SHARP BOUNDS FOR T-HAAR MULTIPLIERS ON L^2

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Abstract. We show that if a weight $w \in C^d_{2t}$ and there is $q > 1$ such that $w^{2t} \in A^d_q$, then the L^2-norm of the t-Haar multiplier of complexity (m, n) associated to w depends on the square root of the C^d_{2t}-characteristic of w times the square root A^d_q-characteristic of w^{2t} times a constant that depends polynomially on the complexity. In particular, if $w \in C^d_{2t} \cap A^d_\infty$ then $w^{2t} \in A^d_q$ for some $q > 1$.

1. Introduction

Recently Tuomas Hytönen settled the A_2-*conjecture* [H]: for *all* Calderón-Zygmund integral singular operators T in $\mathbb{R}^N$, weights $w \in A_p$, there is $C_{p,N,T} > 0$ such that,

$$\|Tf\|_{L^p(w)} \leq C_{p,N,T}[w]_{A_p}^{\max\{1,\frac{1}{p-1}\}}\|f\|_{L^p(w)}.$$

In his proof he developed and used a representation valid for *any* Calderón-Zygmund operator as an average of Haar shift operators of *arbitrary complexity*, paraproducts and their adjoints. See [L1, P4] for surveys of the A_2-conjecture. An important and hard part of the proof was to obtain bounds for Haar shifts operators that depended linearly in the A_2-characteristic and at most polynomially in the complexity.

In this paper we show that if a weight $w \in C^d_{2t} \cap A^d_\infty$, then the L^2-norm of the t-Haar multiplier of complexity (m, n) associated to w depends on the square root of the C^d_{2t}-characteristic of w times the square root A^d_q-characteristic of w^{2t} for some $q > 1$ depending on $t \in \mathbb{R}$ times a constant that depends polynomially on the complexity.

For $t \in \mathbb{R}$, $m, n \in \mathbb{N}$, and a weight w, the *t-Haar multiplier of complexity* (m, n) was introduced in [MoP], and is defined formally by

$$T^{m,n}_{t,w}f(x) = \sum_{L\in\mathcal{D}} \sum_{I\in\mathcal{D}_m(L), J\in\mathcal{D}_n(L)} c^L_{I,J} \frac{w^t(x)}{(m_L w)^t}\langle f, h_I\rangle h_J(x),$$

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where $|c^L_{I,J}| \leq \sqrt{|I|\,|J|}/|L|$, $\mathcal{D}$ denotes the dyadic intervals, $|I|$ the length of interval I, $\mathcal{D}_m(L)$ denotes the dyadic subintervals of L of length $2^{-m}|L|$, h_I is a Haar function associated to $|I|$, and $\langle f, g\rangle$ denotes the L^2-inner product.

When $(m,n) = (0,0)$ we denote the corresponding Haar multiplier by T^t_w, and, if in adittion $t = 1$, simply T_w. The Haar multipliers T_w are closely related to the resolvent of the dyadic paraproduct [P1], and appeared in the study of Sobolev spaces on Lipschitz curves [P3].

A necessary condition for the boundedness of $T^{m,n}_{w,t}$ on $L^2(\mathbb{R})$, when $c^L_{I,J} = \sqrt{|I|\,|J|}/|L|$, is that $w \in C^d_{2t}$, see [MoP], that is,

$$[w]_{C^d_{2t}} := \sup_{I\in\mathcal{D}} \Big(\frac{1}{|I|}\int_I w^{2t}(x)dx\Big)\Big(\frac{1}{|I|}\int_I w(x)dx\Big)^{-2t} < \infty.$$

This condition is sufficient for $t < 0$ and $t \geq 1/2$ and for all t-Haar multipliers $T^{m,n}_{w,t}$. Notice that for $0 \leq t < 1/2$ the condition C^d_{2t} is always fulfilled; in this case, boundedness of $T^{m,n}_{w,t}$ is known when $w \in A^d_\infty$ [MoP, KP]. The first author showed in [Be, Chapter 5], that if $w \in C^d_{2t}$ and $w^{2t} \in A^d_q$ then the L^2-norm of T^t_w, is bounded by a constant times $[w]^{1/2}_{C_{2t}}[w^{2t}]^{1/2}_{A^d_q}$. Here we present a different proof of this result that holds for t-Haar multipliers of complexity (m,n) with polynomial dependence on the complexity.

Theorem 1.1. *Let $w \in C^d_{2t}$ and assume there is $q > 1$ such that $w^{2t} \in A^d_q$, then there is a constant $C_q > 0$ depending only on q, such that*

$$\|T^{m,n}_{t,w} f\|_2 \leq C_q(m+n+2)^3[w]^{\frac12}_{C^d_{2t}}[w^{2t}]^{\frac12}_{A^d_q}\|f\|_2.$$

When $w^{2t} \in A^d_2$, this was proved in [MoP].

Using known properties of weights we can replace the condition $w^{2t} \in A^d_q$, by what may seem to be a more natural condition $w \in C^d_{2t} \cap A^d_\infty$.

Theorem 1.2. *Let $w \in C^d_{2t} \cap A^d_\infty$, then*

(i) *if $0 \leq 2t < 1$, there is $q > 1$ such that $w \in A^d_q$, then $w^{2t} \in A^d_q$, and*

$$\|T^{m,n}_{t,w} f\|_2 \leq C_q(m+n+2)^3[w^{2t}]^{\frac12}_{A^d_q}\|f\|_2 \leq C_q(m+n+2)^3[w]^t_{A^d_q}\|f\|_2.$$

(ii) *If $2t \geq 1$ and $w \in A^d_p$ then for $q = 2t(p-1)+1$, $w^{2t} \in A^d_q$, and*

$$\|T^{m,n}_{t,w} f\|_2 \leq C_q(m+n+2)^3[w]^{\frac12}_{C^d_{2t}}[w^{2t}]^{\frac12}_{A^d_q}\|f\|_2 \leq C_p(m+n+2)^3[w]_{C^d_{2t}}[w]_{A^d_p}\|f\|_2.$$

(iii) *If $t<0$ then for $q=1-2t$, $w^{2t}\in A^d_q$, and the bound becomes linear in the C^d_{2t} characteristic of w,*

$$\|T^{m,n}_{t,w}f\|_2 \le C(m+n+2)^3[w]_{C^d_{2t}}\|f\|_2.$$

The result was known to be optimal when $t=\pm 1/2$ [Be, P2]. The bound in (ii) is not optimal since for $t=1$, the L^2 norm of T_w is bounded by a constant times $[w]_{C^d_2}D(w)$, where $D(w)$ is the doubling constant of w, see [P2]. Here we get the larger norm $C[w]_{C^d_2}[w]_{A^d_p}$.

To prove this theorem we modify the argument in [MoP] that works when $w\in A^d_2$ ($p=2$). In particular we need a couple of new A_p-weight lemmas that are proved using Bellman function techniques: the A_p-Little Lemma, and the $\alpha\beta$-Lemma.

A few open questions remain. In case (i) $0<2t<1$, is $w^{2t}\in A^d_\infty$ a necessary condition for the boundedness of $T^{m,n}_{tw}$? Here we show is sufficient. Is it possible to get an estimate independent of $q>1$ such that $w^{2t}\in A^d_q$? More specifically, can we replace $C_q[w^{2t}]^{1/2}_{A^d_q}$ by $C[w^{2t}]_{A^d_\infty}$? or even better by $CD(w)$? Similarly in case (ii).

The paper is organized as follows. In Section 2 we provide the basic definitions and basic results that are used through-out this paper. In Section 3 we prove the lemmas that are essential for the main result. In Section 4 we prove the main estimate for the t-Haar multipliers with complexity (m,n). In the Appendix we prove the A_p-Little Lemma.

2. Preliminaries

2.1. **Weights, maximal function and dyadic intervals.** A *weight* w is a locally integrable function in $\mathbb{R}$ positive almost everywhere. The w-measure of a measurable set E, denoted by $w(E)$, is $w(E)=\int_E w(x)dx$. For a measure σ, $\sigma(E)=\int_E d\sigma$, and $|E|$ stands for the Lebesgue measure of E. We define $m^\sigma_E f$ to be the integral average of f on E, with respect to σ,

$$m^\sigma_E f := \frac{1}{\sigma(E)}\int_E f(x)d\sigma.$$

When $dx=d\sigma$ we simply write $m_E f$, when $d\sigma = v\,dx$ we write $m^v_E f$.

Given a weight w, a measurable function $f:\mathbb{R}^N\to\mathbb{C}$ is in $L^p(w)$ if and only if $\|f\|_{L^p(w)} := \left(\int_{\mathbb{R}}|f(x)|^p w(x)dx\right)^{1/p}<\infty$.

For a weight v we define the *weighted maximal function of f* by

$$(M_v f)(x) = \sup_{I:x\in I} m^v_I|f|$$

where I is a cube in $\mathbb{R}^N$ with sides parallel to the axis. The operator M_v is bounded in $L^p(v)$ for all $p > 1$ and furthermore

$$\|M_v f\|_{L^p(v)} \leq Cp'\|f\|_{L^q(v)}, \tag{2.1}$$

where p' is the dual exponent of p, that is $1/p + 1/p' = 1$. A proof of this fact can be found in [CrMPz1]. When $v = 1$, M_v is the usual Hardy-Littlewood maximal function, which we will denote by M. It is well-known that M is bounded in $L^p(w)$ if and only if $w \in A_p$ [Mu].

The collection of all *dyadic intervals*, $\mathcal{D}$, is given by: $\mathcal{D} = \cup_{n\in\mathbb{Z}}\mathcal{D}_n$, where $\mathcal{D}_n := \{I \subset \mathbb{R} \ : \ I = [k2^{-n}, (k+1)2^{-n}), \ k \in \mathbb{Z}\}$. For a dyadic interval L , let $\mathcal{D}(L)$ be the collection of its dyadic subintervals, $\mathcal{D}(L) := \{I \subset L \ : \ I \in \mathcal{D}\}$, and let $\mathcal{D}_n(L)$ be the n^{th}-generation of dyadic subintervals of L, $\mathcal{D}_n(L) := \{I \in \mathcal{D}(L) \ : \ |I| = 2^{-n}|L|\}$.

For every dyadic interval $I \in \mathcal{D}_n$ there is exactly one $\widehat{I} \in \mathcal{D}_{n-1}$, such that $I \subset \widehat{I}$, $\widehat{I}$ is called the parent of I. Each dyadic interval I in $\mathcal{D}_n$ has two children in $\mathcal{D}_{n+1}$, the right and left halves, denoted I_+ and I_- respectively.

A weight w is *dyadic doubling* if $w(\widehat{I})/w(I) \leq C$ for all $I \in \mathcal{D}$. The smallest constant C is called the doubling constant of w and is denoted by $D(w)$. Note that $D(w) \geq 2$, and that in fact the ratio between the length of a child and the length of its parent is comparable to one, more precisely, $D(w)^{-1} \leq w(I)/w(\widehat{I}) \leq 1 - D(w)^{-1}$.

2.2. **Dyadic A_p^d, reverse Hölder RH_p^d and C_s^d classes.** A weight w is said to belong to the *dyadic Muckenhoupt A_p^d-class* if and only if

$$[w]_{A_p^d} := \sup_{I\in\mathcal{D}}(m_I w)(m_I w^{\frac{-1}{p-1}})^{p-1} < \infty, \qquad \text{for} \quad 1 < p < \infty,$$

where $[w]_{A_p^d}$ is called the A_p^d-characteristic of the weight. If a weight is in A_p^d then it is dyadic doubling. These classes are nested, $A_p^d \subset A_q^d$ for all $p \leq q$. The class A_∞^d is defined by $A_\infty^d := \bigcup_{p>1} A_p^d$.

A weight w is said to belong to the *dyadic reverse Hölder RH_p^d-class* if and only if

$$[w]_{RH_p^d} := \sup_{I\in\mathcal{D}}(m_I w^p)^{\frac{1}{p}}(m_I w)^{-1} < \infty, \qquad \text{for} \quad 1 < p < \infty,$$

where $[w]_{RH_p^d}$ is called the RH_p^d-characteristic of the weight. If a weight is in RH_p^d then it is not necessarily dyadic doubling (in the non-dyadic setting reverse Hölder weights are always doubling). Also these classes are nested, $RH_p^d \subset RH_q^d$ for all $p \geq q$. The class RH_1^d is defined by $RH_1^d := \bigcup_{p>1} RH_p^d$. In the non-dyadic setting $A_\infty = RH_1$. In the dyadic setting the collection of dyadic doubling weights in RH_1^d is A_∞^d,

hence A^d_∞ is a proper subset of RH^d_1. See [BeRez] for some recent and very interesting results relating these classes.
The following are well-known properties of weights (see [JN]) for (ii)):

Lemma 2.1. *The following hold*

- *If* $0 \le s \le 1$ *and* $w \in A^d_\infty$ *then* $w^s \in A_\infty$. *More precisely, if* $p > 1$ *and* $w \in A^d_p$ *then* $w^s \in A_p$, *and* $[w^s]_{A^d_p} \le [w]^s_{A^d_p}$.
- *If* $s, q > 1$ *then* $w \in RH^d_s \cap A^d_q$ *if and only if* $w^s \in A_{s(q-1)+1}$. *Moreover* $[w^s]_{A_{s(q-1)+1}} \le [w]^s_{RH^d_s}[w]^s_{A^d_q}$, $[w]^s_{A^d_q} \le [w^s]_{A_{s(q-1)+1}}$, *and* $[w]^s_{RH^d_s} \le [w^s]_{A_{s(q-1)+1}}$.
- *If* $p > 1$, *and* $1/p + 1/p' = 1$, *then* $w \in A^d_p$ *if and only if* $w^{-1/p-1} \in A_{p'}$. *Moreover* $[w]_{A^d_p} = [w^{-1/p-1}]^{p-1}_{A^d_{p'}}$.

The following property can be found in [GaRu],

Lemma 2.2. *If* $w \in RH^d_s \cap A^d_q$ *then for all* $E \subset B$,

$$\big(|E|/|B|\big)^q [w]^{-1}_{A^d_q} \le w(E)/w(B) \le \big(|E|/|B|\big)^{1-\frac{1}{s}} [w]_{RH^d_s}.$$

In particular $D(w) \le 2^q [w]_{A^d_q}$.

A weight w satisfies the C^d_s*-condition*, for $s \in \mathbb{R}$, if

$$[w]_{C^d_s} := \sup_{I \in \mathcal{D}} \big(m_I w^s\big) \big(m_I w\big)^{-s} < \infty.$$

The quantity defined above is called the C^d_s-characteristic of w. The class of weights C^d_s was defined in [KP]. Let us analyze this definition. For $0 \le s \le 1$, we have that any weight satisfies the condition with C^d_s-characteristic 1, this is just a consequence of Hölder's Inequality (for $s = 0, 1$ is trivial). When $s > 1$, the condition is analogous to the dyadic reverse Hölder condition and $[w]^{1/s}_{C^d_s} = [w]_{RH^d_s}$. For $s < 0$, we have that $w \in C^d_s$ if and only if $w \in A^d_{1-1/s}$, moreover $[w]_{C^d_s} = [w]^{-s}_{A^d_{1-1/s}}$.

Lemma 2.3. *If* $w \in C^d_s \cap A^d_\infty$ *then the following hold*

- *For all* $0 \le s \le 1$, *there is a* $p > 1$ *such that* $w^s \in A_p$.
- *If* $s > 1$ *then there is* $q > 1$ *such that* $w^s \in A_{s(q-1)+1}$.
- *If* $s < 0$ *then* $w^s \in A_{1-s}$.

The proof of this lemma is a direct application of Lemma 2.1 item by item.

2.3. **Weighted Haar functions.** For a given weight v and an interval I define the *weighted Haar function* as

$$h_I^v(x) = \frac{1}{v(I)}\left(\sqrt{\frac{v(I_-)}{v(I_+)}}\,\chi_{I_+}(x) - \sqrt{\frac{v(I_+)}{v(I_-)}}\,\chi_{I_-}(x)\right), \tag{2.2}$$

where $\chi_I(x)$ is the characteristic function of the interval I.

If v is the Lebesgue measure on $\mathbb{R}$, we will denote the *Haar function* simply by h_I. It is a simple exercise to verify that the weighted and unweighted Haar functions are related linearly as follows,

Proposition 2.4. For any weight v, there are numbers α_I^v, β_I^v such that

$$h_I(x) = \alpha_I^v\, h_I^v(x) + \beta_I^v\, \chi_I(x)/\sqrt{|I|}$$

where (i) $|\alpha_I^v| \leq \sqrt{m_I v}$, (ii) $|\beta_I^v| \leq |\Delta_I v|/m_I v$, $\Delta_I v := m_{I_+} v - m_{I_-} v$.

The family $\{h_I^v\}_{I\in\mathcal{D}}$ is an orthonormal system in $L^2(v)$, with inner product $\langle f, g\rangle_v := \int_{\mathbb{R}} f(x)\,\overline{g(x)}\,v(x)dx$.

2.4. **Carleson sequences.** If v is a weight, a positive sequence $\{\alpha_I\}_{I\in\mathcal{D}}$ is called a *v-Carleson sequence with intensity B* if for all $J \in \mathcal{D}$,

$$\frac{1}{|J|}\sum_{I\in\mathcal{D}(J)} \lambda_I \leq B\, m_J v. \tag{2.3}$$

When $v = 1$ we call a sequence satisfying (2.3) for all $J \in \mathcal{D}$ a *Carleson sequence with intensity B*.

Proposition 2.5. Let v be a weight, $\{\lambda_I\}_{I\in\mathcal{D}}$ and $\{\gamma_I\}_{I\in\mathcal{D}}$ be two v-Carleson sequences with intensities A and B respectively then for any $c, d > 0$ we have that

(i) $\{c\lambda_I + d\gamma_I\}_{I\in\mathcal{D}}$ is a v-Carleson sequence with intensity $cA + dB$.
(ii) $\{\sqrt{\lambda_I}\sqrt{\gamma_I}\}_{I\in\mathcal{D}}$ is a v-Carleson sequence with intensity $\sqrt{AB}$.
(iii) $\{(c\sqrt{\lambda_I} + d\sqrt{\gamma_I})^2\}_{I\in\mathcal{D}}$ is a v-Carleson sequence with intensity $2c^2A + 2d^2B$.

The proof of these statements is quite simple, see [MoP].

3. Main tools

In this section, we state the lemmas and theorems necessary to get the estimate for the t-Haar multipliers of complexity (m, n).

3.1. **Carleson Lemmas.** The Weighted Carleson Lemma we present here is a variation in the spirit of other weighted Carleson embedding theorems that appeared before in the literature [NV, NTV1]. You can find a proof in [MoP].

Lemma 3.1 (Weighted Carleson Lemma). *Let v be a weight, then $\{\alpha_L\}_{L\in\mathcal{D}}$ is a v-Carleson sequence with intensity B if and only if for all non-negative v-measurable functions F on the line,*

$$\sum_{L\in\mathcal{D}} \alpha_L \inf_{x\in L} F(x) \leq B \int_{\mathbb{R}} F(x)\, v(x)\, dx. \tag{3.1}$$

The following lemma we view as a finer replacement for Hölder's inequality: $1 \leq (m_I w)(m_I w^{-1/(p-1)})^{p-1}$.

Lemma 3.2 (A_p-Little Lemma). *Let v be a weight, such that $v^{-1/(p-1)}$ is a a weight as well, and let $\{\lambda_I\}_{I\in\mathcal{D}}$ be a Carleson sequence with intensity Q then $\{\lambda_I/(m_I v^{-1/(p-1)})^{p-1}\}_{I\in\mathcal{D}}$ is a v-Carleson sequence with intensity $4Q$, that is for all $J\in\mathcal{D}$,*

$$\frac{1}{|J|}\sum_{I\in\mathcal{D}(J)} \frac{\lambda_I}{(m_I v^{-1/(p-1)})^{p-1}} \leq 4Q\; m_J v.$$

For $p=2$ this was proved in [Be, Proposition 3.4], or [Be1, Proposition 2.1], using the same Bellman function as in the proof we present in the Appendix.

Lemma 3.3 ([NV]). *Let v be a weight such that $v^{-1/(p-1)}$ is also a weight. Let $\{\lambda_J\}_{J\in\mathcal{D}}$ be a Carleson sequence with intensity B. Let F be a non-negative measurable function on the line. Then*

$$\sum_{J\in\mathcal{D}} \frac{\lambda_J}{(m_J v^{-1/(p-1)})^{p-1}} \inf_{x\in J} F(x) \leq C\; B \int_{\mathbb{R}} F(x)\, v(x)\, dx.$$

Lemma 3.3 is an immediate consequence of Lemma 3.2, and the Weighted Carleson Lemma 3.1. Note that Lemma 3.2 can be deduced from Lemma 3.3 with $F(x)=\chi_J(x)$.

The following lemma , for $v=w^{-1}$, and for $\alpha=1/4$ appeared in [Be], and for $0<\alpha<1/2$, in [NV]. With small modification in their proof, using the Bellman function $B(x,y)=x^\alpha y^\beta$ with domain of definition the first quadrant $x,y>0$, we can accomplish the result below, for a complete proof see [Mo].

Lemma 3.4. ($\alpha\beta$-Lemma) *Let u,v be weights. Then for any $J\in\mathcal{D}$ and any $\alpha,\beta\in(0,1/2)$*

$$\frac{1}{|J|}\sum_{I\in\mathcal{D}(J)} \frac{|\Delta_I u|^2}{(m_I u)^2}|I|(m_I u)^\alpha (m_I v)^\beta \leq C_{\alpha,\beta}(m_J u)^\alpha (m_J v)^\beta. \tag{3.2}$$

The constant $C_{\alpha,\beta} = 36/\min\{\alpha - 2\alpha^2, \beta - 2\beta^2\}$.

From this lemma we immediately deduce the following,

Lemma 3.5. *Let* $1 < q < \infty$*,* $w \in A_q^d$*, then* $\{\mu_I^{q,\alpha}\}_{I\in\mathcal{D}}$*, where*

$$\mu_I^{q,\alpha} := (m_I w)^{\alpha}(m_I w^{\frac{-1}{q-1}})^{\alpha(q-1)}|I|\left(\frac{|\Delta_I w|^2}{(m_I w)^2} + \frac{|\Delta_I w^{\frac{-1}{q-1}}|^2}{(m_I w^{\frac{-1}{q-1}})^2}\right),$$

is a Carleson sequence with Carleson intensity at most $C_\alpha [w]_{A_q}^{\alpha}$ *for any* $\alpha \in \big(0, \max\{1/2, 1/2(q-1)\}\big)$*. Moreover,* $\{\nu_I^q\}_{I\in\mathcal{D}}$*, where*

$$\nu_I^q := (m_I w)(m_I w^{\frac{-1}{q-1}})^{(q-1)}|I|\left(\frac{|\Delta_I w|^2}{(m_I w)^2} + \frac{|\Delta_I w^{\frac{-1}{q-1}}|^2}{(m_I w^{\frac{-1}{q-1}})^2}\right)$$

is a Carleson sequence with Carleson intensity at most $C[w]_{A_q}$.

Proof. Set $u = w$, $v = w^{-\frac{1}{q-1}}$, $\beta = \alpha(q-1)$. By hypothesis $0 < \alpha < 1/2$ and also $0 < \alpha < 1/2(q-1)$ which implies that $0 < \beta < 1/2$, we can now use Lemma 3.4 to show that $\mu_I^{q,\alpha}$ is a Carleson sequence with intensity at most $c_\alpha [w]_{A_q^d}^{\alpha}$. For the second statement suffices to notice that $\nu_I^q \le \mu_I^{q,\alpha}[w]_{A_q^d}^{1-\alpha}$ for all $I \in \mathcal{D}$, for some $\alpha \in \big(0, \max\{1/2, 1/2(q-1)\}\big)$ □

A proof of this lemma for $q = 2$ that works on geometric doubling metric spaces can be found in [NV1, V]. In those papers $\alpha = 1/4$ can be used, and in that case the constant C_α can be replaced by 288.

3.2. **Lift Lemma.**

Given a dyadic interval L, and weights u, v, we introduce a family of stopping time intervals $\mathcal{ST}_L^m$ such that the averages of the weights over any stopping time interval $K \in \mathcal{ST}_L^m$ are comparable to the averages on L, and $|K| \ge 2^m |L|$. This construction appeared in [NV] for the case $u = w$, $v = w^{-1}$. We also present a lemma that lifts w-Carleson sequences on intervals to w-Carleson sequences on "m-stopping intervals". This was used in [NV] for a very specific choice of m-stopping time intervals $\mathcal{ST}_L^m$.

Lemma 3.6 (Lift Lemma [NV])**.** *Let* u *and* v *be weights,* L *be a dyadic interval and* m, n *be fixed positive integers. Let* $\mathcal{ST}_L^m$ *be the collection of maximal stopping time intervals* $K \in \mathcal{D}(L)$*, where the stopping criteria are either* (i) $|\Delta_K u|/m_K u + |\Delta_K v|/m_K v \ge 1/m + n + 2$*, or* (ii) $|K| = 2^{-m}|L|$*. Then for any stopping interval* $K \in \mathcal{ST}_L^m$*,* $e^{-1} m_L u \le m_K u \le e\, m_L u$*, and hence also* $e^{-1} m_L v \le m_K v \le e\, m_L v$*.*

Note that the roles of m and n can be interchanged and we get the family $\mathcal{ST}_L^n$ using the same stopping condition (i) and condition (ii) replaced by $|K| = 2^{-n}|L|$. Notice that $\mathcal{ST}_L^m$ is a partition of L in dyadic subintervals of length at least $2^{-m}|L|$. The following lemma lifts a w-Carleson sequence to m-stopping time intervals with comparable intensity. For the particular m-stopping time $\mathcal{ST}_L^m$ given by the stopping criteria (i) and (ii) in Lemma 3.6, and $w = 1$, this appeared in [NV].

Lemma 3.7. *For each $L \in \mathcal{D}$ let $\mathcal{ST}_L^m$ be a partition of L in dyadic subintervals of length at least $2^{-m}|L|$. Assume $\{\nu_I\}_{I\in\mathcal{D}}$ is a w-Carleson sequence with intensity at most A, let $\nu_L^m := \sum_{K\in\mathcal{ST}_L^m} \nu_K$, then $\{\nu_L^m\}_{L\in\mathcal{D}}$ is a w-Carleson sequence with intensity at most $(m+1)A$.*

For proofs you can see [MoP].

3.3. **Auxiliary quantities.** For a weight v, and a locally integrable function ϕ we define the following quantities,

$$P_L^m\phi := \sum_{I\in\mathcal{D}_m(L)} |\langle \phi, h_I\rangle| \sqrt{|I|/|L|}, \tag{3.3}$$

$$S_L^{v,m}\phi := \sum_{J\in\mathcal{D}_m(L)} |\langle \phi, h_J^v\rangle_v| \sqrt{m_J v}\sqrt{|J|/|L|}, \tag{3.4}$$

$$R_L^{v,m}\phi := \sum_{J\in\mathcal{D}_m(L)} \frac{|\Delta_J v|}{m_J v} m_J(|\phi| v)\ |J|/\sqrt{|L|}, \tag{3.5}$$

Let $w \in A_q^d$, $\mathcal{ST}_L^m$ be an m-stopping time family of subintervals of L, $0 < \alpha < \max\{1/2, 1/2(q-1)\}$, and $\{\mu_K^q = \mu_K^{q,\alpha}\}_{K\in\mathcal{D}}$ be the Carleson sequence with intensity $C_\alpha[w]_{A_q^d}$ defined in Lemma 3.5. For each $m > 0$, we introduce another sequence $\{\mu_L^m\}$, which is Carleson by Lemma 3.7:

$$\mu_L^m := \sum_{K\in\mathcal{ST}_L^m} \mu_K^q \quad \text{with intensity} \quad C_\alpha(m+1)[w]_{A_q^d}.$$

We will use the following estimates for $S_L^{v,m}\phi$ and $R_L^{v,m}\phi$, where $1 < p < 2$ will be dictated by the proof of the theorem.

$$S_L^{v,m}\phi \le \Big(\sum_{J\in\mathcal{D}_m(L)} |\langle \phi, h_J^v\rangle_v|^2\Big)^{\frac{1}{2}} (m_L v)^{\frac{1}{2}}, \tag{3.6}$$

(3.7)

$$R_L^{v,m}\phi \le C\, C_m^n (m_L v^{\frac{-1}{q-1}})^{\frac{-(q-1)}{2}} (m_L v)^{\frac{1}{2}} \inf_{x\in L} \big(M_{w^{-1}}(|g|^p)(x)\big)^{\frac{1}{p}} \sqrt{\mu_L^m},$$

See [NV] for the proof when $q = 2$, slight modification of their argument gives the estimate for $R_L^{v,m}\phi$. Estimating $P_L^n\phi$ is very simple:

$$(P_L^m\phi)^2 \le \sum_{I\in\mathcal{D}_m(L)} |I|/|L| \sum_{I\in\mathcal{D}_m(L)} |\langle\phi, h_I\rangle|^2 = \sum_{I\in\mathcal{D}_m(L)} |\langle\phi, h_I\rangle|^2. \tag{3.8}$$

Remark 3.8. In [NV1], Nazarov and Volberg extend the results that they had in [NV] for Haar shifts to metric spaces with geometric doubling. Following the same modifications in the argument made from [NV] to [NV1], one could obtain the same result as in Theorem 4.1 on a metric space with geometric doubling, see [Mo1].

4. Haar Multipliers

For a weight w, $t \in \mathbb{R}$, and $m, n \in \mathbb{N}$, a *t-Haar multiplier of complexity* (m, n) is the operator defined as

$$T_{t,w}^{m,n} f(x) := \sum_{L\in\mathcal{D}} \sum_{I\in\mathcal{D}_n(L);J\in\mathcal{D}_m(L)} c_{I,J}^L \left(\frac{w(x)}{m_L w}\right)^t \langle f, h_I\rangle h_J(x), \tag{4.1}$$

where $|c_{I,J}^L| \le \sqrt{|I|\,|J|}/|L|$. In [MoP] it is shown that $w \in C_{2t}^d$ is a necessary condition for boundedness of $T_{w,t}^{m,n}$ in $L^2(\mathbb{R})$ when $c_{I,J}^L = \sqrt{|I|\,|J|}/|L|$. It is also shown that the C_{2t}^d-condition is sufficient for a t-Haar multiplier of complexity (m, n) to be bounded in $L^2(\mathbb{R})$ for most t; this was proved in [KP] for the case $m = n = 0$. Here we are concerned not only with the boundedness but also with the dependence of the operator norm on the C_{2t}^d-constant . For T_w^t and $t = 1, \pm 1/2$ this was studied in [P2]. The first author [Be] was able to obtain estimates, under the additional condition on the weight $w^{2t} \in A_q^d$ for some $q > 1$, for T_w^t and for all $t \in \mathbb{R}$. Her results were generalized for $T_{w,t}^{m,n}$ for all t when $w^{2t} \in A_2^d$, see [MoP]. We will show that:

Theorem 4.1. *Let t be a real number and w a weight such that $w^{2t} \in A_q^d$ for some $q > 1$ (i.e. $w^{2t} \in A_\infty^d$), then*

$$\|T_{t,w}^{m,n} f\|_2 \le C_q(m + n + 2)^3 [w]_{C_{2t}^d}^{\frac{1}{2}} [w^{2t}]_{A_q^d}^{\frac{1}{2}} \|f\|_2.$$

Using Lemmas 2.1 and 2.3 we can refine the result as follows:

Theorem 4.2. *Let $t \in \mathbb{R}$, $w \in C^{2t}$, and $C_m^n = n + m + 2$, then*

(i) *If $0 < 2t < 1$ and $w \in A_p^d$ then*

$$\|T_{t,w}^{m,n} f\|_2 \le C_p (C_m^n)^3 [w^{2t}]_{A_p^d}^{\frac{1}{2}} \|f\|_2 \le C_p (C_m^n)^3 [w]_{A_p^d}^t \|f\|_2.$$

(ii) *If $t>1$ and $w \in A_p^d$ then if $q = 2t(p-1)+1$*

$$\|T_{t,w}^{m,n} f\|_2 \leq C_p (C_m^n)^3 [w]_{C_{2t}^d}^{\frac{1}{2}} [w^{2t}]_{A_q^d}^{\frac{1}{2}} \|f\|_2 \leq C_p (C_m^n)^3 [w]_{C_{2t}^d} [w]_{A_p^d}^t.$$

(iii) *If $t<0$ then*

$$\|T_{t,w}^{m,n} f\|_2 \leq C (C_m^n)^3 [w]_{C_{2t}^d} \|f\|_2 = C (C_m^n)^3 [w]_{A_{1-1/2t}^d}^{-2t} \|f\|_2.$$

Remark 4.3. Throughout the proof a constant C_q will be a numerical constant depending only on the parameter $q>1$ that may change from line to line.

Proof of Theorem 4.2. By Lemma 2.3 if $w \in C_{2t}^d \cap A_\infty^d$ then there is $q > 1$ such that $w^{2t} \in A_q^d$, matching cases perfectly. Now use Theorem 4.1. □

Proof of Theorem 4.1. Fix $f,\ g \in L^2(\mathbb{R})$. By duality, it is enough to show that

$$|\langle T_{t,w}^{m,n} f, g\rangle| \leq C(m+n+2)^3 [w]_{C_{2t}^d}^{\frac{1}{2}} [w^{2t}]_{A_q^d}^{\frac{1}{2}} \|f\|_2 \|g\|_2.$$

The inner product on the left-hand-side can be expanded into a double sum, that we now estimate,

$$|\langle T_{t,w}^{m,n} f, g\rangle| \leq \sum_{L\in\mathcal{D}} \sum_{I\in\mathcal{D}_n(L);J\in\mathcal{D}_m(L)} \frac{\sqrt{|I|\,|J|}}{|L|} \frac{|\langle f, h_I\rangle|}{(m_L w)^t}\, |\langle g w^t, h_J\rangle|.$$

Write h_J as a linear combination of a weighted Haar function and a characteristic function, $h_J = \alpha_J h_J^{w^{2t}} + \beta_J \chi_J/\sqrt{|J|}$, where $\alpha_J = \alpha_J^{w^{2t}}$, $\beta_J = \beta_J^{w^{2t}}$, $|\alpha_J| \leq \sqrt{m_J w^{2t}}$, and $|\beta_J| \leq |\Delta_J(w^{2t})|/m_J w^{2t}$. Now break into two terms to be estimated separately so that,

$$|\langle T_{t,w}^{m,n} f, g\rangle| \leq \Sigma_1^{m,n} + \Sigma_2^{m,n},$$

where

$$\Sigma_1^{m,n} := \sum_{L\in\mathcal{D}} \sum_{I\in\mathcal{D}_n(L);J\in\mathcal{D}_m(L)} \frac{\sqrt{|I|\,|J|}}{|L|} \frac{\sqrt{m_J(w^{2t})}}{(m_L w)^t} |\langle f, h_I\rangle|\, |\langle g w^t, h_J^{w^{2t}}\rangle|,$$

$$\Sigma_2^{m,n} := \sum_{L\in\mathcal{D}} \sum_{I\in\mathcal{D}_n(L);J\in\mathcal{D}_m(L)} \frac{|J|\sqrt{|I|}}{|L|(m_L w)^t} \frac{|\Delta_J(w^{2t})|}{m_J(w^{2t})} |\langle f, h_I\rangle|\, m_J(|g| w^t).$$

Let $p = 2 - (C_n^m)^{-1}$ (note that $2 > p > 1$, in fact is getting closer to 2 as m and n increase), and define as in (3.3), (3.4) and (3.5), the quantities $P_L^m\phi$, $S_L^{v,n}\phi$ and $R_L^{v,n}\phi$, we will use here the case $v = w^{2t}$, for appropriate ϕs and corresponding estimates. Note that $1 < p < 2$.

The sequence $\{\eta_I\}_{I\in\mathcal{D}}$ where

$$\eta_I := (m_I w^{2t})\ (m_I w^{\frac{-2t}{q-1}})^{(q-1)}\Big(\frac{|\Delta_I(w^{2t})|^2}{|m_I w^{2t}|^2} + \frac{|\Delta_I(w^{-2t/(q-1)})|^2}{|m_I w^{-2t/(q-1)}|^2}\Big)|I|,$$

is a Carleson sequence with intensity $C_q[w^{2t}]_{A_q^d}$ by Lemma 3.5. The sequence $\{\eta_L^m\}_{I\in\mathcal{D}}$ where

$$\eta_L^m := \sum_{I\in\mathcal{ST}_L^m} \eta_I,$$

and the stopping time $\mathcal{ST}_L^m$ is defined as in Lemma 3.6 but with respect to the weights $u = w^{2t}$, $v = w^{-2t/(q-1)}$, is a Carleson sequence with intensity $C_q(m+1)[w^{2t}]_{A_q^d}$ by Lemma 3.7, .

Observe that on the one hand $\langle gw^t, h_J^{w^{2t}}\rangle = \langle gw^{-t}, h_J^{w^{2t}}\rangle_{w^{2t}}$, and on the other $m_J(|g|w^t) = m_J(|gw^{-t}|w^{2t})$. Therefore,

$$\Sigma_1^{m,n} = \sum_{L\in\mathcal{D}} (m_L w)^{-t} S_L^{w^{2t},n}(gw^{-t})\ P_L^m f,$$

$$\Sigma_2^{m,n} = \sum_{L\in\mathcal{D}} (m_L w)^{-t} R_L^{w^{2t},n}(gw^{-t})\ P_L^m f.$$

Estimates (3.6) and (3.7) hold for $S_L^{w^{2t},m}(gw^{-t})$ and $R_L^{w^{2t},m}(gw^{-t})$ with v and ϕ replaced by w^{2t} and gw^{-t}:

$$\begin{aligned} S_L^{w^{2t},n}(gw^{-t}) &\le (m_L w^{2t})^{\frac{1}{2}}\Big(\sum_{J\in\mathcal{D}_m(L)} |\langle gw^{-t}, h_J^{w^{2t}}\rangle_{w^{2t}}|^2\Big)^{\frac{1}{2}}, \\ R_L^{w^{2t},n}(gw^{-t}) &\le C\, C_m^n (m_L w^{2t})^{\frac{1}{2}} (m_L w^{\frac{2t}{q-1}})^{\frac{-(q-1)}{2}} F^{\frac{1}{2}}(x)\sqrt{\eta_L^m}, \end{aligned}$$

where $F(x) = \inf_{x\in L}\big(M_{w^{2t}}(|gw^{-t}|^p)(x)\big)^{\frac{2}{p}}$.

Estimating $\Sigma_1^{m,n}$: Plug in the estimates for $S_L^{w^{2t},n}(gw^{-t})$ and $P_L^m f$, observe that $(m_L w^{2t})^{\frac{1}{2}}/(m_L w)^t \le [w]_{C_{2t}^d}^{\frac{1}{2}}$, use the Cauchy-Schwarz inequality, to get,

$$\begin{aligned} \Sigma_1^{m,n} &\le \sum_{L\in\mathcal{D}} [w]_{C_{2t}^d}^{\frac{1}{2}} \Big(\sum_{J\in\mathcal{D}_n(L)} |\langle gw^{-t}, h_J^{w^{2t}}\rangle_{w^{2t}}|^2\Big)^{\frac{1}{2}} \Big(\sum_{I\in\mathcal{D}_m(L)} |\langle f, h_I\rangle|^2\Big)^{\frac{1}{2}} \\ &\le [w]_{C_{2t}^d}^{\frac{1}{2}} \|f\|_2 \Big(\sum_{L\in\mathcal{D}}\sum_{J\in\mathcal{D}_n(L)} |\langle gw^{-t}, h_J^{w^{2t}}\rangle_{w^{2t}}|^2\Big)^{\frac{1}{2}} \\ &\le [w]_{C_{2t}^d}^{\frac{1}{2}} \|f\|_2 \|gw^{-t}\|_{L^2(w^{2t})} = [w]_{C_{2t}^d}^{\frac{1}{2}} \|f\|_2 \|g\|_2. \end{aligned}$$

Estimating $\Sigma_2^{m,n}$: Plug in the estimates for $R_L^{w^{2t},n}(gw^{-t})$ and $P_L^m f$, where $F(x) = \big(M_{w^{2t}}(|gw^{-t}|^p)(x)\big)^{2/p}$, use the Cauchy-Schwarz inequality and $(m_L w^{2t})^{\frac{1}{2}}/(m_L w)^t \le [w]_{C_{2t}^d}^{\frac{1}{2}}$ to get

$$\Sigma_2^{m,n} \le C\, C_m^n [w]_{C_{2t}^d}^{\frac{1}{2}} \|f\|_2 \Big(\sum_{L\in\mathcal{D}} (\eta_L^m/(m_L w^{\frac{-2t}{q-1}})^{q-1}) \inf_{x\in L} F(x)\Big)^{\frac{1}{2}}.$$

Now using Weighted Carleson Lemma 3.1 with $\alpha_L = \eta_L^m/(m_L w^{\frac{-2t}{q-1}})^{q-1}$ (which by Lemma 3.2 is a w^{2t}-Carleson sequence with intensity no larger than $C_q(m+1)[w]_{A_q^d}$, $F(x) = \big(M_{w^{2t}}|gw^{-t}|^p(x)\big)^{2/p}$, and $v = w^{2t}$,

$$\Sigma_2^{m,n} \le C_q (C_m^n)^2 [w]_{C_{2t}^d}^{\frac{1}{2}} [w^{2t}]_{A_q^d}^{\frac{1}{2}} \|f\|_2 \Big\| M_{w^{2t}}(|gw^{-t}|^p)\Big\|_{L^{\frac{2}{p}}(w^{2t})}^{\frac{1}{p}}.$$

Using (2.1), that is the boundedness of $M_{w^{2t}}$ in $L^{\frac{2}{p}}(w^{2t})$ for $2/p > 1$,

$$\begin{aligned} \Sigma_2^{m,n} &\le C_q (C_m^n)^2 (2/p)' [w]_{C_{2t}^d}^{\frac{1}{2}} [w^{2t}]_{A_q^d}^{\frac{1}{2}} \|f\|_2 \Big\| |gw^{-t}|^p \Big\|_{L^{\frac{2}{p}}(w^{2t})}^{\frac{1}{p}} \\ &\le C_q (C_m^n)^3 [w]_{C_{2t}^d}^{\frac{1}{2}} [w^{2t}]_{A_q^d}^{\frac{1}{2}} \|f\|_2 \|g\|_2, \end{aligned}$$

Since $(2/p)' = 2/(2-p) = 2C_m^n$. The theorem is proved. □

Appendix

Proof of Lemma 3.2. We will show this inequality using a Bellman function type method. Consider $B(u,v,l) := u - 1/(v^{p-1}(1+l))$ defined on the domain $\mathbb{D} = \{(u,v,l) \in \mathbb{R}^3, u > 0, v > 0, uv^{p-1} > 1 \text{ and } 0 \le l \le 1\}$. Note that $\mathbb{D}$ is convex. Note that

$$0 \le B(u,v,l) \le u \qquad \text{for all} \quad (u,v,l) \in \mathbb{D} \tag{4.2}$$

and

$$(\partial B/\partial l)(u,v,l) \ge 1/4v^{p-1} \qquad \text{for all} \quad (u,v,l) \in \mathbb{D}. \tag{4.3}$$

and also $-(du,dv,dl)d^2B(du,dv,dl)^t$ is non-negative because, it equals

$$-(du,dv,dl)\begin{pmatrix} 0 & 0 & 0 \\ 0 & p(1-p)\frac{v^{-p-1}}{1+l} & (1-p)\frac{v^{-p}}{(l+1)^2} \\ 0 & (1-p)\frac{v^{-p}}{(l+1)^2} & -2\frac{v^{1-p}}{(l+1)^3} \end{pmatrix}\begin{pmatrix} du \\ dv \\ dl \end{pmatrix}$$

$$= p(p-1)\frac{v^{-p-1}}{1+l}(du)^2 + 2(p-1)\frac{v^{-p}}{(l+1)^2}dudv + 2\frac{v^{1-p}}{(l+1)^3}(dv)^2 \ge 0,$$

since all terms are positive for $p > 1$.

Now let us show that if (u_-,v_-,l_-) and (u_+,v_+,l_+) are in $\mathbb{D}$ and we define $(u_0,v_0,l)\in\mathbb{D}$ where l is in between l_+ and l_-, $u_0=(u_-+u_+)/2$, $v_0=(v_-+v_+)/2$, and $l_0=(l_-+l_+)/2$, then

$$B(u_0,v_0,l)-\big(B(u_-,v_-,l_-)+B(u_+,v_+,l_+)\big)/2\geq |l-l_0|/4v_0^{p-1}$$

Write for $-1\leq t\leq 1$, $u(t)=[(t+1)u_+ +(1-t)u_-]/2$, $v(t)=[(t+1)v_+ +(1-t)v_-]/2$, and $l(t)=[(t+1)l_+ +(1-t)l_-]/2$,. Define $b(t):=B(u(t),v(t),l(t))$, then $b(0)=B(u_0,v_0,l_0)$, $b(1)=B(u_+,v_+,l_+)$, $b(-1)=B(u_-,v_-,l_-)$, $du/dt=(u_+-u_-)/2$, $dv/dt=(v_+-v_-)/2$ and $dl/dt=(l_+-l_-)/2$. If (u_+,v_+,l_+) and (u_-,v_-,l_-) are in $\mathbb{D}$ then $(u(t),v(t),l(t))$ is also in $\mathbb{D}$ for all $|t|\leq 1$, since $\mathbb{D}$ is convex. It is a calculus exercise to show that

$$b(0)-\frac{b(1)+b(-1)}{2}=\frac{-1}{2}\int_{-1}^{1}(1-|t|)b''(t)dt \tag{4.4}$$

Also it is easy to check that $-b''(t)=-\big(\frac{du}{dt},\frac{dv}{dt},\frac{dl}{dt}\big)d^2B\big(\frac{du}{dt},\frac{dv}{dt},\frac{dl}{dt}\big)^t$. By the Mean Value Theorem and (4.4),

$$\begin{aligned}B(u_0,v_0,l)&-\frac{B(u_-,v_-,l_-)+B(u_+,v_+,l_+)}{2}\\ &= (l-l_0)\frac{\partial B}{\partial l}(u_0,v_0,l')-\frac{1}{2}\int_{-1}^{1}(1-|t|)b''(t)dt\geq\frac{l-l_0}{4v_0^{p-1}},\end{aligned}$$

where l' is a point between l and $l_0=(l_-+l_+)/2$.

Now we can use the Bellman function argument. Let $u_+=m_{J_+}w$, $u_-=m_{J_-}w$, $v_+=m_{J_+}w^{\frac{-1}{p-1}}$, $v_-=m_{J_-}v^{\frac{-1}{p-1}}$, $l_+=\frac{1}{|J_+|Q}\sum_{I\in\mathcal{D}(J_+)}\lambda_I$ and $l_-=\frac{1}{|J_-|Q}\sum_{I\in\mathcal{D}(J_-)}\lambda_I$. Thus $(u_-,v_-,l_-),(u_+,v_+,l_+)\in\mathbb{D}$ and $u_0=m_Jw$, $v_0=m_Jw^{\frac{-1}{p-1}}$, and $l_0=\frac{1}{|J|Q}\sum_{I\in\mathcal{D}(J)}\lambda_I$. Thus

$$(u_0,v_0,l_0)-\big((u_-+u_+)/2,(v_-+v_+)/2,(l_-+l_+)/2\big)=\big(0,0,\lambda_J/Q|J|\big).$$

Then we can run the usual induction on scale arguments using the properties of the Bellman function,

$$\begin{aligned}|J|m_Jw&\geq |J|B(u_0,v_0,l_0)\\ &\geq |J|\frac{B(u_+,v_+,l_+)}{2}+|J|\frac{B(u_-,v_-,l_-)}{2}+\lambda_J/4Q\big(m_Jw^{\frac{-1}{p-1}}\big)^{p-1}\\ &=|J_+|B(u_+,v_+,l_+)+|J_-|B(u_-,v_-,l_-)+\lambda_J/4Q\big(m_Jw^{\frac{-1}{p-1}}\big)^{p-1}\end{aligned}$$

Iterating, we get

$$m_Jw\geq\frac{1}{4Q|J|}\sum_{I\in\mathcal{D}(J)}\frac{\lambda_I}{(m_Iw^{-1/p-1})^{p-1}}.$$

□

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