (B.6), $\lambda\in Y_{(1/p,\hat\chi)}$. Thus if $\lambda\in([0,1)\setminus Y_{(1/p,\hat\chi)})$ then $a(F^N,\lambda)=0$. To complete the computation of $a(F^N,\lambda)$ let us now consider the case where $\lambda\in Y_{(1/p,\hat\chi)}$ whence, by (B.6) and since p is an integer, $\lambda=k_1\cdot\hat\chi+k_0/p$, where $k_0\in\mathbb{Z},k_1\in\mathbb{Z}^s$. Of course, by (B.74),

$$a(F^N,\lambda)=\sum_{\substack{m\in\mathbb{Z}^s\\ ||m||\leq N}} A^s_{N,m}\sum_{r=0}^{p-1} a(\alpha_m,r/p)a(E_{r/p+m\cdot\hat\chi},\lambda)\ . \tag{B.75}$$

Setting $F\equiv E_{r/p+m\cdot\hat\chi}$ and $c\equiv -k_1\cdot\hat\chi-k_0/p$ in Theorem B.3a we note that $\Lambda_{tot}(E_{r/p+m\cdot\hat\chi})=[0,1)$ whence, by Theorem B.3a, $a(E_{r/p+m\cdot\hat\chi},\lfloor -c\rfloor)=a(E_{r/p+m\cdot\hat\chi}E_c,0)$ so that we compute

$$\begin{aligned}a(E_{r/p+m\cdot\hat\chi},\lambda)&=a(E_{r/p+m\cdot\hat\chi},\lfloor\lambda\rfloor)=a(E_{r/p+m\cdot\hat\chi},\lfloor k_1\cdot\hat\chi+k_0/p\rfloor)\\&=a(E_{r/p+m\cdot\hat\chi}E_{-k_1\cdot\hat\chi-k_0/p},0)=a(E_{(r-k_0)/p+(m-k_1)\cdot\hat\chi},0)\ ,\end{aligned}$$

which implies, by (B.75),

$$a(F^N,\lambda)=\sum_{\substack{m\in\mathbb{Z}^s\\ ||m||\leq N}} A^s_{N,m}\sum_{r=0}^{p-1} a(\alpha_m,r/p)a(E_{(r-k_0)/p+(m-k_1)\cdot\hat\chi},0)\ . \tag{B.76}$$

To simplify the rhs of (B.76) we note, by Theorem B.3a, that if $r=0,...,p-1,m\in\mathbb{Z}^s$ and $a(E_{(r-k_0)/p+(m-k_1)\cdot\hat\chi},0)\neq 0$ then $\lfloor (r-k_0)/p+(m-k_1)\cdot\hat\chi\rfloor=0$ whence an integer k_2 exists such that $(r-k_0)/p+(m-k_1)\cdot\hat\chi=k_2$ so that

$$r-k_0-pk_2+p(m-k_1)\cdot\hat\chi=0\ . \tag{B.77}$$

Recalling from above that $(1,p\hat\chi)$ is nonresonant it follows from (B.77) that

$$m-k_1=0\ ,\qquad r-k_0-pk_2=0\ . \tag{B.78}$$

Using the second equality in (B.78) and since $r=0,...,p-1$ we compute, by (B.64), $\kappa(k_0)=p\lfloor k_0/p\rfloor=p\lfloor (r-pk_2)/p\rfloor=p\lfloor r/p-k_2\rfloor=p\lfloor r/p\rfloor=p(r/p)=r$. We thus have shown that if $r=0,...,p-1,m\in\mathbb{Z}^s$ and $a(E_{(r-k_0)/p+(m-k_1)\cdot\hat\chi},0)\neq 0$ then $m=k_1$ and $r=\kappa(k_0)$ whence (B.76) simplifies, by Theorem B.3a and (B.64) so that, for $N<||k_1||$, $a(F^N,\lambda)=0$ and, for $N\geq ||k_1||$,

$$\begin{aligned}a(F^N,\lambda)&=A^s_{N,k_1}a(\alpha_{k_1},\kappa(k_0)/p)a(E_{(\kappa(k_0)-k_0)/p},0)\\&=A^s_{N,k_1}a(\alpha_{k_1},\kappa(k_0)/p)a(E_{-\kappa'(k_0)},0)=A^s_{N,k_1}a(\alpha_{k_1},\kappa(k_0)/p)a(E_0,0)\\&=A^s_{N,k_1}a(\alpha_{k_1},\kappa(k_0)/p)\ ,\end{aligned} \tag{B.79}$$

where in the third equality we used that $\kappa'(k_0)\in\mathbb{Z}$. With (B.79) we can summarize the computation of $a(F^N,\lambda)$:

$$a(F^N,\lambda)=\begin{cases}0 & \text{if } \lambda\in([0,1)\setminus Y_{(1/p,\hat\chi)})\\ 0 & \text{if } \lambda=k_1\cdot\hat\chi+k_0/p \text{ and } k_0\in\mathbb{Z},k_1\in\mathbb{Z}^s,N<||k_1||\\ A^s_{N,k_1}a(\alpha_{k_1},\kappa(k_0)/p) & \text{if } \lambda=k_1\cdot\hat\chi+k_0/p \text{ and } k_0\in\mathbb{Z},k_1\in\mathbb{Z}^s,N\geq ||k_1||\ .\end{cases} \tag{B.80}$$

We can now finish the second part by applying Theorem B.6a and we first note, by (B.2), that A^s_{N,k_1} converges to 1 as $N \to \infty$ whence, by (B.80), $a(F^N, \lambda)$ converges as $N \to \infty$ and

$$\lim_{N\to\infty} a(F^N,\lambda) = \begin{cases} 0 & \text{if } \lambda \in ([0,1) \setminus Y_{(1/p,\hat{\chi})}) \\ a(\alpha_{k_1}, \kappa(k_0)/p) & \text{if } \lambda = k_1 \cdot \hat{\chi} + k_0/p \text{ and } k_0 \in \mathbb{Z}, k_1 \in \mathbb{Z}^s \end{cases} . \tag{B.81}$$

To apply Theorem B.6a we define the function $G : \mathbb{Z} \to \mathbb{C}$ by $G := F - E_\lambda \lim_{N\to\infty} a(F^N, \lambda)$ and, for $N \in \mathbb{Z}_+$, the function $G^N : \mathbb{Z} \to \mathbb{C}$ by $G^N := F^N - E_\lambda a(F^N, \lambda)$. Note that F^N is bounded since it is a finite sum of bounded functions whence G^N is bounded. Since $\Lambda_{tot}(F^N) = [0,1)$ we also have $\Lambda_{tot}(G^N) = [0,1)$ and we compute, by Theorem B.3a,

$$\begin{aligned} a(G^N,\lambda) &= a(F^N - E_\lambda a(F^N,\lambda),\lambda) = a(F^N,\lambda) - a(E_\lambda a(F^N,\lambda),\lambda) \\ &= a(F^N,\lambda) - a(F^N,\lambda)a(E_\lambda,\lambda) = a(F^N,\lambda) - a(F^N,\lambda) = 0 . \end{aligned} \tag{B.82}$$

To show that G^N converges uniformly to G we first consider the case where $\lambda \in ([0,1) \setminus Y_{(1/p,\hat{\chi})})$. In this case, and by (B.81), $G = F - E_\lambda \lim_{N\to\infty} a(F^N,\lambda) = F$ and, by (B.80), $G^N = F^N - E_\lambda a(F^N,\lambda) = F^N$ whence, and since F^N converges uniformly to F, G^N converges uniformly to G on $\mathbb{Z}$ as $N \to \infty$. We now consider the case where $\lambda \in Y_{(1/p,\hat{\chi})}$. In this case, and by (B.81), $G = F - E_\lambda \lim_{N\to\infty} a(F^N,\lambda) = F - E_\lambda a(\alpha_{k_1}, \kappa(k_0)/p)$. Also in this case F^N converges uniformly to F and, by (B.81), $a(F^N,\lambda)$ converges to $a(\alpha_{k_1},\kappa(k_0)/p)$ whence G^N converges uniformly to $F - E_\lambda a(\alpha_{k_1},\kappa(k_0)/p) = G$ on $\mathbb{Z}$ as $N \to \infty$. Thus we have shown, for every $\lambda \in [0,1)$, that G and G^N satisfy the conditions of Theorem B.6a whence, by Theorem B.6a and for every $\lambda \in [0,1)$, $\lambda \in \Lambda_{tot}(G)$, i.e., $\Lambda_{tot}(G) = [0,1)$ and that $a(G,\lambda) = 0$. Therefore in the case, where $\lambda \in ([0,1) \setminus Y_{(1/p,\hat{\chi})})$, we have $\lambda \in \Lambda_{tot}(G) = \Lambda_{tot}(F)$ and $a(F,\lambda) = 0$. Moreover in the case, where $\lambda \in Y_{(1/p,\hat{\chi})}$, we have $\lambda \in \Lambda_{tot}(G) = \Lambda_{tot}(F - E_\lambda a(\alpha_{k_1},\kappa(k_0)/p))$ whence, by Theorem B.3a, $\lambda \in \Lambda_{tot}(F)$ and $0 = a(G,\lambda) = a(F - E_\lambda a(\alpha_{k_1},\kappa(k_0)/p),\lambda)$ so that $a(F,\lambda) = a(\alpha_{k_1},\kappa(k_0)/p)a(E_\lambda,\lambda) = a(\alpha_{k_1},\kappa(k_0)/p)$ which completes the proof that $\Lambda_{tot}(F) = [0,1)$. Note that we also got

$$a(F,\lambda) = \begin{cases} 0 & \text{if } \lambda \in ([0,1) \setminus Y_{(1/p,\hat{\chi})}) \\ a(\alpha_{k_1}, \kappa(k_0)/p) & \text{if } \lambda = k_1 \cdot \hat{\chi} + k_0/p \text{ and } k_0 \in \mathbb{Z}, k_1 \in \mathbb{Z}^s \end{cases} , \tag{B.83}$$

which completes the second part.

In the third part we first note, by (B.64), (B.83) and for $m \in \mathbb{Z}^m, r = 0, \cdots, p-1$, that $a(\alpha_m, r/p) = a(\alpha_m, \kappa(r)/p) = a(F, m \cdot \hat{\chi} + r/p)$ whence, by (B.74) and for $N \in \mathbb{Z}_+$,

$$F^N = \sum_{\substack{m\in\mathbb{Z}^s \\ ||m||\le N}} A^s_{N,m} \sum_{r=0}^{p-1} a(F, m\cdot\hat{\chi} + r/p) E_{r/p+m\cdot\hat{\chi}} . \tag{B.84}$$

If $\Lambda(F) = \emptyset$ then, by Definition B.2c and for $\mu \in [0,1)$, we get $a(F,\mu) = 0$ whence, by (B.84) and for $N \in \mathbb{Z}_+$, $F^N = 0$ which implies, by the convergence of F^N to F, that $F = 0$. □

Proof of Theorem B.7b: Let $F : \mathbb{Z} \to \mathbb{C}$ be χ–quasiperiodic where $\chi \in \mathbb{R}^d$ and let $Z \in GL(d,\mathbb{Z})$. Let also $H \in \mathcal{C}(\mathbb{R}^d,\mathbb{C})$ be a χ-generator of F. Then we define $h \in \mathcal{C}(\mathbb{R}^d,\mathbb{C})$ by $h(\phi) := H(Z^{-1}\phi)$. Since H is 2π-periodic in its arguments and $Z^{-1} \in \mathbb{Z}^{d\times d}$, we note that h is 2π-periodic in its arguments. We also compute $F(n) = H(2\pi n\chi) = h(2\pi n Z\chi)$ whence h is a $Z\chi$-generator of F so that F is $Z\chi$–quasiperiodic. □

Proof of Theorem B.7c: Let χ be in $\mathbb{R}^d \setminus \mathbb{Q}^d$ and let $(1, \chi)$ be resonant. The set M_χ in (B.63), i.e., $M_\chi = \{m \in \mathbb{Z}^d : m \cdot \chi \in \mathbb{Z}\}$ is the vehicle which allows us to construct from χ the matrix Z_1 and the Smith Normal Form of the latter then gives us Z. This all is accomplished by the $\mathbb{Z}$-module structure of M_χ and so our proof has two parts. In the first part we show that M_χ is an s-dimensional $\mathbb{Z}$-submodule of $\mathbb{Z}^d$ where $0 < s \leq d$ and this allows us to pick, in the second part, a basis of M_χ which gives us the matrix Z_1 and thus will allow us to show that $0 < s < d$ and to find the matrix Z for completing the proof.

An additively written Abelian group M' equipped with a scalar multiplication by integers is called a "$\mathbb{Z}$-module" if for all $x, y \in M', z, z' \in \mathbb{Z}$

$$z(x + y) = zx + zy \ , \ (z + z')x = zx + z'x \ , \ (zz')x = z(z'x) \ , \ 1x = x \ , \ z0 = 0 \ . \quad \text{(B.85)}$$

Clearly $\mathbb{Z}^d$ and M_χ are $\mathbb{Z}$-modules by using the addition and scalar multiplication from $\mathbb{R}^d$. Every $\mathbb{Z}$-module is an Abelian group and, conversely, every Abelian group is a $\mathbb{Z}$-module in a natural way since, according to (B.85), nx can be defined inductively via the Abelian group structure by $(n+1)x := nx+x$ and by $(n-1)x := nx-x$. Thus on every Abelian group the $\mathbb{Z}$-module structure allows one to do Linear Algebra. If M', M'' are $\mathbb{Z}$-modules and if $M'' \subset M'$ and if the addition and scalar multiplication in M'' are the restricitions from M' then M'' is called a $\mathbb{Z}$-submodule of M'. Clearly M_χ is a $\mathbb{Z}$-submodule of $\mathbb{Z}^d$. If r is a positive integer and $x^1, x^2, ..., x^r$ are elements of a $\mathbb{Z}$-module M' then they are called "linearly independent" iff for every choice of integers $z_1, ..., z_r$ the equality $0 = z_1 x^1 + ... + z_r x^r$ implies that $z_1, ..., z_r$ vanish. More generally, the elements of a subset L of M' are called "linearly independent" (and L is called "linearly independent") if the elements of every finite nonempty subset of L are linearly independent. Defining $e_d^1, ..., e_d^d \in \mathbb{Z}^d$ by $e_d^1 := (1, 0, ..., 0)^t, ..., e_d^d := (0, ..., 0, 1)^t$ it is clear that $e_d^1, ..., e_d^d$ are linearly independent. If r is a positive integer and $x^1, x^2, ..., x^r$ are elements in M' then the elements $z_1 x^1 + ... + z_r x^r$ form a $\mathbb{Z}$–module, say M'', if one varies the $z_1, ..., z_r$ over the integers. One calls M'' the $\mathbb{Z}$-module generated by $x^1, x^2, ..., x^r$. More generally, if L is a nonempty subset of M' then the $\mathbb{Z}$-module generated by L is defined by the union of all $\mathbb{Z}$-modules generated by the finite nonempty subsets of L (and one says that the elements of L "generate" M'). Of course the $\mathbb{Z}$-module generated by a nonempty subset of M' is a $\mathbb{Z}$-submodule of M'. Clearly $\mathbb{Z}^d$ is generated by $e_d^1, ..., e_d^d$. A nonempty subset L of a $\mathbb{Z}$-module M' is called a "basis" of M' if it is linearly independent and if it generates M'. Of course $\{e_d^1, ..., e_d^d\}$ is a basis of the $\mathbb{Z}$-module $\mathbb{Z}^d$. A $\mathbb{Z}$-module M' is called "trivial" if $M' = \{0\}$. If a $\mathbb{Z}$-module M' has a finite basis of, say m elements, then (see for example Section 3.6 in [AW]) every of its bases has m elements and M' is called m-dimensional and one writes $dim(M') = m$. If a $\mathbb{Z}$-module M' is trivial, then it has no basis and it then is called 0-dimensional and one writes $dim(M') = 0$. Clearly $dim(\mathbb{Z}^d) = d$. In general, $\mathbb{Z}$-modules are neither trivial nor have a basis and this demonstrates that $\mathbb{Z}$-modules are more delicate than vector spaces. A simple example is the Abelian group $\mathbb{Z}_2$ of two elements 0 and b where the natural $\mathbb{Z}$-module structure of $\mathbb{Z}_2$ gives us, for $z \in \mathbb{Z}$, $z0 = 0, (2z)b = 0, (2z + 1)b = b$ where z is an integer. Clearly $\mathbb{Z}_2$ is not trivial and it is easy to check that $\mathbb{Z}_2$ has no linearly independent subset whence it has no basis. In contrast $\mathbb{Z}^d$ does not have this pathology because it has the basis $\{e_d^1, ..., e_d^d\}$. Most importantly M_χ does not have this pathology either. In fact since the $\mathbb{Z}$-module $\mathbb{Z}^d$ is d-dimensional every of its $\mathbb{Z}$-submodules, e.g., M_χ is s-dimensional where $0 \leq s \leq d$ (see, e.g., [Jac] or [Hun, Section IV.6]). If s would vanish then M_χ would be trivial, i.e., $M_\chi = \{0\}$ whence, by (B.63), $(1, \chi)$ would be nonresonant, a contradiction. Thus $s > 0$ whence $0 < s \le d$ which completes the first part of the proof. In the second part we now use the fact that M_χ is s–dimensional and $0 < s \le d$ whence M_χ has a finite basis of s elements, $k^1, ..., k^s$. Thus we can do Linear Algebra by defining the matrix $Z_1 \in \mathbb{Z}^{s\times d}$ by $Z_1 := [k^1, ..., k^s]^t$, i.e., the j-th row of Z_1 is $(k^j)^t$. Clearly Z_1 satisfies

$$M_\chi = \{Z_1^t k : k \in \mathbb{Z}^s\} \ , \qquad Z_1\chi \in \mathbb{Z}^s \ , \tag{B.86}$$

where in the equality we used (B.63). By the Linear Algebra of $\mathbb{Z}$–modules (see for example [Jac, Sec. III.8] or [Hun, Section VII.2]) the matrix Z_1 can be factorized via

$$Z_1 = Z_2 Z_3 Z \ , \tag{B.87}$$

where $Z_2 \in GL(s, \mathbb{Z})$ and $Z \in GL(d, \mathbb{Z})$ and where the $Z_3 \in \mathbb{Z}^{s\times d}$ is of "Smith Normal Form", i.e., $Z_3 = [l_1 e_d^1, ..., l_s e_d^s]^t$ with nonzero integers $l_1, ..., l_s$. The integers $l_1, ..., l_s$ also satisfy a divisibility condition (which however is not needed in our proof). For the notion of "Smith Normal Form" see also [AW, Section 5.3]. We define $\check\chi \in \mathbb{R}^d$ and $\tilde\chi \in \mathbb{R}^s$ by

$$(\check\chi_1, ..., \check\chi_d)^t = \check\chi := Z\chi \ , \qquad \tilde\chi := (\check\chi_1, ..., \check\chi_s)^t \ . \tag{B.88}$$

To show that $\tilde\chi \in \mathbb{Q}^s$ and that $0 < s < d$ we compute, by (B.87),(B.88),

$$Z_2 Z_3 \check\chi = Z_2 Z_3 Z\chi = Z_1\chi \in \mathbb{Z}^s \ . \tag{B.89}$$

We also note, by (B.88),

$$Z_3\check\chi = (l_1\check\chi_1, ..., l_s\check\chi_s)^t = (l_1\tilde\chi_1, ..., l_s\tilde\chi_s)^t \ . \tag{B.90}$$

Because $Z_2 \in GL(s, \mathbb{Z})$, (B.89) yields $Z_3\check\chi \in \mathbb{Z}^s$ whence, by (B.90) and since $l_1, ..., l_s$ are nonzero integers, we obtain that $\tilde\chi \in \mathbb{Q}^s$. To show that $0 < s < d$ we first note that $\check\chi \notin \mathbb{Q}^d$ since otherwise $Z^{-1}\check\chi$ would belong to $\mathbb{Q}^d$, i.e., χ would belong to $\mathbb{Q}^d$, a contradiction. It is now easy to see that $s < d$ because the equality: $s = d$ would imply $\check\chi = \tilde\chi$ whence $\check\chi$ would belong to $\mathbb{Q}^d$, a contradiction. Thus $0 < s < d$. We now define $\hat\chi \in \mathbb{R}^{d-s}$ by

$$\hat\chi := (\check\chi_{s+1}, ..., \check\chi_d)^t \ , \tag{B.91}$$

whence, by (B.88),

$$\check\chi = (\tilde\chi^t, \hat\chi^t)^t \ , \tag{B.92}$$

so it remains to be shown that $(1, \hat\chi)$ is nonresonant. We first compute, by (B.63), (B.86), (B.87),(B.88),

$$\begin{aligned}
M_{\check\chi} &= \{m \in \mathbb{Z}^d : m\cdot\check\chi \in \mathbb{Z}\} = \{m \in \mathbb{Z}^d : m\cdot(Z\chi) \in \mathbb{Z}\} = \{m \in \mathbb{Z}^d : (Z^t m)\cdot\chi \in \mathbb{Z}\} \\
&\quad = \{(Z^t)^{-1}m : m \in \mathbb{Z}^d, m\cdot\chi \in \mathbb{Z}\} = \{(Z^t)^{-1}m : m \in M_\chi\} \\
&= \{(Z^t)^{-1}Z_1^t k : k \in \mathbb{Z}^s\} = \{(Z^t)^{-1}(Z_2 Z_3 Z)^t k : k \in \mathbb{Z}^s\} = \{Z_3^t Z_2^t k : k \in \mathbb{Z}^s\} \\
&\quad = \{Z_3^t k : k \in \mathbb{Z}^s\} \ ,
\end{aligned} \tag{B.93}$$

where in the fourth equality we used that $Z \in GL(d, \mathbb{Z})$ and in the ninth equality we used that $Z_2 \in GL(s, \mathbb{Z})$. We are now in a position to show that $(1, \hat\chi)$ is nonresonant. By the

remarks after (B.63), $(1, \hat\chi)$ is nonresonant iff $M_{\hat\chi} = \{0\}$. To show the latter let $\hat m \in M_{\hat\chi}$ whence, by (B.63), $\hat m \cdot \hat\chi \in \mathbb{Z}$ so that, by (B.92), $(0, \hat m) \cdot \check\chi \in \mathbb{Z}$. Thus, by (B.63), $(0, \hat m) \in M_{\check\chi}$ whence, by (B.93), $(0, \hat m) = Z_3^t k$ for some $k = (k_1, ..., k_s)^t \in \mathbb{Z}^s$. On the other hand, by the definition of Z_3, $Z_3^t k = k_1 l_1 e_d^1 + \cdots k_s l_s e_d^s$ whence $(0, \hat m) = k_1 l_1 e_d^1 + \cdots k_s l_s e_d^s$ so that $k_1 = \cdots = k_s$ and $\hat m = 0$ which implies that $M_{\hat\chi} = \{0\}$, i.e., that $(1, \hat\chi)$ is nonresonant. □
Proof of Theorem B.7d: By Theorem B.7b, F is $Z\chi$-quasiperiodic whence F is $(\tilde\chi, \hat\chi)$–quasiperiodic. Thus, by Theorem B.7a, $\Lambda_{tot}(F) = [0, 1)$. If $\Lambda(F) = \emptyset$ then, since F is $(\tilde\chi, \hat\chi)$–quasiperiodic and by by Theorem B.7a, $F = 0$. □

With the Second, Third and Fourth Spectral Theorems we have shown, for every quasiperiodic function F, that $\Lambda_{tot}(F) = [0, 1)$ and that, in the case $\Lambda(F) = \emptyset$, $F = 0$ (and we will apply these facts in Appendices B.5 and B.6 below). However we have accomplished even more since we have found, by (B.50),(B.55), (B.84), how to explicitly express F in terms of the data $a(F, \lambda)$. The latter circumstance is very interesting for future work on spin-orbit systems in $\mathcal{SOS}(d, \mathcal{P}[\omega])$ since it opens the road towards generalizing theorems which hold off orbital resonance (e.g., Theorem 3.3a and the Uniqueness Theorem) to theorems which even hold on orbital resonance.
Remark:

(3) Let $(\mathcal{P}[\omega], A) \in \mathcal{ACB}(d, \mathcal{P}[\omega])$ and $\nu \in \Xi(\mathcal{P}[\omega], A)$. Let also (Z, S) be a particle-spin-vector trajectory of $(\mathcal{P}[\omega], A)$ and let S_j denote the j-th component of S. We recall from Remark 2 above that S_j is (ω, ν)-quasiperiodic whence, by the Second, Third and Fourth Spectral Theorems, $\Lambda(S_j) \subset Y_{(\omega,\nu)}$. This inclusion will be sharpened in Appendix B.5 below to the spectral formula (7.28). □

B.5 Proof of the spectral formula (7.28)

The following theorem proves the main claims made in Section 7.2 about the spectral approach to spin tunes. In particular it proves (7.28).

Theorem B.8 *Let (Z, S) be a particle-spin-vector trajectory of $(\mathcal{P}[\omega], A) \in \mathcal{ACB}(d, \mathcal{P}[\omega])$ and let $S_j(n)$ denote the j-th component of $S(n)$. Then $\Lambda_{tot}(S_j) = [0, 1)$ and, for every $\lambda \in [0, 1)$, $a_N(S_j, \lambda)$ converges to $a(S_j, \lambda)$ as $N \to \infty$. Moreover (7.28) holds, i.e.,*

$$\Lambda(S_j) \subset \Xi(\mathcal{P}[\omega], A) \cup \{l \cdot \omega + n : l \in \mathbb{Z}^d, n \in \mathbb{Z}\} .$$

Proof of Theorem B.8: We will prove the claims in two parts. In the first part we prove, by using the Second, Third and Fourth Spectral Theorems, that $\Lambda_{tot}(S_j) = [0, 1)$ and in the second part we use the First Spectral Theorem. Following Remark 2 above, we pick a $\nu \in \Xi(\mathcal{P}[\omega], A)$ and a transfer field T from $(\mathcal{P}[\omega], A)$ to $(\mathcal{P}[\omega], A_{d,\nu})$ (recall from Remark 2 that T is a uniform IFF of $(\mathcal{P}[\omega], A)$). We also pick a $\phi_0 \in \mathbb{R}^d$ such that $\pi_d(\phi_0) = Z(0)$ and we define the function $t : \mathbb{Z} \to SO(3)$ by (B.24) which gives us (B.25) whence, by (7.5),

$$\Psi[\mathcal{P}[\omega], A](n; Z(0)) = t(n) \exp(2\pi \mathcal{J} n\nu) t^t(0) . \tag{B.94}$$

Defining

$$\Delta_\pm := \frac{1}{2}\begin{pmatrix} 1 & \pm i & 0 \\ \mp i & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} , \qquad \Delta_0 := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} , \tag{B.95}$$

we have, by (6.7),

$$\exp(\mathcal{J}\nu 2\pi n) = \Delta_+ \exp(2\pi i n\nu) + \Delta_- \exp(-2\pi i n\nu) + \Delta_0 \, . \tag{B.96}$$

By (B.5),(B.94) and (B.96) we have, for all n,

$$\begin{aligned}\Psi[\mathcal{P}[\omega], A](n; Z(0)) &= t(n)\Big(\Delta_+ \exp(2\pi i n\nu) + \Delta_- \exp(-2\pi i n\nu) + \Delta_0\Big)t^t(0) \\ &=: t^+(n)E_\nu(n) + t^-(n)E_{-\nu}(n) + t^0(n) \, ,\end{aligned} \tag{B.97}$$

where

$$t^+(n) := t(n)\Delta_+ t^t(0) \, , \quad t^-(n) := t(n)\Delta_- t^t(0) \, , \quad t^0(n) := t(n)\Delta_0 t^t(0) \, . \tag{B.98}$$

It follows from (2.35) that $S(n) = \Psi[\mathcal{P}[\omega], A](n; Z(0))S(0)$ whence, by (B.97),

$$S_j = \sum_{k=1}^{3}\Big(t^+_{jk}E_\nu + t^-_{jk}E_{-\nu} + t^0_{jk}\Big)S_k(0) \, , \tag{B.99}$$

where t^+_{jk}, t^-_{jk} and t^0_{jk} denote the jk-matrix elements of t^+, t^- and t^0. We recall from Remark 2 above that each matrix element of t is an ω-quasiperiodic function whence, by (B.98) and Remark 1 above, t^+_{jk}, t^-_{jk} and t^0_{jk} are ω-quasiperiodic so that, by the Second, Third and Fourth Spectral Theorems in Appendix B.4, $\Lambda_{tot}(t^+_{jk}) = \Lambda_{tot}(t^-_{jk}) = \Lambda_{tot}(t^0_{jk}) = [0, 1)$ which implies, by Theorem B.3a,

$$\Lambda_{tot}(t^+_{jk}E_\nu) = [0, 1) \, , \quad \Lambda_{tot}(t^-_{jk}E_{-\nu}) = [0, 1) \, , \quad \Lambda_{tot}(t^0_{jk}) = [0, 1) \, . \tag{B.100}$$

We conclude from (B.99),(B.100) that

$$\Lambda_{tot}(S_j) = [0, 1) \, , \tag{B.101}$$

which completes the first part of the proof. Of course (B.101) implies, by Definition B.2c, that $a_N(S_j, \lambda)$ converges to $a(S_j, \lambda)$ as $N \to \infty$. We now prove (7.28) by using the First Spectral Theorem in Appendix B.2. We note, by (B.101) and Definition B.2c, that the Cesàro spectrum of S_j is well-defined so let $\lambda \in \Lambda(S_j)$. Then, by Definition B.2c, $a(S_j, \lambda) \neq 0$ whence, by (B.99), there exist $j, k \in \{1, 2, 3\}$ such that either $a(t^+_{jk}E_\nu, \lambda) \neq 0$ or $a(t^-_{jk}E_{-\nu}, \lambda) \neq 0$ or $a(t^0_{jk}, \lambda) \neq 0$ so that we have to consider three cases, i.e., we will show for each case that λ is an element of the set on the rhs of (7.28). In the case where $a(t^+_{jk}E_\nu, \lambda) \neq 0$ it follows from Theorem B.3a that $a(t^+_{jk}, \lfloor\lambda - \nu\rfloor) \neq 0$ whence, by Definition B.2c, $\lfloor\lambda - \nu\rfloor \in \Lambda(t^+_{jk})$ so that, by the First Spectral Theorem in Appendix B.2 and since t^+_{jk} is ω-quasiperiodic,

$$\lfloor\lambda - \nu\rfloor \in Y_\omega \, . \tag{B.102}$$

It follows from (B.6),(B.102) that $\lambda - \nu = m \cdot \omega + l$ where $m \in \mathbb{Z}^d, l \in \mathbb{Z}$ whence, by Theorem 7.3f and since $\nu \in \Xi(\mathcal{P}[\omega], A)$, we get $\lambda \in \Xi(\mathcal{P}[\omega], A)$ so that indeed λ is an element of the set on the rhs of (7.28). In the case where $a(t^-_{jk} E_{-\nu}, \lambda) \neq 0$ it follows from Theorem B.3a that $a(t^-_{jk}, \lfloor \lambda + \nu \rfloor) \neq 0$ whence, by Definition B.2c, $\lfloor \lambda + \nu \rfloor \in \Lambda(t^-_{jk})$ so that, by the First Spectral Theorem in Appendix B.2 and since t^-_{jk} is ω-quasiperiodic,

$$\lfloor \lambda + \nu \rfloor \in Y_\omega \; . \tag{B.103}$$

It follows from (B.6),(B.103) that $\lambda + \nu = m \cdot \omega + l$ where $m \in \mathbb{Z}^d, l \in \mathbb{Z}$ whence, by Theorem 7.3f and since $\nu \in \Xi(\mathcal{P}[\omega], A)$, we get $\lambda \in \Xi(\mathcal{P}[\omega], A)$ so that indeed λ is an element of the set on the rhs of (7.28). In the case where $a(t^0_{jk}, \lambda) \neq 0$ it follows from Definition B.2c that $\lambda \in \Lambda(t^0_{jk})$ whence, by the First Spectral Theorem in Appendix B.2 and since t^0_{jk} is ω-quasiperiodic, $\lambda \in Y_\omega$ so that, by (B.6), $\lambda = m \cdot \omega + l$ where $m \in \mathbb{Z}^d, l \in \mathbb{Z}$ which implies that λ is an element of the set on the rhs of (7.28). This completes the proof of the theorem. $\square$

By (B.6) and Theorem 7.3f, $\Xi(\mathcal{P}[\omega], A) \cup \{l \cdot \omega + n : l \in \mathbb{Z}^d, n \in \mathbb{Z}\} \subset Y_{(\omega,\nu)}$ whence the spectral formula (7.28) sharpens the inclusion: $\Lambda(S_j) \subset Y_{(\omega,\nu)}$ from Remark 3 above.

B.6 Remarks on the absence of spin tunes

Let $(\mathcal{P}[\omega], A) \in \mathcal{ACB}(d, \mathcal{P}[\omega])$, i.e., by Theorem 7.3b, let $(\mathcal{P}[\omega], A)$ have spin tunes. Let also (Z, S) be a particle-spin-vector trajectory of $(\mathcal{P}[\omega], A)$ and let S_j denote the j-th component of S. Then, by Remark 3 above, every S_j is quasiperiodic and satisfies: $\Lambda_{tot}(S_j) = [0, 1)$ and $\Lambda(S_j) \subset Y_{(\omega,\nu)}$. Denoting the union over the $\Lambda(S_j)$ by Λ where (Z, S) varies over all particle-spin-vector trajectories of $(\mathcal{P}[\omega], A)$ we get $\Lambda \subset \bigcup_{\nu \in \Xi[\mathcal{P}[\omega],A]} Y_{(\omega,\nu)}$ whence, by (B.6) and Theorem 7.3f, Λ is a countable set.

These properties of the S_j are not satisfied in general by spin-orbit systems which do not have spin tunes. One example is the 2-snake model $(\mathcal{P}[1/2], A_{2S})$ from Section 3.3. In fact one can show for $(\mathcal{P}[1/2], A_{2S})$ that, while Λ is well-defined, it is an uncountable set [He1]. This implies, by the above, that $(\mathcal{P}[1/2], A_{2S})$ has no spin tunes. Of course this is no surprise since we know from Section 3.3 that $(\mathcal{P}[1/2], A_{2S})$ has no ISF whence, by Remark 5 in Chapter 4, $(\mathcal{P}[\omega], A) \notin \mathcal{ACB}(d, \mathcal{P}[\omega])$, i.e., by Theorem 7.3b, $(\mathcal{P}[\omega], A)$ has no spin tunes. Another example is the spin-orbit system $(\mathcal{P}[\omega], A)$ of Theorem 7.6 in the case where $d = 1$ and $N \neq 0$. In that case let us consider the particle-spin-vector trajectory (Z, S) of $(\mathcal{P}[\omega], A)$ for which $Z(0) = \pi_1(0)$ and $S(0) = (1, 0, 0)^t$. It is easy to show, by (2.35),(2.36), $S_1(n) = \cos(\pi \omega N n(n-1))$ which implies (see for example Weyl's Theorem in Section III.19 of [Kor]) that $\Lambda_{tot}(S_1) = [0, 1)$ and $\Lambda(S_1) = \emptyset$. However the latter implies, by the Second, Third and Fourth Spectral Theorems and since S_1 is not the zero function, that S_1 is not quasiperiodic. Therefore, by the above, $(\mathcal{P}[\omega], A)$ has no spin tunes. Of course this is no surprise since we know from Theorem 7.6 that $(\mathcal{P}[\omega], A)$ has no spin tunes.

C Further proofs

C.1 Proof of Theorem 2.5

We first state and prove Theorem C.1 since we need it to prove Theorem 2.5a. For $\varepsilon \in (0,\infty)$ and $\phi \in \mathbb{R}^d$ we define

$$B_d[\phi,\varepsilon] := (\phi_1-\varepsilon,\phi_1+\varepsilon)\times\cdots\times(\phi_d-\varepsilon,\phi_d+\varepsilon) \subset \mathbb{R}^d\,, \tag{C.1}$$

and we also abbreviate

$$\tilde{\tau}_{\mathbb{R}^d} := \{B_d[\phi,\varepsilon] : \varepsilon\in(0,\infty), \phi\in\mathbb{R}^d\}\,, \tag{C.2}$$
$$\tilde{\tau}_d^{fin} := \{\pi_d(B_d[\phi,\varepsilon]) : \varepsilon\in(0,\infty), \phi\in\mathbb{R}^d\}\,. \tag{C.3}$$

Theorem C.1 *a) Let $A\subset\mathbb{R}^d$. Then*

$$\pi_d^{-1}(\pi_d(A)) = \bigcup_{N\in\mathbb{Z}^d} \hat{\mathcal{P}}[N](A)\,, \tag{C.4}$$

where $\hat{\mathcal{P}}[N]$ is defined in Section 2.2.
b) Denoting the natural topology of $\mathbb{R}^d$ by $\tau_{\mathbb{R}^d}$ we have

$$\tau_d^{fin} = \{\pi_d(A) : A\in\tau_{\mathbb{R}^d}\}\,. \tag{C.5}$$

c) $\tilde{\tau}_{\mathbb{R}^d}$ is a base of the topology $\tau_{\mathbb{R}^d}$ and $\tilde{\tau}_d^{fin}$ is a base of the topology τ_d^{fin}.
d) For every $\phi\in\mathbb{R}^d$ and $\varepsilon\in(0,\infty)$

$$\pi_d(\phi) = (\pi_1(\phi_1),\cdots,\pi_1(\phi_d))^t\,, \tag{C.6}$$
$$\pi_d(B_d[\phi,\varepsilon]) = \pi_1(B_1[\phi_1,\varepsilon])\times\cdots\times\pi_1(B_1[\phi_d,\varepsilon])\,. \tag{C.7}$$

Moreover the topological space $(\mathbb{T}^d,\tau_d^{fin})$ is Hausdorff.
e) τ_d is the subspace topology from $\tau_{\mathbb{R}^{2d}}$, i.e., $\tau_d = \{B\cap\mathbb{T}^d : B\in\tau_{\mathbb{R}^{2d}}\}$. Moreover the topological space $(\mathbb{T}^d,\tau_d)$ is compact.
f) $\tau_d^{fin}\subset\tau_d$.

Proof of Theorem C.1a: If $\phi\in\mathbb{R}^d$ then, by Definition 2.4 and by the definition of $\hat{\mathcal{P}}[N]$ in Section 2.2, we compute $\pi_d^{-1}(\pi_d(\{\phi\})) = \{\phi+2\pi N : N\in\mathbb{Z}^d\} = \bigcup_{N\in\mathbb{Z}^d}\hat{\mathcal{P}}[N](\{\phi\})$ whence, for $A\subset\mathbb{R}^d$,

$$\pi_d^{-1}(\pi_d(A)) = \pi_d^{-1}(\pi_d(\bigcup_{\phi\in A}\{\phi\})) = \pi_d^{-1}(\bigcup_{\phi\in A}\pi_d(\{\phi\})) = \bigcup_{\phi\in A}\pi_d^{-1}(\pi_d(\{\phi\}))$$
$$= \bigcup_{\phi\in A}\bigcup_{N\in\mathbb{Z}^d}\hat{\mathcal{P}}[N](\{\phi\}) = \bigcup_{N\in\mathbb{Z}^d}\bigcup_{\phi\in A}\hat{\mathcal{P}}[N](\{\phi\}) = \bigcup_{N\in\mathbb{Z}^d}\hat{\mathcal{P}}[N](\bigcup_{\phi\in A}\{\phi\}) = \bigcup_{N\in\mathbb{Z}^d}\hat{\mathcal{P}}[N](A)\,,$$

which proves the claim. □

Proof of Theorem C.1b: "$\subset$": Let $V \in \tau_d^{fin}$ whence, by the definition of τ_d^{fin} in Section 2.3, $A := \pi_d^{-1}(V)$ belongs to $\tau_{\mathbb{R}^d}$. Since π_d is onto $\mathbb{T}^d$ we get $V = \pi_d(\pi_d^{-1}(V)) = \pi_d(A)$ whence V belongs to the set on the rhs of (C.5) so that

$$\tau_d^{fin} \subset \{\pi_d(A) : A \in \tau_{\mathbb{R}^d}\} \,. \tag{C.8}$$

"$\supset$": Let $A \in \tau_{\mathbb{R}^d}$. To show that $\pi_d(A)$ belongs to τ_d^{fin} we note, for every $N \in \mathbb{Z}^d$ and since $\hat{\mathcal{P}}[N] \in \mathrm{Homeo}(\mathbb{R}^d)$, that $\hat{\mathcal{P}}[N](A) \in \tau_{\mathbb{R}^d}$ whence the union $\bigcup_{N\in\mathbb{Z}^d} \hat{\mathcal{P}}[N](A)$ also belongs to $\tau_{\mathbb{R}^d}$ so that, by Theorem C.1a, $\pi_d^{-1}(\pi_d(A)) \in \tau_{\mathbb{R}^d}$ which implies, by the definition of τ_d^{fin}, that $\pi_d(A) \in \tau_d^{fin}$. Thus $\tau_d^{fin} \supset \{\pi_d(V) : V \in \tau_{\mathbb{R}^d}\}$ whence the claim follows from (C.8). $\square$

Proof of Theorem C.1c: Because $\tau_{\mathbb{R}^d}$ is the metric topology on $\mathbb{R}^d$ (from the Euclidean metric on $\mathbb{R}^d$) it is clear that every $A \in \tau_{\mathbb{R}^d}$ is a union of sets of the form $B_d[\phi, \varepsilon]$, i.e., that $\tilde{\tau}_{\mathbb{R}^d}$ is a base of the topology $\tau_{\mathbb{R}^d}$. To show the second claim let $V \in \tau_d^{fin}$, i.e., $A := \pi_d^{-1}(V)$ belongs to $\tau_{\mathbb{R}^d}$. Since $\tilde{\tau}_{\mathbb{R}^d}$ is a base of the topology $\tau_{\mathbb{R}^d}$ there exists a set Λ and, for every $\lambda \in \Lambda$, an $\phi_\lambda \in \mathbb{R}^d$ and $\varepsilon_\lambda \in (0, \infty)$ such that $A = \bigcup_{\lambda\in\Lambda} B_d[\phi_\lambda, \varepsilon_\lambda]$ whence $V = \pi_d(\pi_d^{-1}(V)) = \pi_d(A) = \pi_d(\bigcup_{\lambda\in\Lambda} B_d[\phi_\lambda, \varepsilon_\lambda]) = \bigcup_{\lambda\in\Lambda} \pi_d(B_d[\phi_\lambda, \varepsilon_\lambda])$ so that V is a union of sets of the form $\pi_d(B_d[\phi, \varepsilon])$ which implies, by (C.5), that $\tilde{\tau}_d^{fin}$ is a base of τ_d^{fin}. $\square$

Proof of Theorem C.1d: We first note that (C.6) follows from the definition (2.19) of π_d and that (C.7) follows from (C.1) and (C.6). To prove the last claim we first consider the case $d = 1$ so let $z, z' \in \mathbb{T}$ and $z \neq z'$. We choose $\phi, \phi' \in (-\pi, \pi]$ such that $z = \pi_1(\phi), z' = \pi_1(\phi')$ and without loss of generality we assume that $\phi < \phi'$ and we define ε such that 3ε is the minimum of ϕ and $\phi' - \phi$. Clearly $\varepsilon \in (0, \infty)$ whence, by Theorem C.1b, $\pi_1(B_1[\phi, \varepsilon])$ is an open neighborhood of z and $\pi_1(B_1[\phi', \varepsilon])$ is an open neighborhood of z'. Moreover $\phi + \varepsilon < \phi' - \varepsilon, \phi' + \varepsilon < \phi - \varepsilon + 2\pi$ whence $\pi_1(B_1[\phi, \varepsilon]) \cap \pi_1(B_1[\phi', \varepsilon]) = \emptyset$ so that $(\mathbb{T}, \tau_1^{fin})$ is Hausdorff. Let now $z, z' \in \mathbb{T}^d$ and $z \neq z'$. We choose $\phi, \phi' \in (-\pi, \pi]^d$ such that $z = \pi_d(\phi), z' = \pi_d(\phi')$. Because $z \neq z'$ we have $z_j \neq z'_j$ for some $j = 1, ..., d$ and without loss of generality we assume that $\phi_j < \phi'_j$ and we define ε such that 3ε is the minimum of ϕ_j and $\phi'_j - \phi_j$. Clearly $\varepsilon \in (0, \infty)$ whence, by Theorem C.1b, $\pi_d(B_d[\phi, \varepsilon])$ is an open neighborhood of z and $\pi_d(B_d[\phi', \varepsilon])$ is an open neighborhood of z'. Moreover $\phi_j + \varepsilon < \phi'_j - \varepsilon, \phi'_j + \varepsilon < \phi_j - \varepsilon + 2\pi$ whence $\pi_1(B_1[\phi_j, \varepsilon]) \cap \pi_1(B_1[\phi'_j, \varepsilon]) = \emptyset$ so that, by Theorem C.1d, $\pi_d(B_d[\phi, \varepsilon]) \cap \pi_d(B_d[\phi', \varepsilon]) = \emptyset$ which implies that the topological space $(\mathbb{T}^d, \tau_d^{fin})$ is Hausdorff. $\square$

Proof of Theorem C.1e: Since μ_d is the restriction of the Euclidean metric of $\mathbb{R}^{2d}$ it follows and since $\tau_{\mathbb{R}^{2d}}$ is the metric topology from the Euclidean metric it is easy to show, by using the base $\tilde{\tau}_{\mathbb{R}^{2d}}$ of $\tau_{\mathbb{R}^{2d}}$, that τ_d is the subspace topology from $\tau_{\mathbb{R}^{2d}}$ (see also Section 2.10 in [Mu]). To prove the second claim we define the function $f \in \mathcal{C}(\mathbb{R}^{2d}, \mathbb{R}^d)$ by $f(z_1, z_2, \cdots, z_{2d-1}, z_{2d}) := (z_1^2 + z_2^2, \cdots, z_{2d-1}^2 + z_{2d}^2)^t$ and note that $\mathbb{T}^d$ is the inverse image of $(1, \cdots, 1)^t$ under f. Since the singleton $\{(1, \cdots, 1)^t\}$ is a closed subset of $\mathbb{R}^d$ and since f is continuous it follows that $\mathbb{T}^d$ is a closed subset of $\mathbb{R}^{2d}$. Moreover $\mathbb{T}^d$ is bounded (w.r.t. the Euclidean metric on $\mathbb{R}^{2d}$) whence, by the Heine-Borel Theorem (see, e.g., Corollary 2.12 in [Hu, Chapter III]), $\mathbb{T}^d$ is a compact subset of $\mathbb{R}^{2d}$ so that, since $\mathbb{T}^d$ is a subspace of $\mathbb{R}^{2d}$, the topological space $(\mathbb{T}^d, \tau_d)$ is compact. $\square$

Proof of Theorem C.1f: To show that $\tau_d^{fin} \subset \tau_d$ we note, by Theorem C.1c, that $\tilde{\tau}_d^{fin}$ is a base of the topology τ_d^{fin} whence we are done if we show that $\tilde{\tau}_d^{fin} \subset \tau_d$. We first consider the case $d = 1$ so let $\phi \in (-\pi, \pi]$ and $\varepsilon \in (0, \infty)$ whence our aim is to show that $\pi_1(B_1[\phi, \varepsilon]) \in \tau_1$.

We define the wedge $W := \{r(\cos(\psi), \sin(\psi))^t : r \in (0,\infty), \phi - \varepsilon < \psi < \phi + \varepsilon\}$ and observe that $W \in \tau_{\mathbb{R}^2}$ as can be easily checked. Moreover $\pi_1(B_1[\phi,\varepsilon]) = W \cap \mathbb{T}$ whence $\pi_1(B_1[\phi,\varepsilon]) \in \tau_1$. We now consider the general case so let $\phi \in (-\pi,\pi]^d$ and $\varepsilon \in (0,\infty)$. To show that $\pi_d(B_d[\phi,\varepsilon]) \in \tau_d$ we define, for $j = 1, ..., d$, $W_j := \{r(\cos(\psi), \sin(\psi))^t : r \in (0,\infty), \phi_j - \varepsilon < \psi < \phi_j + \varepsilon\}$ and recall from above that $W_j \in \tau_{\mathbb{R}^2}$. On the other hand, by Theorem C.1c, $\tilde{\tau}_{\mathbb{R}^{2d}}$ is a base of $\tau_{\mathbb{R}^{2d}}$ whence $W_1 \times \cdots \times W_d \in \tau_{\mathbb{R}^{2d}}$ as can be easily checked. Moreover $\pi_1(B_1[\phi_j,\varepsilon]) = W_j \cap \mathbb{T}$ so we compute, by (2.15) and Theorem C.1d, $\pi_d(B_d[\phi,\varepsilon]) = \pi_1(B_1[\phi_1,\varepsilon]) \times \cdots \times \pi_1(B_1[\phi_d,\varepsilon]) = (W_1 \cap \mathbb{T}) \times \cdots \times (W_d \cap \mathbb{T}) = (W_1 \times \cdots \times W_d) \cap (\mathbb{T} \times \cdots \times \mathbb{T}) = (W_1 \times \cdots \times W_d) \cap \mathbb{T}^d$ whence, and since $W_1 \times \cdots \times W_d \in \tau_{\mathbb{R}^{2d}}$, $\pi_d(B_d[\phi,\varepsilon])$ belongs to the subspace topology from $\tau_{\mathbb{R}^{2d}}$. However, by Theorem C.1e, τ_d is the subspace topology from $\tau_{\mathbb{R}^{2d}}$ whence $\pi_d(B_d[\phi,\varepsilon]) \in \tau_d$. □

We are now ready to prove Theorem 2.5.

Proof of Theorem 2.5a: To prove that $\tau_d = \tau_d^{fin}$ we first note that this equality holds iff $id_{\mathbb{T}^d}$ is a homeomorphism from $(\mathbb{T}^d, \tau_d)$ to $(\mathbb{T}^d, \tau_d^{fin})$. Since $id_{\mathbb{T}^d}$ is a bijection we can use the fact from Proposition 2.6 in [Hu, Chapter III] that $id_{\mathbb{T}^d}$ is a homeomorphism if it is continuous and if τ_d^{fin} is Hausdorff and τ_d is compact. Thus, by Theorems C.1d and C.1e above, $id_{\mathbb{T}^d}$ is a homeomorphism if it is continuous. Of course since $id_{\mathbb{T}^d}$ is the identity function on $\mathbb{T}^d$ it is a continuous function from $(\mathbb{T}^d, \tau_d)$ to $(\mathbb{T}^d, \tau_d^{fin})$ iff $\tau_d^{fin} \subset \tau_d$ whence, by Theorem C.1f above, $id_{\mathbb{T}^d}$ is a continuous function. This completes the proof of the first claim, i.e., that $\tau_d = \tau_d^{fin}$. To show that π_d is continuous let $V \in \tau_d$ whence we are done if we show that $\pi_d^{-1}(V)$ is open. In fact, by the first claim, $V \in \tau_d^{fin}$ whence $\pi_d^{-1}(V)$ is open as was to be shown. □

Proof of Theorem 2.5b: Let $F \in \mathcal{C}(\mathbb{R}^d, X)$ be 2π-periodic in its arguments. If $\phi \in \mathbb{R}^d$ then, by Definition 2.4, there exists an $N(\phi) \in \mathbb{Z}^d$ such that $\phi + 2\pi N(\phi) = \mathrm{Arg}(\pi_d(\phi))$ whence $F(\mathrm{Arg}(\pi_d(\phi))) = F(\phi + 2\pi N(\phi)) = F(\phi)$. Let $f : \mathbb{T}^d \to X$ be defined by $f(z) := F(\mathrm{Arg}(z))$. If $\phi \in \mathbb{R}^d$ then, by the above $f(\pi_d(\phi)) = F(\mathrm{Arg}(\pi_d(\phi))) = F(\phi)$ whence $f \circ \pi_d = F$. To see that f is continuous we need to show that for every open subset V of X the inverse image $f^{-1}(V)$ is open, the latter meaning that $f^{-1}(V) \in \tau_d$. Because of Theorem 2.5a we just need to show that $f^{-1}(V) \in \tau_d^{fin}$. In fact we compute $\pi_d^{-1}(f^{-1}(V)) = (f \circ \pi_d)^{-1}(V) = F^{-1}(V)$. However F is continuous whence $F^{-1}(V)$ is open so that $\pi_d^{-1}(f^{-1}(V))$ is open so that $f^{-1}(V) \in \tau_d^{fin}$ which implies that indeed f is continuous. Let $g \in \mathcal{C}(\mathbb{T}^d, X)$ such that $g \circ \pi_d = F$. To show that $f = g$ we compute $f(\pi_d(\phi)) = F(\mathrm{Arg}(\pi_d(\phi))) = g(\pi_d(\mathrm{Arg}(\pi_d(\phi)))) = g(\pi_d(\phi))$ where in the third equality we used Definition 2.4. Thus indeed $f = g$. Let $G \in \mathcal{C}(\mathbb{R}^d, X)$ be 2π-periodic in its arguments and $F(\mathrm{Arg}(z)) = G(\mathrm{Arg}(z))$ and let $\phi \in \mathbb{R}^d$. Thus, by the first claim, $F(\phi) = F(\mathrm{Arg}(\pi_d(\phi))) = G(\mathrm{Arg}(\pi_d(\phi))) = G(\phi)$ so that $F = G$.

Let conversely $h \in \mathcal{C}(\mathbb{T}^d, X)$ and $H = h \circ \pi_d$. By Theorem 2.5a π_d is continuous whence $H \in \mathcal{C}(\mathbb{R}^d, X)$. Moreover $H(\mathrm{Arg}(z)) = h(\pi_d(\mathrm{Arg}(z))) = h(z)$ where in the second equality we used Definition 2.4. □

Proof of Theorem 2.5c: Let $F \in \mathcal{C}(\mathbb{R}^k, \mathbb{T}^d)$. Since $\mathbb{R}^k$ is simply connected one can show, e.g., as in [Br, Section III.4] or [Mu, Section 8.4], that a $\hat{f} \in \mathcal{C}(\mathbb{R}^k, \mathbb{R}^d)$ exists such that $F = \pi \circ \hat{f}$.

To prove the second claim let first of all $G \in \mathcal{C}(\mathbb{R}^k, \mathbb{T}^d)$ be constant valued with value $\pi_d(0, \cdots, 0) = (1, 0, \cdots, 1, 0)^t$ and let $\hat{g} \in \mathcal{C}(\mathbb{R}^k, \mathbb{R}^d)$ be such that $G = \pi_d \circ \hat{g}$. Then, by Definition 2.4, $\hat{g}(\phi) = 2\pi N(\phi)$ where $N(\phi) \in \mathbb{Z}^d$ whence, and since $\hat{g}$ is continuous, $N(\phi)$ is independent of ϕ. To finish the proof of the second claim let $\hat{f}_1, \hat{f}_2 \in \mathcal{C}(\mathbb{R}^k, \mathbb{R}^d)$ such that $\pi_d \circ \hat{f}_1 =$

$\pi_d \circ \hat{f}_2$ whence, by Definition 2.4, $\exp(\mathcal{J}_{\hat{f}_1(\phi)})(1,0,\cdots,1,0)^t = \pi_d(\hat{f}_1(\phi)) = \pi_d(\hat{f}_2(\phi)) = \exp(\mathcal{J}_{\hat{f}_2(\phi)})(1,0,\cdots,1,0)^t$ so that $\pi_d(\hat{f}_1(\phi) - \hat{f}_2(\phi)) = \exp(\mathcal{J}_{\hat{f}_1(\phi)-\hat{f}_2(\phi)})(1,0,\cdots,1,0)^t = \exp(\mathcal{J}_{\hat{f}_1(\phi)})\exp(-\mathcal{J}_{\hat{f}_2(\phi)})(1,0,\cdots,1,0)^t = (1,0,\cdots,1,0)^t$ which implies, by the above, that setting $\hat{g} \equiv \hat{f}_1 - \hat{f}_2$ we get $\hat{g}(\phi) = 2\pi N$ with $N \in \mathbb{Z}^d$, i.e., $\hat{f}_1(\phi) = \hat{f}_2(\phi) + 2\pi N$. This proves the second claim. □

Proof of Theorem 2.5d: To prove the first claim let us first assume that $\hat{f} \in \mathrm{Fun}_d$ whence, for every $N \in \mathbb{Z}^d$ and recalling Section 2.2, there exists an $\tilde{N} \in \mathbb{Z}^d$ such that $\hat{f}(\phi + 2\pi N) = \hat{f}(\phi) + 2\pi\tilde{N}$ so that, by the periodicity of π_d, $\pi_d(\hat{f}(\phi + 2\pi N)) = \pi_d(\hat{f}(\phi) + 2\pi\tilde{N}) = \pi_d(\hat{f}(\phi))$ which implies that $\pi_d \circ \hat{f}$ is 2π-periodic in its arguments. Conversely let $\hat{f} \in \mathcal{C}(\mathbb{R}^d, \mathbb{R}^d)$ and $\pi_d \circ \hat{f}$ be 2π-periodic in its arguments then, for $N \in \mathbb{Z}^d$, $\pi_d \circ \hat{f}(\cdot + 2\pi N)$ is continuous and equal to $\pi_d \circ \hat{f}$ whence, by the Baby Lift Theorem, an $\tilde{N} \in \mathbb{Z}^d$ exists such that $\hat{f}(\phi + 2\pi N) = \hat{f}(\phi) + 2\pi\tilde{N}$ so that $\hat{f} \in \mathrm{Fun}_d$.

The second claim follows from the first claim and the definition, (2.11), of Map_d.
To prove the third claim let $\hat{f} \in \mathrm{Fun}_d$. Then, by the first claim, $\pi_d \circ \hat{f}$ is 2π-periodic in its arguments whence, by Theorem 2.5b and since $\pi_d \circ \hat{f}$ is continuous, we conclude that $\pi_d \circ \hat{f} \circ \mathrm{Arg}$ belongs to $\mathcal{C}(\mathbb{T}^d, \mathbb{T}^d)$. □

Proof of Theorem 2.5e: The proof is a multiple application of Theorem 2.5b. To prove the first claim we define the function $A : \mathbb{R}^d \to SO(3)$ by $A(z) := \hat{A}(\mathrm{Arg}(z))$. Since $\hat{A}$ is 2π-periodic in its arguments and continuous and setting $X = SO(3)$, we conclude from Theorem 2.5b that $A \in \mathcal{C}(\mathbb{T}^d, SO(3))$.

To prove the second claim we have to show that $\pi_d \circ \hat{j} \circ \mathrm{Arg}$ belongs to $\mathrm{Homeo}(\mathbb{T}^d)$ so we define the function $j : \mathbb{T}^d \to \mathbb{T}^d$ by $j := \pi_d \circ \hat{j} \circ \mathrm{Arg}$. To show that j is continuous we note by Theorem 2.5d that $\pi_d \circ \hat{j}$ is 2π-periodic in its arguments because $\hat{j} \in \mathrm{Map}_d$. Also, by Theorem 2.5a, π_d is continuous whence, and since $\hat{j}$ is continuous, $\pi_d \circ \hat{j}$ is continuous too. We thus have shown that $\pi_d \circ \hat{j}$ is 2π-periodic in its arguments and continuous whence, by Theorem 2.5b, $\pi_d \circ \hat{j} \circ \mathrm{Arg}$ is continuous, i.e., $j \in \mathcal{C}(\mathbb{T}^d, \mathbb{T}^d)$. To complete the proof that j is a homeomorphism we first need to show that it has an inverse. Since $j = \pi_d \circ \hat{j} \circ \mathrm{Arg}$, a natural candidate for the inverse of j is the function $g : \mathbb{T}^d \to \mathbb{T}^d$, defined by $g := \pi_d \circ \hat{j}^{-1} \circ \mathrm{Arg}$. To show that g indeed is the inverse of j we compute $j \circ g = \pi_d \circ \hat{j} \circ \mathrm{Arg} \circ \pi_d \circ \hat{j}^{-1} \circ \mathrm{Arg} = \pi_d \circ \hat{j} \circ \hat{j}^{-1} \circ \mathrm{Arg} = \pi_d \circ \mathrm{Arg} = id_{\mathbb{T}^d}$ where in the second equality we used Theorem 2.5b and the periodicity of $\pi_d \circ \hat{j}$ and where in the fourth equality we used Definition 2.4. Analogously we compute $g \circ j = \pi_d \circ \hat{j}^{-1} \circ \mathrm{Arg} \circ \pi_d \circ \hat{j} \circ \mathrm{Arg} = \pi_d \circ \hat{j}^{-1} \circ \hat{j} \circ \mathrm{Arg} = \pi_d \circ \mathrm{Arg} = id_{\mathbb{T}^d}$ where in the second equality we used Theorem 2.5b and the periodicity of $\pi_d \circ \hat{j}^{-1}$ and where in the fourth equality we used Definition 2.4. Thus indeed g is the inverse of j, i.e., $g = j^{-1}$. To complete the proof that j is a homeomorphism we now show that j^{-1} is continuous in the same way as we proved above that j is continuous. Thus we first note by Theorem 2.5d that $\pi_d \circ \hat{j}^{-1}$ is 2π-periodic in its arguments because $\hat{j}^{-1} \in \mathrm{Map}_d$. Also, by Theorem 2.5a, π_d is continuous whence, and since $\hat{j}^{-1}$ is continuous, $\pi_d \circ \hat{j}^{-1}$ is continuous too. We thus have shown that $\pi_d \circ \hat{j}^{-1}$ is 2π-periodic in its arguments and continuous whence, by Theorem 2.5b, $\pi_d \circ \hat{j}^{-1} \circ \mathrm{Arg}$ is continuous, i.e., $j^{-1} \in \mathcal{C}(\mathbb{T}^d, \mathbb{T}^d)$.

To prove the third claim we proceed as with our claim after Definition 2.4 so let $z = \pi_d(\phi), \mathcal{S} = S$ and let $\phi' := \hat{j}(\phi), z' := j(z)$ as well as $S' := \hat{A}(\phi)S, \mathcal{S}' := A(z)\mathcal{S}$. To show that $z' = \pi_d(\phi')$ we compute $z' = j(z) = j(\pi_d(\phi)) = (\pi_d \circ \hat{j} \circ \mathrm{Arg} \circ \pi_d)(\phi) = (\pi_d \circ \hat{j})(\phi) = \pi_d(\phi')$ where in the fourth equality we used Theorem 2.5b and the periodicity of $\pi_d \circ \hat{j}$ whence indeed $z' = \pi_d(\phi')$. To show that $\mathcal{S}' = S'$ we compute $\mathcal{S}' = A(z)\mathcal{S} = A(\pi_d(\phi))S = ((\hat{A} \circ \mathrm{Arg} \circ \pi_d)\phi)S = \hat{A}(\phi)S = S'$ where in the fourth equality we used Theorem 2.5b and the periodicity of $\hat{A}$ whence indeed $\mathcal{S}' = S'$ which completes the proof of the third claim. □

C.2 Proof of Theorem 2.6

Proof of Theorem 2.6a: We define the function $F : \mathbb{R}^d \to \mathbb{T}^d$ by $F := f \circ \pi_d$ whence, by Theorem 2.5b, F is 2π-periodic in its arguments. Thus, by Theorem 2.5c, there exists a $\hat{f} \in \mathcal{C}(\mathbb{R}^d, \mathbb{R}^d)$ such that $F = \pi_d \circ \hat{f}$ whence $f \circ \pi_d = \pi_d \circ \hat{f}$ so that, by Definition 2.4, $f = f \circ \pi_d \circ \mathrm{Arg} = \pi_d \circ \hat{f} \circ \mathrm{Arg}$ which completes the proof of the first claim. To prove the second claim let $\hat{f}_1, \hat{f}_2$ be in Fun_d and such that $\pi_d \circ \hat{f}_1 \circ \mathrm{Arg} = f = \pi_d \circ \hat{f}_2 \circ \mathrm{Arg}$. By Theorem 2.5d $\pi_d \circ \hat{f}_1$ and $\pi_d \circ \hat{f}_2$ are 2π-periodic in their arguments. Using Theorem 2.5b and the periodicity of $\pi_d \circ \hat{f}_1$ and the periodicity of $\pi_d \circ \hat{f}_2$ we compute $\pi_d \circ \hat{f}_1 = \pi_d \circ \hat{f}_1 \circ \mathrm{Arg} \circ \pi_d = \pi_d \circ \hat{f}_2 \circ \mathrm{Arg} \circ \pi_d = \pi_d \circ \hat{f}_2$ whence, by Theorem 2.5c and since $\hat{f}_1, \hat{f}_2$ are continuous, there exists indeed a constant $N \in \mathbb{Z}^d$ such that $\hat{f}_1(\phi) = \hat{f}_2(\phi) + 2\pi N$. □

Proof of Theorem 2.6b: To prove the first claim we note, by Theorem 2.6a and since $j \in \mathrm{Homeo}(\mathbb{T}^d)$, that there exists a $\hat{j}$ and $\hat{g}$ in $\mathcal{C}(\mathbb{R}^d, \mathbb{R}^d)$ such that $\pi_d \circ \hat{j}$ and $\pi_d \circ \hat{g}$ are 2π-periodic in their arguments and such that $j = \pi_d \circ \hat{j} \circ \mathrm{Arg}$ and $j^{-1} = \pi_d \circ \hat{g} \circ \mathrm{Arg}$. Thus to complete the proof of the first claim we have to show that $\hat{j} \in \mathrm{Map}_d$, i.e., that $\hat{j} \in \mathrm{Homeo}(\mathbb{R}^d)$ and that $\pi_d \circ \hat{j}$ and $\pi_d \circ \hat{j}^{-1}$ are 2π-periodic in their arguments. Of course the latter are already established whence we have just to show that $\hat{j}$ is a homeomorphism so we compute

$$id_{\mathbb{T}^d} = j \circ j^{-1} = \pi_d \circ \hat{j} \circ \mathrm{Arg} \circ \pi_d \circ \hat{g} \circ \mathrm{Arg} = \pi_d \circ \hat{j} \circ \hat{g} \circ \mathrm{Arg} \; , \tag{C.9}$$

where in the third equality we used Theorem 2.5b and the periodicity of $\pi_d \circ \hat{j}$. It follows from (C.9) that

$$\pi_d = id_{\mathbb{T}^d} \circ \pi_d = \pi_d \circ \hat{j} \circ \hat{g} \circ \mathrm{Arg} \circ \pi_d \; . \tag{C.10}$$

To get rid of the factor $\mathrm{Arg} \circ \pi_d$ on the rhs of (C.10) we note, by Theorem 2.5b and the periodicity of $\pi_d \circ \hat{j}$, that

$$j \circ \pi_d = \pi_d \circ \hat{j} \circ \mathrm{Arg} \circ \pi_d = \pi_d \circ \hat{j} \; , \tag{C.11}$$

whence, by (C.10),

$$\pi_d = j \circ \pi_d \circ \hat{g} \circ \mathrm{Arg} \circ \pi_d \; . \tag{C.12}$$

By Theorem 2.5b and the periodicity of $\pi_d \circ \hat{g}$ we conclude from (C.12) that

$$\pi_d = j \circ \pi_d \circ \hat{g} \circ \mathrm{Arg} \circ \pi_d = j \circ \pi_d \circ \hat{g} = \pi_d \circ \hat{j} \circ \hat{g} \; , \tag{C.13}$$

where in the third equality we used (C.11). By Theorem 2.5c and (C.13) a $N \in \mathbb{Z}^d$ exists such that $\hat{j} \circ \hat{g} = id_{\mathbb{T}^d} + 2\pi N$, i.e.,

$$\hat{j}(\hat{g}(\phi)) = \phi + 2\pi N \; , \tag{C.14}$$

whence $\hat{j}$ is onto $\mathbb{R}^d$. To show that $\hat{j}$ is one-one we repeat the argumentation which led us from (C.9) to (C.14) so we first compute

$$id_{\mathbb{T}^d} = j^{-1} \circ j = \pi_d \circ \hat{g} \circ \mathrm{Arg} \circ \pi_d \circ \hat{j} \circ \mathrm{Arg} = \pi_d \circ \hat{g} \circ \hat{j} \circ \mathrm{Arg} \ , \tag{C.15}$$

where in the third equality we used Theorem 2.5b and the periodicity of $\pi_d \circ \hat{g}$. It follows from (C.15) that

$$\pi_d = id_{\mathbb{T}^d} \circ \pi_d = \pi_d \circ \hat{g} \circ \hat{j} \circ \mathrm{Arg} \circ \pi_d \ . \tag{C.16}$$

To get rid of the factor $\mathrm{Arg} \circ \pi_d$ on the rhs of (C.16) we note, by Theorem 2.5b and the periodicity of $\pi_d \circ \hat{g}$, that

$$g \circ \pi_d = \pi_d \circ \hat{g} \circ \mathrm{Arg} \circ \pi_d = \pi_d \circ \hat{g} \ , \tag{C.17}$$

whence, by (C.16),

$$\pi_d = g \circ \pi_d \circ \hat{j} \circ \mathrm{Arg} \circ \pi_d \ . \tag{C.18}$$

By Theorem 2.5b and the periodicity of $\pi_d \circ \hat{g}$ we conclude from (C.18) that

$$\pi_d = g \circ \pi_d \circ \hat{j} \circ \mathrm{Arg} \circ \pi_d = g \circ \pi_d \circ \hat{j} = \pi_d \circ \hat{g} \circ \hat{j} \ , \tag{C.19}$$

where in the third equality we used (C.17). By Theorem 2.5c and (C.19) a $M \in \mathbb{Z}^d$ exists such that $\hat{g} \circ \hat{j} = id_{\mathbb{T}^d} + 2\pi M$, i.e.,

$$\hat{g}(\hat{j}(\phi)) = \phi + 2\pi M \ , \tag{C.20}$$

whence $\hat{j}$ is one-one. Thus we have shown that $\hat{j}$ is a bijection. To show that $\hat{j}$ is a homeomorphism we need to show that its inverse is continuous so we define the function $\hat{h} \in \mathcal{C}(\mathbb{R}^d, \mathbb{R}^d)$ by $\hat{h}(\phi) := \hat{g}(\phi) - 2\pi M$ and compute, by (C.20), $\hat{h}(\hat{j}(\phi)) = \hat{g}(\hat{j}(\phi)) - 2\pi M = \phi + 2\pi M - 2\pi M = \phi$ whence $\hat{h} \circ \hat{j} = id_{\mathbb{R}^d}$ so that, and since $\hat{j}$ is a bijection, $\hat{h}$ is the inverse of $\hat{j}$. Thus the continuous function $\hat{h}$ is the inverse of the continuous function $\hat{j}$ whence $\hat{j} \in \mathrm{Homeo}(\mathbb{R}^d)$ so that indeed $\hat{j} \in \mathrm{Map}_d$.

To prove the second claim let $\hat{j} \in \mathrm{Map}_d$ such that $j = \pi_d \circ \hat{j} \circ \mathrm{Arg}$ and let us define, for fixed but arbitrary N, the function $\hat{g} \in \mathcal{C}(\mathbb{R}^d, \mathbb{R}^d)$ by $\hat{g}(\phi) := \hat{j}(\phi) + 2\pi N$. Since π_d is 2π-periodic in its arguments it is clear that $j = \pi_d \circ \hat{g} \circ \mathrm{Arg}$ whence it remains to be shown that $\hat{g} \in \mathrm{Map}_d$. We first note that $\hat{j} \in \mathrm{Map}_d$ whence $\pi_d \circ \hat{j}$ is 2π-periodic in its arguments. Also π_d is 2π-periodic in its arguments whence $\pi_d \circ \hat{g}$ is 2π-periodic in its arguments. To show that $\pi_d \circ \hat{g}^{-1}$ is 2π-periodic in its arguments we have to find $\hat{g}^{-1}$. We again note that $\hat{j} \in \mathrm{Map}_d$ whence $\hat{j}$ is a bijection which implies that $\hat{g}$ is a bijection so that $\hat{g}^{-1}$ exists. To find $\hat{g}^{-1}$ we compute $\hat{g}(\hat{j}^{-1}(\phi)) = \hat{j}(\hat{j}^{-1}(\phi)) + 2\pi N = \phi + 2\pi N$ which suggests to define the function $\hat{f} \in \mathcal{C}(\mathbb{R}^d, \mathbb{R}^d)$ by $\hat{f}(\phi) := \hat{j}^{-1}(\phi - 2\pi N)$ and we readily obtain $\hat{g}(\hat{f}(\phi)) = \hat{j}(\hat{f}(\phi)) + 2\pi N = \hat{j}(\hat{j}^{-1}(\phi - 2\pi N)) + 2\pi N = \phi$ whence $\hat{g} \circ \hat{f} = id_{\mathbb{R}^d}$ so that, and since $\hat{g}$ is a bijection, $\hat{f}$ is the inverse of $\hat{g}$. To show that $\pi_d \circ \hat{g}^{-1}$ is 2π-periodic in its arguments we note again that $\hat{j} \in \mathrm{Map}_d$ whence $\pi_d \circ \hat{j}^{-1}$ is 2π-periodic in its arguments. On the other hand, $\pi_d(\hat{g}^{-1}(\phi)) = \pi_d(\hat{f}(\phi)) = \pi_d(\hat{j}^{-1}(\phi - 2\pi N)) = \pi_d(\hat{j}^{-1}(\phi))$ whence $\pi_d \circ \hat{g}^{-1} = \pi_d \circ \hat{j}^{-1}$ so that $\pi_d \circ \hat{g}^{-1}$ is 2π-periodic in its arguments since $\pi_d \circ \hat{j}^{-1}$ is 2π-periodic in its arguments.

To show that $\hat{g}$ is a homeomorphism we note that the continuous function $\hat{f}$ is the inverse of the continuous function $\hat{g}$ whence $\hat{g} \in \mathrm{Homeo}(\mathbb{R}^d)$.

The third claim follows readily from Theorem 2.6a.

To prove the fourth claim let $A \in \mathcal{C}(\mathbb{T}^d, SO(3))$ and let us define $\hat{A} \in \mathcal{C}(\mathbb{R}^d, SO(3))$ by $\hat{A} := A \circ \pi_d$. Clearly $\hat{A}$ is 2π-periodic in its arguments and, by Definition 2.4, $\hat{A} \circ \mathrm{Arg} = A \circ \pi_d \circ \mathrm{Arg} = A$. If furthermore $\hat{B} \in \mathcal{C}(\mathbb{R}^d, SO(3))$ is 2π-periodic in its arguments and $\hat{B} \circ \mathrm{Arg} = A$ then, by Theorem 2.5b, $\hat{A} = \hat{B}$. □

C.3 Proof of Theorem 7.6

We first state and prove Theorem C.2 since we need its part c) to prove Theorem 7.6.

Theorem C.2 *Let $\mathcal{P}[\omega]$ be topologically transitive, i.e., $(1, \omega)$ be nonresonant and let $A \in \mathcal{C}(\mathbb{T}^d, SO(3))$ be of the form (7.22), i.e.,*

$$A(z) = \exp(\mathcal{J}(N \cdot \phi)) ,$$

where $\pi_d(\phi) = z$ and where $N \in \mathbb{Z}^d$. Then the following hold.
a) Let T be a transfer field from $(\mathcal{P}[\omega], A)$ to $(\mathcal{P}[\omega], A_{d,\nu})$ and let either T or $T\mathcal{J}'$ be $SO(2)$-valued. Then $N = 0$.
Remark: Note, by Definition 7.2 and (7.21), that ν is a spin tune and T is a uniform IFF of $(\mathcal{P}[\omega], A)$.
b) Let $\nu \in [0, 1)$ and let $(\mathcal{P}[\omega], A)$ and $(\mathcal{P}[\omega], A_{d,\nu})$ be equivalent. Then a transfer field T from $(\mathcal{P}[\omega], A)$ to $(\mathcal{P}[\omega], A_{d,\nu})$ exists such that either T or $T\mathcal{J}'$ is $SO(2)$-valued.
Remark: Note, by (7.21), that T is a uniform IFF of $(\mathcal{P}[\omega], A)$.
c) Let $\nu \in [0, 1)$ and let $(\mathcal{P}[\omega], A)$ and $(\mathcal{P}[\omega], A_{d,\nu})$ be equivalent. Then $N = 0$.

Proof of Theorem C.2a: We first consider the case where T is $SO(2)$-valued. Thus, by Theorem 6.3b,

$$T(z) = \exp(\mathcal{J}[M \cdot \phi + 2\pi g(z)]) , \tag{C.21}$$

where $\pi_d(\phi) = z$ and where $M \in \mathbb{Z}^d$ and $g \in \mathcal{C}(\mathbb{T}^d, \mathbb{R})$. Using (2.25),(7.4),(7.22), (C.21) and the fact that T is a transfer field from $(\mathcal{P}[\omega], A)$ to $(\mathcal{P}[\omega], A_{d,\nu})$ we compute

$$\begin{aligned} &\exp\Big(-\mathcal{J}[M \cdot (\phi + 2\pi\omega) + 2\pi g(\pi_d(\phi + 2\pi\omega))]\Big) \exp(\mathcal{J}(N \cdot \phi)) \\ &\quad \exp\Big(\mathcal{J}[M \cdot \phi + 2\pi g(\pi_d(\phi))]\Big) = T^t(\pi_d(\phi + 2\pi\omega)) A(\pi_d(\phi)) T(\pi_d(\phi)) \\ &\quad = T^t(\mathcal{P}[\omega](z)) A(z) T(z) = A_{d,\nu} = \exp(2\pi\nu\mathcal{J}) , \end{aligned}$$

i.e.,

$$\exp\Big(\mathcal{J}[-2\pi(M \cdot \omega) - 2\pi g(\pi_d(\phi + 2\pi\omega)) + N \cdot \phi + 2\pi g(\pi_d(\phi)) - 2\pi\nu]\Big) = I_{3\times3} ,$$

whence, by Theorem 6.3b, an integer n exists such that, for $\phi \in \mathbb{R}^d$,

$$2\pi\Big(-M\cdot\omega - g(\pi_d(\phi + 2\pi\omega)) + g(\pi_d(\phi)) - \nu \Big) = -N\cdot\phi + 2\pi n \ . \tag{C.22}$$

Since $g\circ\pi_d$ is 2π-periodic in its arguments, the lhs of (C.22) is 2π-periodic in all components of ϕ whence, by (C.22), the rhs of (C.22) is 2π-periodic in all components of ϕ so that $N = 0$.

We now consider the case when $T\mathcal{J}'$ is $SO(2)$-valued so we define $R \in \mathcal{C}(\mathbb{T}^d, SO(3))$ by $R := T\mathcal{J}'$. Thus, by Theorem 6.3b, $M \in \mathbb{Z}^d$ and $g \in \mathcal{C}(\mathbb{T}^d, \mathbb{R})$ exist such that, when $\pi_d(\phi) = z$, $R(z) = \exp(\mathcal{J}[M\cdot\phi + 2\pi g(z)])$ whence

$$T(z) = T(z)\mathcal{J}'\mathcal{J}' = R(z)\mathcal{J}' = \exp(\mathcal{J}[M\cdot\phi + 2\pi g(z)])\mathcal{J}' \ , \tag{C.23}$$

where in the first equality of (C.23) we used (B.32). Using (2.25), (7.4),(7.22), (B.33),(C.23) and Definition 4.1 and the fact that T is a transfer field from $(\mathcal{P}[\omega], A)$ to $(\mathcal{P}[\omega], A_{d,\nu})$ we compute

$$\begin{aligned}
&\exp\Big(\mathcal{J}[2\pi(M\cdot\omega) + 2\pi g(\pi_d(\phi+2\pi\omega)) - N\cdot\phi - 2\pi g(\pi_d(\phi))]\Big) \\
&= \mathcal{J}'\exp\Big(\mathcal{J}[-2\pi(M\cdot\omega) - 2\pi g(\pi_d(\phi+2\pi\omega)) + N\cdot\phi + 2\pi g(\pi_d(\phi))]\Big)\mathcal{J}' \\
&\quad = \mathcal{J}'\exp\Big(-\mathcal{J}[M\cdot(\phi+2\pi\omega) + 2\pi g(\pi_d(\phi+2\pi\omega))]\Big)\exp(\mathcal{J}(N\cdot\phi)) \\
&\quad \exp\Big(\mathcal{J}[M\cdot\phi + 2\pi g(\pi_d(\phi))]\Big)\mathcal{J}' = T^t(\pi_d(\phi+2\pi\omega))A(\pi_d(\phi))T(\pi_d(\phi)) \\
&\quad = T^t(\mathcal{P}[\omega](z))A(z)T(z) = A_{d,\nu} = \exp(2\pi\nu\mathcal{J}) \ ,
\end{aligned}$$

i.e.,

$$\exp\Big(\mathcal{J}[2\pi(M\cdot\omega) + 2\pi g(\pi_d(\phi+2\pi\omega)) - N\cdot\phi - 2\pi g(\pi_d(\phi)) - 2\pi\nu]\Big) = I_{3\times 3} \ ,$$

whence, by Theorem 6.3b, an integer n exists such that, for $\phi \in \mathbb{R}^d$,

$$2\pi\Big(M\cdot\omega + g(\pi_d(\phi + 2\pi\omega)) - g(\pi_d(\phi)) - \nu \Big) = N\cdot\phi + 2\pi n \ . \tag{C.24}$$

Since $g\circ\pi_d$ is 2π-periodic in its arguments, the lhs of (C.24) is 2π-periodic in all components of ϕ whence, by (C.24), the rhs of (C.24) is 2π-periodic in all components of ϕ so that $N = 0$.
$\square$

Proof of Theorem C.2b: By Definition 4.2 we have a transfer field T' from $(\mathcal{P}[\omega], A)$ to $(\mathcal{P}[\omega], A_{d,\nu})$ whence our task is to construct, out of T', a transfer field T from $(\mathcal{P}[\omega], A)$ to $(\mathcal{P}[\omega], A_{d,\nu})$ such that either T or $T\mathcal{J}'$ is $SO(2)$-valued. Since T' is a transfer field from $(\mathcal{P}[\omega], A)$ to $(\mathcal{P}[\omega], A_{d,\nu})$ it follows from (7.4) and Definition 4.1 that

$$A(z)T'(z) = T'(\mathcal{P}[\omega](z))\exp(2\pi\nu\mathcal{J}) \ . \tag{C.25}$$

Defining $t \in \mathcal{C}(\mathbb{T}^d, \mathbb{R}^3)$ by $t(z) := T'(z)(0,0,1)^t$, we conclude from (C.25) and (6.7) that

$$\begin{aligned} A(z)t(z) &= A(z)T'(z)(0,0,1)^t = T'(\mathcal{P}[\omega](z))\exp(2\pi\nu\mathcal{J})(0,0,1)^t = T'(\mathcal{P}[\omega](z))(0,0,1)^t \\ &= t(\mathcal{P}[\omega](z)) \; . \end{aligned} \tag{C.26}$$

Note that $|t(z)| = 1$ because $T'(z) \in SO(3)$. Because $t \in \mathcal{C}(\mathbb{T}^d, \mathbb{R}^3)$, and for $j = 1, 2, 3$, the j-th component t_j of t belongs to $\mathcal{C}(\mathbb{T}^d, \mathbb{R})$. Since A is $SO(2)$-valued it follows from (C.26),(6.5) and (6.7) that

$$t_3(z) = t_3(\mathcal{P}[\omega](z)) \; . \tag{C.27}$$

Because $\mathcal{P}[\omega]$ is topologically transitive we conclude from (C.27) and Theorem 3.3a that t_3 is a constant function so that, since $|t_3| \le |t| = 1$, only the following three cases can occur: Case (i) where, for all $z \in \mathbb{T}^d$, $t_3(z) = 1$, Case (ii) where, for all $z \in \mathbb{T}^d$, $t_3(z) = -1$, Case (iii) where, for all $z \in \mathbb{T}^d$, $|t_3(z)| < 1$.

We first consider Case (i). Since $|t| = 1$, in the present case $t_1(z) = t_2(z) = 0$ whence $t = t_3(0,0,1)^t = (0,0,1)^t$, i.e., $T'(0,0,1)^t = t = (0,0,1)^t$. Due to Theorem 6.2a, we obtain that T' is $SO(2)$-valued whence T' is an $SO(2)$-valued transfer field from $(\mathcal{P}[\omega], A)$ to $(\mathcal{P}[\omega], A_{d,\nu})$.

We now consider Case (ii). Since $|t| = 1$, in the present case $t_1(z) = t_2(z) = 0$ whence $t = t_3(0,0,1)^t = (0,0,-1)^t$, i.e., $T'(0,0,1)^t = t = (0,0,-1)^t$ so that, by (B.32), $T'\mathcal{J}'(0,0,1)^t = (0,0,1)^t$ which implies, by Theorem 6.2a, that $T'\mathcal{J}'$ is $SO(2)$-valued whence T' is a transfer field from $(\mathcal{P}[\omega], A)$ to $(\mathcal{P}[\omega], A_{d,\nu})$ for which $T'\mathcal{J}'$ is $SO(2)$-valued.

We now consider Case (iii). Note that if T_1 or $T_1\mathcal{J}'$ would be $SO(2)$-valued then, by (6.5),(6.7), (B.32), the third column t of T_1 would either be $(0,0,1)^t$-valued or $(0,0,-1)^t$-valued whence $|t_3|$ would be 1-valued which of course is impossible in the present case. Thus, unlike to Cases (i) and (ii), the transfer field T we are looking for is different from T_1 so we have to do some work. In fact this work is rather easy since the (33)-matrix element t_3 of T_1 is a constant function which allows us to factorize T_1 into three simple and continuous functions in (C.32) below. To accomplish all that we first note, because $|t_3| < 1$, that the function $t_0 \in \mathcal{C}(\mathbb{T}^d, \mathbb{R})$, defined by $t_0(z) := \sqrt{t_1^2(z) + t_2^2(z)} = \sqrt{1 - t_3^2(z)}$ has only positive values. Note that t_0 is a constant function because t_3 is a constant function. Since t_0 has only nonzero values we can define $T_2 : \mathbb{T}^d \to SO(3)$ by

$$T_2(z) := \begin{pmatrix} \frac{t_1(z)}{t_0(z)} & -\frac{t_2(z)}{t_0(z)} & 0 \\ \frac{t_2(z)}{t_0(z)} & \frac{t_1(z)}{t_0(z)} & 0 \\ 0 & 0 & 1 \end{pmatrix} \; . \tag{C.28}$$

Note, by (C.28) and Theorem 6.2a, that T_2 is $SO(2)$-valued which implies that $T_2 \in \mathcal{C}(\mathbb{T}^d, SO(3))$. It also follows from (C.28) that

$$T_2^t T_1(0,0,1)^t = T_2^t t = T_2^t(t_1, t_2, t_3)^t = ((t_1^2 + t_2^2)/t_0, 0, t_3)^t = t_0(1,0,0)^t + t_3(0,0,1)^t \; . \tag{C.29}$$

Since t_0 and t_3 are constant functions and since $|t_0(1,0,0)^t + t_3(0,0,1)^t| = 1$ there exists a C in $SO(3)$ such that $C(0,0,1)^t = t_0(1,0,0)^t + t_3(0,0,1)^t$ whence, by (C.29),

$$C^t T_2^t T_1(0,0,1)^t = C^t(t_0(1,0,0)^t + t_3(0,0,1)^t) = (0,0,1)^t \; . \tag{C.30}$$

We thus define the function $T_3 \in \mathcal{C}(\mathbb{T}^d, SO(3))$ by

$$T_3(z) := C^t T_2^t(z) T_1(z) \; , \tag{C.31}$$

and observe, by (C.30), that $T_3(z)(1,0,0)^t = (1,0,0)^t$ whence, by Theorem 6.2a, T_3 is $SO(2)$-valued. With (C.31), we get the following factorization of T_1:

$$T_1(z) = T_2(z) C T_3(z) \; . \tag{C.32}$$

To obtain T from T_1 we first generalize T_1 by defining, for fixed but arbitrary $B \in SO(3)$, the function $T_B \in \mathcal{C}(\mathbb{T}^d, SO(3))$ by

$$T_B(z) := T_2(z) B T_3(z) \; . \tag{C.33}$$

Indeed $T_1 = T_C$ whence T_1 is a special case of T_B as can be easily checked by (C.32),(C.33). We now show that every T_B is a transfer field from $(\mathcal{P}[\omega], A)$ to $(\mathcal{P}[\omega], A_{d,\nu})$ whence we have to compute $T_B^t(\mathcal{P}[\omega](z))A(z)T_B(z)$. We first compute $T_2^t(\mathcal{P}[\omega](z))A(z)T_2(z)$ and we note, by (C.26),(C.29),

$$\begin{aligned} A(z)T_2(z)[t_0(1,0,0)^t + t_3(0,0,1)^t] &= A(z)T_2(z)[t_0(z)(1,0,0)^t + t_3(z)(0,0,1)^t] = A(z)t(z) \\ &= t(\mathcal{P}[\omega](z)) = T_2(\mathcal{P}[\omega](z))[t_0(\mathcal{P}[\omega](z))(1,0,0)^t + t_3(\mathcal{P}[\omega](z))(0,0,1)^t] \\ &= T_2(\mathcal{P}[\omega](z))[t_0(1,0,0)^t + t_3(0,0,1)^t] \; , \end{aligned} \tag{C.34}$$

where in the first and fifth equalities we used that t_0 and t_3 are constant functions. Multiplying (C.34) from the left by $T_2^t(\mathcal{P}[\omega](z))$ we get

$$T_2^t(\mathcal{P}[\omega](z))A(z)T_2(z)[t_0(1,0,0)^t + t_3(1,0,0)^t] = [t_0(1,0,0)^t + t_3(1,0,0)^t] \; . \tag{C.35}$$

Since $A(z), T_2(z)$ and $T_2^t(\mathcal{P}[\omega](z))$ belong to $SO(2)$, their product belongs to $SO(2)$ whence, by Theorem 6.2a,

$$T_2^t(\mathcal{P}[\omega](z))A(z)T_2(z)t_3(0,0,1)^t = t_3 T_2^t(\mathcal{P}[\omega](z))A(z)T_2(z)(0,0,1)^t = t_3(0,0,1)^t \; ,$$

so that, by (C.35),

$$t_0 T_2^t(\mathcal{P}[\omega](z))A(z)T_2(z)(1,0,0)^t = T_2^t(\mathcal{P}[\omega](z))A(z)T_2(z)t_0(1,0,0)^t = t_0(1,0,0)^t \; , \tag{C.36}$$

which implies, since t_0 is nonzero, that

$$T_2^t(\mathcal{P}[\omega](z))A(z)T_2(z)(1,0,0)^t = (1,0,0)^t \; . \tag{C.37}$$

Recalling from the remarks after (C.35), that $T_2^t(\mathcal{P}[\omega](z))A(z)T_2(z)$ belongs to $SO(2)$, we conclude from Theorem 6.2a that $T_2^t(\mathcal{P}[\omega](z))A(z)T_2(z)(0,0,1)^t = (0,0,1)^t$ whence, by (C.37),

$$T_2^t(\mathcal{P}[\omega](z))A(z)T_2(z) = I_{3\times 3} \; . \tag{C.38}$$

It follows from (C.33),(C.38) that

$$\begin{aligned} T_B^t(\mathcal{P}[\omega](z))A(z)T_B(z) &= T_3^t(\mathcal{P}[\omega](z))B^t T_2^t(\mathcal{P}[\omega](z))A(z)T_2(z)BT_3(z) \\ &= T_3^t(\mathcal{P}[\omega](z))B^t I_{3\times 3} B T_3(z) = T_3^t(\mathcal{P}[\omega](z))T_3(z) \; . \end{aligned} \tag{C.39}$$

We recall from the remarks after (C.33) that $T_1 = T_C$ whence in that case (C.39) results in

$$A_{d,\nu}(z) = T_1^t(\mathcal{P}[\omega](z))A(z)T_1(z) = T_C^t(\mathcal{P}[\omega](z))A(z)T_C(z) = T_3^t(\mathcal{P}[\omega](z))T_3(z) \; , \quad \text{(C.40)}$$

where in the first equality of (C.40) we used that T_1 is a transfer field T from $(\mathcal{P}[\omega], A)$ to $(\mathcal{P}[\omega], A_{d,\nu})$. It follows from (C.39),(C.40) that, for every $B \in SO(3)$,

$$A_{d,\nu}(z) = T_B^t(\mathcal{P}[\omega](z))A(z)T_B(z) \; , \quad \text{(C.41)}$$

whence indeed every T_B is a transfer field from $(\mathcal{P}[\omega], A)$ to $(\mathcal{P}[\omega], A_{d,\nu})$. Thus our task of finding an $SO(2)$-valued transfer field T from $(\mathcal{P}[\omega], A)$ to $(\mathcal{P}[\omega], A_{d,\nu})$ boils down to find a $B \in SO(3)$ such that T_B is $SO(2)$-valued. In fact since T_2 and T_3 are $SO(2)$-valued we pick $B = I_{3\times3}$ and observe, by (C.33), that $T_{I_{3\times3}}$ is $SO(2)$-valued whence $T := T_{I_{3\times3}} = T_2T_3$ is an $SO(2)$-valued transfer field T from $(\mathcal{P}[\omega], A)$ to $(\mathcal{P}[\omega], A_{d,\nu})$. $\square$

Proof of Theorem C.2c: Since $(\mathcal{P}[\omega], A)$ and $(\mathcal{P}[\omega], A_{d,\nu})$ are equivalent we conclude from Theorem C.2b that a transfer field T from $(\mathcal{P}[\omega], A)$ to $(\mathcal{P}[\omega], A_{d,\nu})$ exists such that either T or $T\mathcal{J}'$ is $SO(2)$-valued which implies, by Theorem C.2a, that $N = 0$. $\square$

Proof of Theorem 7.6: It is clear, by Definition 6.1, that $T = A_{d,0}$ is an IFF of $(\mathcal{P}[\omega], A)$ whence, by Theorem 6.2c, $(\mathcal{P}[\omega], A)$ has an ISF (recall from the remarks after (3.4) that $A_{d,0}(z) = I_{3\times3}$). To prove the final claim let first of all $N = 0$. Then $A = A_{d,0}$ whence, by Definition 7.2, $(\mathcal{P}[\omega], A)$ has spin tunes. Let now $N \neq 0$. Then $(\mathcal{P}[\omega], A)$ has no uniform IFF as follows. In fact if a uniform IFF would exist then, by (7.21), a $\nu \in [0, 1)$ and a transfer field T from $(\mathcal{P}[\omega], A)$ to $(\mathcal{P}[\omega], A_{d,\nu})$ would exist whence, by Definition 4.2, $(\mathcal{P}[\omega], A)$ and $(\mathcal{P}[\omega], A_{d,\nu})$ would be equivalent which, by Theorem C.2c above, would imply that $N = 0$, a contradiction. Thus if $N \neq 0$. Then $(\mathcal{P}[\omega], A)$ has no uniform IFF whence, by Remark 6 in Chapter 7, $(\mathcal{P}[\omega], A)$ has no spin tune. $\square$

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