# Multiwavelet characterization of function spaces adapted to the Navier-Stokes equations

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## ABSTRACT

We use wavelets based on a modification of the Geronimo-Hardin-Massopust construction to define localized extension/restriction operators from half-spaces to their full spaces/boundaries respectively. These operations are continuous in Sobolev and Morrey space norms. We also prove estimates for multiresolution projections of pointwise products of functions in these spaces. These are two of the key steps in extending results of Federbush and of Cannone and Meyer concerning solutions of Navier-Stokes with initial data in Sobolev and Morrey spaces to the case of half spaces and, ultimately, to more general domains with boundary.

**Keywords:** multiwavelets, Sobolev spaces, Morrey spaces, Navier-Stokes

## 1. INTRODUCTION

The Navier-Stokes equations for the velocity $v(x,t)$ of a fluid at the point $x$ and time $t$ are

$$\frac{\partial v}{\partial t} - \Delta v + (v \cdot \nabla)\, v + \nabla p = 0; \nabla \cdot v = 0;\; v(0) = v_0 \tag{1}$$

where $p$ denotes the fluid pressure and the condition $\nabla \cdot v = 0$ reflects the *incompressibility* of the fluid. Cannone and Meyer[2] developed a method for obtaining so-called mild solutions of (1) with initial values belonging to function spaces defined by relatively relaxed regularity properties in terms of smoothness and/or order of local integrability. They did so by developing a notion under which a function space is adapted to the bilinear estimates they required for applying a Picard iteration method to solve the variational form of Navier-Stokes,

$$v(t) = S(t)v_0 - \int_0^t \mathbf{P}S(t-s)\nabla \cdot (v \otimes v)(s)ds. \tag{2}$$

Here $S(t) = \exp(t\Delta)$ is the heat semigroup and $\mathbf{P}$ is the singular integral operator that projects a vector field onto its divergence-free component. The differential form (1) is easily obtained from the variational form (2) by differentiation in time modulo the fact that the pressure depends on the velocity. While we do not wish to overplay the subtleties tied to the sense in which the separate forms converge, it is useful to note that the form (2) does not impose any pointwise time differentiability on $v$. This allows solutions to persist for some time in function space norms that impose little regularity. Such spaces include $L^2$-Sobolev spaces of small order as well as Morrey-Campanato spaces.[12]

To prove time continuity of solutions one needs estimates for the bilinear operator family $B_t : (f,g) \mapsto \mathbf{P}S(t)\nabla \cdot (f \otimes g)$. Cannone and Meyer used Littlewood-Paley theory to analyze this family so it was important that the function space possess a norm expressible in terms of a Littlewood-Paley square function. A Littlewood-Paley projection is essentially a projection of a function onto the part whose frequencies belonging to a certain dyadic band. Cannone

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and Meyer called a function norm ‘adapted’ to $B_t$ when certain decay estimates pertained to the norm of the operator when restricted to functions living in frequency bands. In a sense, Meyer’s invention of bandlimited wavelets was tailored to expressing Littlewood-Paley norms as coefficient norms. The corresponding discretization of function norms then led to more elementary techniques for answering questions about continuity of complicated operators on the corresponding function spaces.[11]

On the negative side, Fourier based techniques could only be applied to function spaces defined on the whole Euclidean space or to periodic functions.[13] Moreover, the results of Federbush[6] that solve Navier-Stokes in Morrey space norms using divergence-free wavelets and that partially motivated the more general Littlewood-Paley techniques of Cannone and Meyer could only be applied to function spaces defined on all of Euclidean space because those wavelets have global support. In order to extend the techniques of Federbush and of Cannone and Meyer to domains in Euclidean space one needs several key ingredients. The most important of these are: (1) formulas or, at least, estimates for the heat semigroup for the domain; (2) a domain specific analogue of the Littlewood-Paley theory; and (3) bilinear estimates in terms of local Littlewood-Paley projections.

In this paper we will focus on those issues related to localizing the Littlewood-Paley theory. The by now standard method of doing so is based on wavelets.[11] After we briefly review the construction of the symmetric multiwavelets[9] ,[10] we will show the role that they play in analyzing function spaces on domains where symmetry can be exploited. To make the analysis transparent we will restrict to half-spaces although we can handle rectangles now and future modifications will yield wavelets for domains with polygonal boundaries. The function spaces that we will consider include $L^2$-Sobolev spaces and Morrey-Campanato spaces. The specific problems pertaining to our wavelets and these function spaces that will be addressed here are: (1) defining continuous extension operators from a half space to the whole space and restriction operators from a half space to its boundary for the Sobolev and Morrey spaces in terms of wavelets and (2) showing that these spaces are adapted to the bilinear estimates of Cannone and Meyer when one uses wavelets instead of standard Littlewood-Paley theory. These are two of the three main issues surrounding the question of solvability of Navier-Stokes on bounded domains. Estimates for the heat semigroup will complete the analysis, but this is left for future work.

The work of Cohen-Daubechies-Vial[4] was a first step in trying to adapt wavelets to analysis on domains. While their adapted wavelets also led to fast algorithms, their wavelets living at the boundary of the interval were conceptually complicated. We started a program[9] of using DGHM-type multiwavelets with symmetry properties, originally developed by Geronimo, Hardin and Massopust,[5] to answer questions about function spaces on bounded domains – at least intervals and rectangles at present – in a much simpler fashion. Furthermore, we showed how to build derivatives and antiderivatives into the multiresolution structure in the multiwavelet context, using some ideas of Strela, to obtain very simple characterizations of Sobolev spaces on the interval in terms of the size of the wavelet coefficients. This technique also allowed us to build divergence-free multiwavelets which will play an important role in solving Navier-Stokes, although it is not necessary to discuss them further in this paper. In the present work the symmetry properties of the wavelets will play a crucial role, as will the shortness of their supports and their Lipschitz regularity in addressing some of the difficulties that must be overcome when solving problems like Navier-Stokes on rectangles using wavelets.

Here is an outline of the paper. To make it reasonably self-contained, in the next section we will briefly review the multiwavelets,[9] including their symmetry and smoothness properties and how they give rise to biorthogonal families related by differentiation. In Section 3 we will show how the wavelets can be used to define local extension operators from Sobolev spaces defined on a half-space to all of Euclidean space. The main issue here is that the wavelets abutting the boundary of a half line happen to be overcomplete for the functions they span so one wants to know which boundary wavelets yield suitable expansions. In Section 4 we will review what it means for function norms to be adapted to bilinear operators when those operators are expressed in terms of their matrices in the wavelet basis and the norms are expressed in wavelet coordinates as well. Finally in Section 5 we will summarize the correponding extension and adaptation results for the Morrey spaces.

## 2. MULTIWAVELETS

The specific multiwavelets that we use are a modification of the DGHM wavelets.[5] The plots in Figure 1 were produced using Strela’s Matlab based multiwavelet package (`http://www.math.dartmouth.edu/~strela`). One only needs to know the coefficients. For this one notes that the scaling vector $\Phi(x) = [\phi_1(x)\ \phi_2(x)]^T$ and wavelet vector $\Psi(x) = [\psi_1(x)\ \psi_2(x)]^T$ are solutions to the $2 \times 2$ matrix equations

$$\Phi(x) = \sum_{k=-2}^{1} 2C_k\Phi(2x-k) \text{ and } \Psi(x) = \sum_{k=-2}^{1} 2D_k\Phi(2x-k)$$

having coefficient matrices as in Table 1:

**Table 1.** Symmetric scaling and wavelet coefficients

| k | scaling coefficient $C_k(s)$ | wavelet coefficient $D_k(s)$ |
|---|---|---|
| $-2$ | $\frac{1}{24}\begin{bmatrix} 0 & -\sqrt{2}(1+2s) \\ 0 & 0 \end{bmatrix}$ | $\frac{1}{24}\begin{bmatrix} 0 & -\sqrt{2}(1+2s) \\ 0 & -2(1+2s) \end{bmatrix}$ |
| $-1$ | $\frac{1}{24}\begin{bmatrix} -2+8s & \sqrt{2}(5-2s) \\ 0 & 0 \end{bmatrix}$ | $\frac{1}{24}\begin{bmatrix} -2+8s & \sqrt{2}(5-2s) \\ \sqrt{2}(8s-2) & 10-4s \end{bmatrix}$ |
| $0$ | $\frac{1}{24}\begin{bmatrix} 12 & \sqrt{2}(5-2s) \\ 0 & 8+4s \end{bmatrix}$ | $\frac{1}{24}\begin{bmatrix} -12 & \sqrt{2}(5-2s) \\ 0 & 4s-10 \end{bmatrix}$ |
| $1$ | $\frac{1}{24}\begin{bmatrix} -2+8s & -\sqrt{2}(1+2s) \\ 8\sqrt{2}(1-s) & 8+4s \end{bmatrix}$ | $\frac{1}{24}\begin{bmatrix} -2+8s & -\sqrt{2}(1+2s) \\ \sqrt{2}(2-8s) & 2+4s \end{bmatrix}$ |

These scaling vectors and wavelets depend on a parameter $s$ so we will write $\Phi = \Phi_s$ and $\Psi = \Psi_s$ when clarification of the role of $s$ is needed. In particular, for different values of $s$ one obtains different scaling functions and wavelets and when $\tilde{s} = \frac{1+2s}{5s-2}$ the multiresolution analysis generated by $\Phi_s$ and $\Psi_s$ is biorthogonal or *dual* to that generated by $\Phi_{\tilde{s}}$ and $\Psi_{\tilde{s}}$. In other words one has, for example, $\left\langle \psi^i_{sjk}, \psi^{i'}_{\tilde{s}j'k'} \right\rangle = \delta_{ii'}\delta_{jj'}\delta_{kk'}$ where $i = 1, 2$ and $j, k \in \mathbf{Z}$. The smoothness of the wavelets depends on the parameter $s$. In particular it is known that $\Phi_s$ and $\Psi_s$ are Lipschitz continuous provided $s \in [-1/2, 0]$ in which case we also have $\tilde{s} \in [-1/2, 0]$. *Henceforth we shall always assume that $s$ lies in this interval.* In the particular case where $\tilde{s} = -1/5 = s$ one has the DGHM orthonormal multiwavelets.

One word about symmetry of the wavelets is in order. The reader who is familiar with the cascade algorithm will recognize that symmetry or antisymmetry of the filter coefficients translates into symmetry and antisymmetry of the scaling functions and wavelets. Furthermore the supports are related to the indices of the coefficients. For each $s$, $\phi_1$ will have support in $[-1, 1]$ and will be symmetric with respect to the origin while $\phi_2$ will be supported in $[-1, 1]$ and will be symmetric with respect to $x = 1/2$. On the other hand, $\psi_1, \psi_2$ will both be supported in $[-1, 1]$ but $\psi_1$ will be symmetric with respect to $x = 0$ while $\psi_2$ will be antisymmetric.

The combination of symmetry and short support will play an incredibly important role in what follows. Because we are using biorthogonality and multiwavelets, keeping track of different terms can be a notational nightmare. To keep matters simple, first we will subdue the role of $s$. In fact, for most purposes we can assume that the wavelets themselves correspond to the orthogonal case $s = -1/5$. Thus we will not retain any notation to indicate that the analyzing and symthesizing wavelets could in fact be coming from a pair of biorthogonal MRAs rather than a single MRA. To keep track of symmetry we need to separate the behavior of the scaling functions. First, instead of writing $\phi_1$ we will write $\phi^{\mathrm{e}}$ to emphasize that this component is even with respect to $x = 0$ while we shall write $\phi_2 = \phi^{\mathrm{i}}$ to emphasize that this scaling term lives inside the interval $[0, 1]$. Thus $\Phi = [\phi^{\mathrm{e}}\ \phi^{\mathrm{i}}]^T$. On the other hand, we can write $\Psi = [\psi^{\mathrm{e}}\ \psi^{\mathrm{o}}]^T$, now referring to the fact that the first wavelet component is even and the second odd with respect to $x = 0$.

Restricting to the case where $\Phi$ is Lipschitz continuous, as mentioned above, one easily sees that the derivative of $\Phi$ is bounded and has compact support. It turns out that if one differentiates one and antidifferentiates the other of a biorthogonal pair of wavelets the new smoothed and roughened wavelets still come from a multiresolution analysis. This fact will play an important role in the analysis of (1)[9] but will not be addressed further here.

# 3. WAVELETS ON HALF SPACES

While the analysis in this section can be carried out with no trouble in higher dimensions, we will stick to the cases of one and two dimensions for illustrative purposes. The main issue that we want to address here is that of the sense in which one can use wavelets to extend functions from within a half space to functions on the whole space. For simplicity we will restrict our attention to the case of Sobolev spaces.

DEFINITION 1. *For $\alpha \in \mathbf{R}$ the Sobolev space $H^\alpha(\mathbf{R}^n)$ consists of those distributions $f$ whose Fourier transforms are locally square integrable and satisfy $\|f\|^2_{H^\alpha} = \int |\hat{f}(\xi)|^2(1+|\xi|^2)^\alpha d\xi < \infty$. If $\Omega$ is a domain in $\mathbf{R}^n$ then $H^\alpha(\boldsymbol{\Omega}) = \left\{f : f = \tilde{f}|_\Omega \text{ for some } \tilde{f} \in H^\alpha(\mathbf{R}^n)\right\}$. The $H^\alpha(\boldsymbol{\Omega})$ norm of $f$ is $\inf_{\tilde{f}} \left\|\tilde{f}\right\|_{H^\alpha}$.*

REMARK 1. *Given any basis of compactly supported wavelets in $C^\alpha(\mathbf{R}^n)$, or really Lipschitz of order $\alpha \geq 0$, a function $f = \sum_{j,k} c_{jk}\psi_{jk}$ belongs to $H^\alpha(\mathbf{R}^n)$ if and only if $f \in L^2(\mathbf{R}^n)$ and $\sum_{j=1}^{\infty} 2^{2j\alpha} \sum_{k\in\mathbf{Z}^n} |c_{jk}|^2 < \infty$*[8]

In this section $V_j$ and $W_j$ are the usual multiresolution spaces generated by $\mathbf{\Phi}_s$ and $\mathbf{\Psi}_s$ for a specific value of $s$ such as $s = 0$ or $s = -1/5$. In Tables 2 and 3 we describe multiresolution spaces on the half line and half space as the $L^2$-closure of the span of the scaling and wavelet terms having the appropriate support properties.

**Table 2.** Multiresolution spaces for $[0, \infty)$

| Multiresolution space | spanned by or sum of | type |
|---|---|---|
| $V_j^{\text{int}}(\mathbf{R}_+)$ | $\left\{\phi^{\text{e}}_{jk}\right\}_{k=1}^{\infty} \cup \left\{\phi^{\text{i}}_{jk}\right\}_{k=0}^{\infty}$ | interior |
| $W_j^{\text{int}}(\mathbf{R}_+)$ | $\left\{\psi^{\text{e}}_{jk}\right\}_{k=1}^{\infty} \cup \left\{\psi^{\text{o}}_{jk}\right\}_{k=1}^{\infty}$ | interior |
| $V_j^{\partial}(\mathbf{R}_+)$ | $\phi^{\text{e}}_{j0}\chi_{[0,\infty)}$ | boundary |
| $W_j^{\partial}(\mathbf{R}_+)$ | $\psi^{\text{e}}_{j0}\chi_{[0,\infty)},\ \psi^{\text{o}}_{j0}\chi_{[0,\infty)}$ | boundary |
| $L_2^{\text{int}}(\mathbf{R}_+)$ | $V_0^{\text{int}}(\mathbf{R}_+) \oplus \left[\oplus_{j\geq 0} W_j^{\text{int}}(\mathbf{R}_+)\right]$ | interior |
| $L_2^{\partial}(\mathbf{R}_+)$ | $V_0^{\partial}(\mathbf{R}_+) \oplus \left[\oplus_{j\geq 0} W_j^{\partial}(\mathbf{R}_+)\right]$ | boundary |

As indicated, the spaces represent the parts of the multiresolution spaces in $\mathbf{R}$ and $\mathbf{R}^2$ that live inside $\mathbf{R}_+$ and $\mathbf{R}^2_+$ and at the interior boundary of $\mathbf{R}_+$ and $\mathbf{R}^2_+$ respectively. The corresponding $L^2$-spaces are orthogonal or oblique direct sums depending on whether the wavelets are orthogonal or biorthogonal.

One remark concerning the interior boundary spaces is in order. It is not true that the functions $\phi^{\text{e}}\chi_{[0,\infty)}$ together with the $\psi^{\text{e}}_{j0}\chi_{[0,\infty)}$ and $\psi^{\text{o}}_{j0}\chi_{[0,\infty)}$ form a basis for $L_2^{\partial}(\mathbf{R}_+)$ for example. In fact, any element of $L_2^{\partial}(\mathbf{R}_+)$ can be written in terms of either the even scaling and wavelet terms or the odd wavelet terms. This is because the sum of the odd terms inside must equal the sum of the even terms inside, since these sums cancel for almost all $x < 0$. On the other hand, while this argument applies to any expansion converging in $L^2$, it is not clear whether rewriting the odd expansions in terms of the even or vice-versa respects convergence in a more stringent function norm such as a Sobolev norm. To give a rough idea of what can go wrong, consider the wavelets in Figure 1 with $s = 0$. In this

**Table 3.** Multiresolution spaces for $\mathbf{R}^2_+$

| Multiresolution space | tensor product/direct sum description | type |
|---|---|---|
| $V_j^{\text{int}}(\mathbf{R}^2_+)$ | $V_j(\mathbf{R}) \otimes V_j^{\text{int}}(\mathbf{R}_+)$ | interior |
| $W_j^{\text{int}}(\mathbf{R}^2_+)$ | $[V_j(\mathbf{R}) \otimes W_j^{\text{int}}(\mathbf{R}_+)] \oplus [W_j(\mathbf{R}) \otimes V_j^{\text{int}}(\mathbf{R}_+)] \oplus [W_j(\mathbf{R}) \otimes W_j^{\text{int}}(\mathbf{R}_+)]$ | interior |
| $V_j^{\partial}(\mathbf{R}^2_+)$ | $V_j(\mathbf{R}) \otimes V_j^{\partial}(\mathbf{R}_+)$ | boundary |
| $W_j^{\partial}(\mathbf{R}^2_+)$ | $[V_j(\mathbf{R}) \otimes W_j^{\partial}(\mathbf{R}_+)] \oplus [W_j(\mathbf{R}) \otimes V_j^{\partial}(\mathbf{R}_+)] \oplus [W_j(\mathbf{R}) \otimes W_j^{\partial}(\mathbf{R}_+)]$ | boundary |
| $L_2^{\text{int}}(\mathbf{R}_+)$ | $V_0^{\text{int}}(\mathbf{R}^2_+) \oplus \left[\oplus_{j>0} W_j^{\text{int}}(\mathbf{R}^2_+)\right]$ | interior |
| $L_2^{\partial}(\mathbf{R}_+)$ | $V_0^{\partial}(\mathbf{R}^2_+) \oplus \left[\oplus_{j\geq 0} W_j^{\partial}(\mathbf{R}^2_+)\right]$ | boundary |

case $\psi^{\mathrm{e}}$ is piecewise linear so that for small positive $x$ we may write $\psi^{\mathrm{e}}(x) = ax + b, b \neq 0$. For correspondingly large values of the scale parameter $j \geq j_0$, where $j_0$ is chosen so that $\psi^{\mathrm{e}}(x) = ax + b$ on $[0, 1/2^{j_0}]$, one estimates

$$\begin{aligned} \left\langle \psi^{\mathrm{e}}, \tilde{\psi}^{\mathrm{o}}_j \right\rangle &= \int_0^1 (ax+b)\tilde{\psi}^{\mathrm{o}}_j(x)dx = \int_0^1 (ax+b)2^{\frac{j}{2}}\tilde{\psi}^{\mathrm{o}}(2^j x)dx \\ &\approx c2^{-\frac{j}{2}} + d2^{-\frac{3j}{2}} \end{aligned}$$

where $c = b\int_0^1 \tilde{\psi}^{\mathrm{o}}(x)dx$ and $d = a\int_0^1 au\tilde{\psi}^{\mathrm{o}}(u)du$. We note that $c \neq 0$, though this is not completely evident from the graph of $\tilde{\psi}^{\mathrm{o}}$. It follows that the coefficients of $\psi^{\mathrm{e}}\chi_{[0,\infty)}$, when expanded in terms of the functions $\psi^{\mathrm{o}}_{j0}\chi_{[0,\infty)}$ decay like $c2^{-\frac{j}{2}}$. This indicates that $\sum_{j=1}^{\infty} 2^{2j\alpha}\left|\left\langle \psi^{\mathrm{e}}\chi_{[0,\infty)}, \tilde{\psi}^{\mathrm{o}}_j \right\rangle\right|^2$ fails to converge whenever $\alpha \geq 1/2$. Therefore, in view of Remark 1 which applies to the present wavelets one sees that the expansion of $\psi^{\mathrm{e}}\chi_{[0,\infty)}$ in terms of the odd wavelets fails to identify $\psi^{\mathrm{e}}\chi_{[0,\infty)}$ as the restriction to $[0,\infty)$ of a function in the Sobolev space $H^{\alpha}(\mathbf{R})$ when $\alpha \in [1/2, 1]$. This dichotomy is expected because the space $H^{\alpha}_0[0,\infty)$ – the closure in the Sobolev norm of $C^{\infty}_c(0,\infty)$ – and the space $H^{\alpha}[0,\infty) = \left\{f\chi_{[0,\infty)} : f \in H^{\alpha}(\mathbf{R})\right\}$ agree for $\alpha < 1/2$ but differ for $\alpha \geq 1/2$.[3] In summary, one has to be careful about which frame expressions identify convergence in a Sobolev norm.

For any $\alpha \in [0,1]$ it follows from the Lipschitz property of the wavelets that $H^{\alpha}(\mathbf{R}) \cap W^{\mathrm{int}}_j(\mathbf{R}_+) \subset H^{\alpha}_0(\mathbf{R}_+)$. A similar statement holds for the scaling space and this works in higher dimensions as well. Our first goal is to build, in the case $\alpha \in [0,1]$ and $\Omega$ is a half-space, a particular continuous extension mapping from $H^{\alpha}(\Omega)$ to $H^{\alpha}(\mathbf{R}^n)$ that uses the wavelets described above. We just consider the case of the half line for now. First we recall the following theorem about extensions of Sobolev spaces. Let $f$ be a function defined on $\mathbf{R}^n_+ = \{x = (x_1, ..., x_n) : x_n \geq 0\}$. We define the even extensions of $f$ to be $f^{\mathrm{e}}(x_1, ..., -x_n) = f(x_1, ..., x_n)$ when $x_n \geq 0$.

THEOREM 3.1. [3] *The even extension mapping $f \mapsto f^e$ is continuous from $H^{\alpha}(\mathbf{R}^n_+)$ to $H^{\alpha}(\mathbf{R}^n)$ whenever $0 \leq \alpha \leq 1$.*

COROLLARY 3.2. *If $f \in L^{\partial}_2(\mathbf{R}_+) \cap H^{\alpha}(\mathbf{R}_+)$ $(0 \leq \alpha \leq 1)$ then the frame expansion $f = c_0\phi^e_{10}\chi_{[0,\infty)} + \sum_{j=1}^{\infty} c_j\psi^e_{j0}\chi_{[0,\infty)}$ satisfies $\sum_{j=1}^{\infty} 2^{2j\alpha}|c_j|^2 < \infty$. That is, this frame expansion converges to $f$ in $H^{\alpha}(\mathbf{R}_+)$.*

*Proof.* First we recall that such a frame expansion that converges to $f$ in $L^2$ is possible by Remark 1. Now consider the mapping $f \mapsto f^{\mathrm{e}}$. Since $f^{\mathrm{e}}$ is even one has $\left\langle f^{\mathrm{e}}, \tilde{\psi}^{\mathrm{o}}_j \right\rangle = 0$ for all $j = 1, 2, ...$ and because $f \in L^{\partial}_2(\mathbf{R}_+)$, the symmetry properties of the wavelets guarantee that $\left\langle f^{\mathrm{e}}, \tilde{\phi}_{1k} \right\rangle = 0 = \left\langle f^{\mathrm{e}}, \tilde{\psi}_{jk} \right\rangle$ whenever $k < 0$. Therefore $f^{\mathrm{e}} = c_0\phi^{\mathrm{e}}_{10} + \sum_{j=1}^{\infty} c_j\psi^{\mathrm{e}}_{j0}$ where $c_j = \left\langle f^{\mathrm{e}}, \tilde{\psi}^{\mathrm{e}}_{j0} \right\rangle$ by the properties of biorthogonal basis expansions for $L^2(\mathbf{R})$. But by Theorem 3.1, $f^{\mathrm{e}} \in H^{\alpha}(\mathbf{R})$ so that $\sum_{j=1}^{\infty} 2^{2j\alpha}|c_j|^2 < \infty$. Restricting $f$ back to $[0,\infty)$ gives the desired conclusion. ◻

The same sort of argument serves to prove the following even wavelet extension result for functions in $H^{\alpha}(\mathbf{R}^2_+)$:

COROLLARY 3.3. *If $f \in L^{\partial}_2(\mathbf{R}^2_+) \cap H^{\alpha}(\mathbf{R}^2_+)$ $(0 \leq \alpha \leq 1)$ then the frame expansion*

$$\begin{aligned} f &= \left[\sum_k c^e_{0k}\phi^e_{1k} + c^{int}_{0k}\phi^{int}_{1k}\right] \otimes \phi^e_{10}\chi_{[0,\infty)} + \sum_{j=1}^{\infty}\left[\sum_k c^e_{jk}\phi^e_{jk} + c^{int}_{jk}\phi^{int}_{jk}\right] \otimes \psi^e_{j0}\chi_{[0,\infty)} \\ &+ \sum_{j=1}^{\infty}\left[\sum_k d^e_{jk}\psi^e_{jk} + d^o_{jk}\psi^o_{jk}\right] \otimes \phi^e_{j0}\chi_{[0,\infty)} + \sum_{j=1}^{\infty}\left[\sum_k e^e_{jk}\psi^e_{jk} + e^o_{jk}\psi^o_{jk}\right] \otimes \psi^e_{j0}\chi_{[0,\infty)} \end{aligned}$$

*satisfies*

$$\sum_k \left[|c^e_{0k}|^2 + |c^{int}_{0k}|^2\right] + \sum_{j=1}^{\infty} 2^{2j\alpha}\left[|c^e_{jk}|^2 + |c^{int}_{jk}|^2 + |d^e_{jk}|^2 + |d^o_{jk}|^2 + |e^e_{jk}|^2 + |e^o_{jk}|^2\right] < \infty.$$

*That is, this frame expansion converges to $f$ in $H^{\alpha}(\mathbf{R}^2_+)$.*

Next we move on to the question of what the boundary values of $H^{\alpha}(\mathbf{R}^2_+)$ functions look like. The results we present are classical,[37] and wavelet techniques play an important role in their most general formulation, but the present purpose is to give very concrete proofs based on the use of symmetric wavelets.

Corollary 3.4. *If $f \in L_2^{\partial}(\mathbf{R}_+^2) \cap H^{\alpha}(\mathbf{R}_+^2)$ $(0 \le s \le 1)$ then the restriction of $f$ to $\mathbf{R}$ belongs to $H^{\alpha-\frac{1}{2}}(\mathbf{R})$ with convergence in the weak sense if $\alpha \le 1/2$.*

*Proof.* By the previous corollary we know that the 'even' frame expansion converges to $f$ so restricting $f$ to $\mathbf{R}$ is the same as evaluating the expansion terms in the vertical variable at $y = 0$. This gives

$$\begin{aligned} f|_{\mathbf{R}} &= \phi_{10}^{\mathrm{e}}(0)\left[\sum_k c_{0k}^{\mathrm{e}}\phi_{1k}^{\mathrm{e}} + c_{0k}^{\mathrm{int}}\phi_{1k}^{\mathrm{int}}\right] + \psi_{j0}^{\mathrm{e}}(0)\sum_{j=1}^{\infty}\left[\sum_k c_{jk}^{\mathrm{e}}\phi_{jk}^{\mathrm{e}} + c_{jk}^{\mathrm{int}}\phi_{jk}^{\mathrm{int}}\right] \\ &\quad + \phi_{j0}^{\mathrm{e}}(0)\sum_{j=1}^{\infty}\left[\sum_k d_{jk}^{\mathrm{e}}\psi_{jk}^{\mathrm{e}} + d_{jk}^{\mathrm{o}}\psi_{jk}^{\mathrm{o}}\right] + \psi_{j0}^{\mathrm{e}}(0)\sum_{j=1}^{\infty}\left[\sum_k e_{jk}^{\mathrm{e}}\psi_{jk}^{\mathrm{e}} + e_{jk}^{\mathrm{o}}\psi_{jk}^{\mathrm{o}}\right] \\ &= \phi_{10}^{\mathrm{e}}(0)\left[\sum_k c_{0k}^{\mathrm{e}}\phi_{1k}^{\mathrm{e}} + c_{0k}^{\mathrm{int}}\phi_{1k}^{\mathrm{int}}\right] + \psi^{\mathrm{e}}(0)\sum_{j=1}^{\infty}2^{j/2}\left[\sum_k c_{jk}^{\mathrm{e}}\phi_{jk}^{\mathrm{e}} + c_{jk}^{\mathrm{int}}\phi_{jk}^{\mathrm{int}}\right] \\ &\quad + \phi^{\mathrm{e}}(0)\sum_{j=1}^{\infty}2^{j/2}\left[\sum_k d_{jk}^{\mathrm{e}}\psi_{jk}^{\mathrm{e}} + d_{jk}^{\mathrm{o}}\psi_{jk}^{\mathrm{o}}\right] + \psi^{\mathrm{e}}(0)\sum_{j=1}^{\infty}2^{j/2}\left[\sum_k e_{jk}^{\mathrm{e}}\psi_{jk}^{\mathrm{e}} + e_{jk}^{\mathrm{o}}\psi_{jk}^{\mathrm{o}}\right] \end{aligned}$$

since $\psi_{j0}^{\mathrm{e}}(0) = 2^{j/2}\psi^{\mathrm{e}}(0)$ and $\phi_{j0}^{\mathrm{e}}(0) = 2^{j/2}\phi^{\mathrm{e}}(0)$. It follows from the characterization of $H^{\alpha}(\mathbf{R})$ in terms of wavelet coefficients, together with Corollary 3.3, that the last two terms above converge in $H^{\alpha-\frac{1}{2}}(\mathbf{R})$ – keeping in mind that the convergence is in the weak sense if $\alpha \le 1/2$. One just needs to check that $\sum_{j=1}^{\infty} 2^{j/2}\left[\sum_k c_{jk}^{\mathrm{e}}\phi_{jk}^{\mathrm{e}} + c_{jk}^{\mathrm{int}}\phi_{jk}^{\mathrm{int}}\right]$ converges in $H^{\alpha-\frac{1}{2}}(\mathbf{R})$. Since symmetry properties play no role in this last argument we will pretend we are back in the case of 'ordinary' wavelets just to make the notation simpler. The same argument holds when one works with biorthogonal multiwavelets – only the notation is complicated by this fact. Since the scaling function $\phi$ satisfies $\|\phi_{jk}\|_{H^{\alpha}} \le C2^{j\alpha}$, by the multiresolution decomposition and the coefficient characterization of the Sobolev norm we can then write $\phi_{jk} = \sum_{\nu<j}\sum_l \beta_{jk\nu l}\psi_{\nu l}$ where $\sum_{\nu<j} 2^{\nu\alpha}\sum_l |\beta_{jk\nu l}|^2 \le C2^{2j\alpha}$. It follows that if

$$f = \sum_{j=1}^{\infty} 2^{j/2}\left[\sum_k c_{jk}\phi_{jk}\right] = \sum_{j=1}^{\infty} 2^{j/2}\left[\sum_k c_{jk}\sum_{\nu<j}\sum_l \beta_{jk\nu l}\psi_{\nu l}\right]$$

then

$$\begin{aligned} \|f\|_{H^{\alpha-1/2}}^2 &\le C\sum_{\nu} 2^{2\nu(\alpha-\frac{1}{2})}\sum_l \left|\sum_{j>\nu}^{\infty} 2^{j/2}\sum_k c_{jk}\beta_{jk\nu l}\right|^2 \\ &\le C\sum_{\nu} 2^{2\nu(\alpha-\frac{1}{2})}\sum_l \sum_{j>\nu,k}^{\infty} 2^j|c_{jk}|^2 \sum_{j>\nu,k}^{\infty}|\beta_{jk\nu l}|^2 \\ &= C\sum_{j,k}^{\infty} 2^j|c_{jk}|^2\sum_{\nu<j} 2^{2\nu(\alpha-\frac{1}{2})}\sum_l |\beta_{jk\nu l}|^2 \\ &= C\sum_{j,k}^{\infty} 2^j|c_{jk}|^2 2^{2j(\alpha-\frac{1}{2})} = C\sum_{j,k}^{\infty} 2^{2j\alpha}|c_{jk}|^2 < \infty \end{aligned}$$

by the previous corollary. Combining this estimate with the immediate ones for the other terms proves the corollary. □

## 4. ADAPTED SPACES

In this section we wish to investigate the sense in which the Sobolev spaces $H^{\alpha}$ are adapted to certain bilinear singular integral operators. Specifically we will obtain Sobolev estimates for wavelet subspace projections of pointwise products and give a brief indication of how these estimates can be applied to special bilinear singular integrals such as the one arising in the variational form of Navier-Stokes. To streamline the presentation we will no longer adapt the analysis to domains. We note in passing, however, that the analysis in this section can easily be combined with the half space analysis in the previous section to obtain Sobolev estimates for products on half spaces. The techniques presented here will apply to any wavelet basis in which the wavelets have compact support and Lipschitz continuity. Here we will assume for convenience that the wavelets are *orthogonal*. There are some not so trivial issues attached to our goal of solving Navier-Stokes when one uses biorthogonal wavelets – but again we will be content to let those issues lurk in the background. Finally we will use the notation of 'ordinary' wavelets – not multiwavelets – for simplicity.

If $T$ is any linear operator defined on finite linear combinations of wavelets then formally we can write $T = \sum_{i,j} Q_i T Q_j$ where $Q_j$ is the operator now of orthogonal projection onto the wavelet space $W_j$. The reason for writing $T$ this way is that then one can express $T$ in terms of its wavelet matrix and obtain estimates for $T$ in terms of this matrix. If $T(\psi_{jk}) = \sum_{im} c_{imjk}\psi_{im}$ then the matrix of $T$ in the wavelet coordinates is $M_T = \{c_{imjk}\}$. In order to make full use of the wavelet decomposition one must hope for the possibility of obtaining estimates for the wavelet projections of pointwise products in terms of the norms of the factors, that is, for estimates of the form

$$\|Q_j(fg)\|_X \leq \eta_j \|f\|_X \|g\|_X . \tag{3}$$

Such estimates enable one to control $T$ in turn by the sizes of the separate rows of its wavelet matrix. To extoll the virtues of matrix analysis we first consider the simplest type of operator that one can envision when expressed in wavelet coordinates: one that maps $W_j$ back into $W_j$.

DEFINITION 2. *(i) A linear operator $T$ that maps $W_j$ back into $W_j$ is called a wavelet subspace multiplier.*

*(ii) A wavelet subspace multiplier $T$ is adapted to pointwise products on $X$ provided $\sum \eta_j \left\|T|_{W_j}\right\|_{X\to X} < \infty$ where $\eta_j$ is as in (3).*

COROLLARY 4.1. *If a wavelet subspace multiplier $T$ is adapted to pointwise products on $X$ then $(f,g) \mapsto T(fg)$ is strongly continuous from $X \times X \to X$.*

*Proof.* This is clear because $T = \sum_{i,j} Q_i T Q_j$ so that

$$T(fg) = \sum_{i,j} Q_i T Q_j(fg) = \sum_{i,j} Q_i Q_j T Q_j(fg) = \sum_j Q_j T Q_j(fg)$$

by the subspace multiplier property. But then

$$\|T(fg)\|_X \leq \sum_j \|TQ_j(fg)\|_X \leq \sum_j \left\|T|_{W_j}\right\|_{X\to X} \|Q_j(fg)\|_X \leq \|f\|_X \|g\|_X \sum_j \eta_j \left\|T|_{W_j}\right\|_{X\to X}$$

This proves the corollary. □

The problem with our definition of wavelet multipliers is that it depends on the particular wavelet basis chosen. On the other hand, unlike Fourier multipliers the operators in which we are ultimately interested are not convolutions because they operate on functions defined on proper domains of $\mathbf{R}^n$. Wavelet multipliers are a suitable first approximation to operators having the property that the norm of $Q_i T Q_j$ decays sufficiently rapidly with $|i-j|$. More generally,

DEFINITION 3. *A singular integral operator $T$ satisfying $\sum_{i,j=1}^{\infty} \eta_j \|Q_i T Q_j\|_{X\to X} < \infty$ where $\eta_j$ is as in (3) is said to be adapted to pointwise products on $X$.*

Essentially the same argument as above once again gives boundedness of $(f,g) \mapsto T(fg)$. This illustrates why one wants estimates on projections of pointwise products. The operator $\mathbf{P}S(t)\nabla$ in (2) is adapted to pointwise products for the Sobolev spaces[2] because the net smoothing imposed by the heat semigroup effectively pushes energy from higher to lower scales in the wavelet decomposition.

At this stage we will go into a little bit of the detail concerning how to obtain estimates of the form (3). The main result of this section is:

THEOREM 4.2. *Let $Q_j$ denote the projection onto the $j$-th wavelet space of a multiresolution analysis of $L^2(\mathbf{R})$ where the wavelets are orthogonal and Lipschitz with compact support. Then for $0 \leq \alpha \leq 1$ and $j \geq 1$,*

$$\|Q_j(fg)\|_{H^\alpha} \leq C 2^{j(\frac{1}{2}-\alpha)} \|f\|_{H^\alpha} \|g\|_{H^\alpha}$$

The same argument applies to obtain a corresponding estimate in higher dimensions, where $\eta_j = C2^{j(\frac{n}{2}-\alpha)}$.

The first stage of the argument is based on the paraproduct decomposition. This decomposition defines the pointwise product when it converges. In terms of multiresolution projections, one writes $f = P_0 f + \sum_{j=0}^{\infty} Q_j f$ and similarly for $g$ and, upon rearranging sums in the multiresolution decompositions of functions or distributions $f, g$ one obtains

$$fg = \sum_{j=0}^{\infty} (Q_j f)\,(P_{j-2} g) + \sum_{j=0}^{\infty} (Q_j g)\,(P_{j-2} f) + \sum_{|i-j|\le 2} Q_i f Q_j g + P_0 f P_0 g. \tag{4}$$

Naturally we mean $P_{j-2} = 0$ when $j < 2$. Now one can estimates $Q_j(fg)$ by applying $Q_j$ to each term in the paraproduct expansion.

## 4.1. First term: $Q_i\left[(Q_j f)\,(P_{j-2} g)\right]$

When the $P_j, Q_j$ are traditional Littlewood-Paley operators one benefits from the fact that, for example, $Q_j g P_{j-2} f$ is the inverse Fourier transform of the convolution of two functions supported in $[-2^{j+1}, 2^{j+1}]$ to conclude that $Q_k\left[Q_j g P_{j-2} f\right] = 0$ if, say, $k > j+2$. When the $P_j, Q_j$ conform to wavelet projections, however, one cannot automatically throw away such terms. Nevertheless, one can still estimate the norm of the bilinear operator $(f, g) \mapsto Q_i\left[Q_j g P_{j-2} f\right]$. In order to do so, we look at a typical wavelet coefficient in this expansion.

### 4.1.1. The case $j < i$

For $i > j$, pretending $\operatorname{supp}\phi, \psi \subset [-1, 1]$:

$$\begin{aligned} \langle \psi_{jk_1}\phi_{j-2,k_2}, \psi_{i0}\rangle &= 2^{(i-2)/2}\int_{-1/2^i}^{1/2^i} 2^j \psi(2^j x - k_1)\phi(2^{j-2}x - k_2)\psi(2^i x)dx \\ &= 2^{(i-2)/2} 2^j \int_{-1/2^i}^{1/2^i} [\psi(2^j x - k_1) - \psi(x_{jk_1})][\phi(2^{j-2}x - k_2) - \phi(x_{j-2,k_2})]\psi(2^i x)dx. \end{aligned}$$

Here we have used orthogonality of the scaling functions and wavelets together with the fact that $\psi$ has integral zero. Also, $x_{jk} = k/2^j$ is thought of as the center of the support of $\psi_{jk}$. If one worked instead with biorthogonal wavelets one would need to keep track of decay of terms like $\langle \psi_{jk_1}, \psi_{i0}\rangle$. The result of Theorem 4.2 still holds but the book-keeping is more intense. At this stage we apply the Lipschitz continuity of $\psi$ to conclude that the integral is at most a constant times $2^{2j-3i}$ so that, altogether, we have

$$|\langle \psi_{jk_1}\phi_{j-2,k_2}, \psi_{i0}\rangle| \le c2^{i/2}2^{3(j-i)}.$$

Now the coefficient of $\psi_{im}$ in $Q_i\left[Q_j g P_{j-2} f\right]$ will be

$$\sum_{k_1,k_2} c_{jk_1} d_{j-2,k_2} \left\langle \psi_{jk_1}\phi_{j-2,k_2}, \psi_{im}\right\rangle .$$

Because of the support properties of the wavelets, for each $m$ there will be at most a fixed finite number of values of $k_1$ and $k_2$ such that $\langle \psi_{jk_1}\phi_{j-2,k_2}, \psi_{im}\rangle \neq 0$. Therefore it does no harm to write

$$Q_i\left[Q_j g P_{j-2} f\right] = \sum_m c_{jk_1(m)} d_{j-2,k_2(m)} \left\langle \psi_{jk_1}\phi_{j-2,k_2}, \psi_{im}\right\rangle$$

and by the wavelet characterization of the Sobolev spaces we have

$$\begin{aligned} \|Q_i\left[Q_j g P_{j-2} f\right]\|_{H^\alpha}^2 &\le C2^{2i\alpha}\sum_m \left|c_{jk_1(m)} d_{j-2,k_2(m)} \left\langle \psi_{jk_1}\phi_{j-2,k_2}, \psi_{im}\right\rangle\right|^2 \\ &\le C2^{2i(\alpha+1/2)}2^{5(j-i)}\sum_m \left|c_{jk_1(m)} d_{j-2,k_2(m)}\right|^2 \end{aligned}$$

$$
\begin{aligned}
&\leq C 2^{5j+2i(\alpha-2)} \left[\sum_m \left|c_{jk_1(m)}\right|^2\right] \left[\sum_m \left|d_{j-2,k_2(m)}\right|^2\right] \\
&\leq C 2^{5j+2i(\alpha-2)} \left[\sum_k |c_{jk}|^2\right] \left[\sum_k |d_{j-2,k}|^2\right] \\
&\leq C 2^{5j+2i(\alpha-2)-4j\alpha} \|f\|^2_{H^\alpha} \|g\|^2_{H^\alpha}
\end{aligned}
$$

and this estimate applies whenever $j < i$. In the second inequality we used the bound for $\langle \psi_{jk_1}\phi_{j-2,k_2}, \psi_{im}\rangle$ together with the fact that the coefficient $c_{jk_1}d_{j-2,k_2}$ can appear on the order of $2^{i-j}$ times in the sum. Summing these estimates over all $i < j$ yields

$$
\begin{aligned}
\left\| Q_i \left[\sum_{0\leq j<i} Q_j g P_{j-2} f\right]\right\|_{H^\alpha} &\leq \sum_{0\leq j<i} \|Q_i [Q_j g P_{j-2} f]\|_{H^\alpha} \\
&\leq C \|f\|_{H^\alpha} \|g\|_{H^\alpha} \sum_{0\leq j<i} 2^{5j/2+i(\alpha-2)-2j\alpha} \\
&\leq C \|f\|_{H^\alpha} \|g\|_{H^\alpha} 2^{-i\alpha} \sum_{0\leq j<i} 2^{5j/2-2i+2\alpha(i-j)} \\
&\leq C \|f\|_{H^\alpha} \|g\|_{H^\alpha} 2^{i(\frac{1}{2}-\alpha)}
\end{aligned}
$$

which is valid for $\alpha < 5/4$. Next we proceed to estimating those terms $Q_i \left[\sum_{i<j} Q_j g P_{j-2} f\right]$.

### 4.1.2. The case $j > i$

Here we wish to estimate $\|Q_i [Q_j g P_{j-2} f]\|_{H^\alpha}$ where $j > i$. In this case the term $\psi_{i0}$ is the term to which a Lipschitz estimate can be applied to give

$$
\begin{aligned}
\langle \psi_{jk_1}\phi_{j-2,k_2}, \psi_{i0}\rangle &= 2^{(i-2)/2} \int_{(k_1-1)/2^j}^{(k_1+1)/2^j} 2^j \psi(2^j x - k_1)\phi(2^{j-2}x - k_2)[\psi(2^i x) - \psi(2^{i-j}k_1)]dx \\
&\leq C 2^{3i/2-j},
\end{aligned}
$$

where this time we have used $\langle \psi_{jk_1}, \phi_{j-2,k_2}\rangle = 0$ together with the Lipschitz estimate on $\psi$. Therefore we can estimate

$$
\begin{aligned}
\|Q_i [Q_j g P_{j-2} f]\|^2_{H^\alpha} &\leq C 2^{2i\alpha} \sum_m |c_{jk_1} d_{j-2,k_2} \langle \psi_{jk_1}\phi_{j-2,k_2}, \psi_{im}\rangle|^2 \\
&\leq C 2^{2i\alpha} 2^{3i-2j} 2^{j-i} \sum_{k_1} \sum_{k_2\sim k_1} |c_{jk_1} d_{j-2,k_2}|^2 \\
&\leq C 2^{2i(\alpha+1)-j} \left[\sum_k |c_{jk}|^2\right] \left[\sum_k |d_{j-2,k}|^2\right] \\
&\leq C 2^{2i(\alpha+1)} 2^{-j-4j\alpha} \|f\|^2_{H^\alpha} \|g\|^2_{H^\alpha}
\end{aligned}
$$

So this time we have

$$
\begin{aligned}
\left\| Q_i \sum_{j>i} [Q_j g P_{j-2} f]\right\|_{H^\alpha} &\leq C 2^{i(\alpha+1)} \|f\|_{H^\alpha} \|g\|_{H^\alpha} \sum_{j>i} 2^{-j/2-2j\alpha} \\
&\leq C 2^{i(\alpha+1)} 2^{-i/2-2i\alpha} \|f\|_{H^\alpha} \|g\|_{H^\alpha} \\
&= C 2^{i(\frac{1}{2}-\alpha)} \|f\|_{H^\alpha} \|g\|_{H^\alpha}
\end{aligned}
$$

This together with a corresponding estimate for the case $i = j$ completes the estimate of the first term in the paraproduct expansion. The estimate of the second term is exactly the same with the roles of $f, g$ reversed.

## 4.2. Estimate of the third term

Here we need to estimate the final term $Q_j(\sum_{k\geq j} Q_k f Q_k g)$ that arises when we consider the paraproduct decomposition (4). In Cannone-Meyer one uses the fact that the Littlewood-Paley projections are convolutions to apply Young's inequality and reduce the estimate to that of the $L^1$-norm of $F = \sum_{k\geq j} Q_k f Q_k g$. But in the present case we can express $Q_j F$ by integration of $F$ against the kernel $K_j(x,y) = \sum_m \psi_{jm}(x)\psi_{jm}(y)$. Then

$$\begin{aligned}\|Q_j F\|_2^2 &= \int \left| \int K_j(x,y)F(y)dy \right|^2 dx \\ &= \int F(y) \int F(z) \int K_j(x,y)K_j(x,z)dxdydz\end{aligned}$$

where Fubini is justified by the fact that the integrals are all absolutely convergent so in conclusion

$$\|Q_j F\|_2^2 \leq \sup_{y,z} \left| \int K_j(x,y)K_j(x,z)dx \right| \|F\|_1^2 .$$

But here we note that

$$|K_j(x,y)K_j(x,z)| = \left| \sum_{m,n} \psi_{jm}(x)\psi_{jn}(x)\psi_{jm}(y)\psi_{jn}(z) \right| .$$

Now we use our fantasy that $\psi$ is supported in $[-1,1]$ to control this sum and the domain of the integration variable $x$ independent of $y, z$. In particular, for each value of $y, z$ there are at most three values of $m$ and of $n$ such that $\psi_{jm}(y) \neq 0 \neq \psi_{jn}(z)$. Then we have at most nine terms contributing to the sum so it is easy to see that

$$\begin{aligned}|K_j(x,y)K_j(x,z)| &\leq \|\psi\|_\infty \, 2^j \left| \sum_{m,n=-1}^{1} \psi_{jm}(x)\psi_{jn}(x) \right| \\ &\leq \|\psi\|_\infty^2 \, 2^j \sum_{n=-1}^{1} |\psi_{jm}(x)|^2\end{aligned}$$

by Cauchy-Schwarz. We conclude that

$$\sup_{y,z} \left| \int K_j(x,y)K_j(x,z)dx \right| \leq C2^j \, \|\psi\|_2^2$$

so the conclusion of Young's inequality is preserved as is the remainder of the argument of Cannone and Meyer[2] from which one concludes that

$$\left\| Q_j(\sum_{k\geq j} Q_k f Q_k g) \right\|_{L^2} \leq C2^{j(1/2-\alpha)} \, \|f\|_{H^\alpha} \, \|g\|_{H^\alpha}$$

and the desired Sobolev estimate follows. This, together with corresponding estimates for the wavelet projections of $P_0 f P_0 g$ – obtained by similar methods – proves Theorem 4.2. We note in passing that estimates could be obtained for larger values of $\alpha$ by subtracting higher moments if necessary if one has more regularity and vanishing moments of the wavelets themselves.

# 5. THE MORREY SPACES

Much the same results that hold for the Sobolev spaces hold for the scale of Morrey spaces. These spaces are function spaces where local regularity is measured by the type of peaks or the order of energy localization that a function can have. In this sense they provide appropriate norms for measuring, for example, the strength of a vortex. Federbush[6] used descriptions of the Morrey norms in terms of wavelet/Carleson coefficient norms together with divergence-free wavelets to solve the variational form of Navier-Stokes in $\mathbf{R}^3$.

DEFINITION 4. *Let $\Omega$ be a domain in $\mathbf{R}^n$. The Morrey space $M^\lambda(\mathbf{\Omega})$ is defined for $0 \le \lambda \le n$ as the space of locally square integrable functions such that*

$$\|f\|^2_{M^\lambda(\Omega)} = \sup_{x;R\le 1} \frac{1}{R^\lambda}\int_{B(x,R)\cap\Omega} |f|^2 < \infty.$$

Note that $M^0(\mathbf{R}^n) = L^2(\mathbf{R}^n)$ and $M^n(\mathbf{R}^n) = L^\infty(\mathbf{R}^n)$. One can extend the definition to $\lambda > n$ by subtracting moments but we will not pursue this aspect. There are other useful variations of the basic definition that have not been adequately exploited. For example, one can consider those functions having derivatives in the Morrey space. That is, we can define $M^{\lambda,m}(\mathbf{\Omega}) = \{f : \partial^\alpha f \in M^\lambda(\mathbf{\Omega}), |\alpha| \le m\}$. Unlike the Sobolev spaces, it is not immediately clear how to define Morrey spaces with fractional derivatives, but it turns out that wavelets actually provide a means for this. The motivation for doing so is provided by the following result whose proof is simple, provided one has an adequate notion of how things converge.

THEOREM 5.1.[1] *Let $\{\psi_{jk}\}$ be a basis of orthonormal, compactly supported wavelets for $\mathbf{R}$ and let $Q_{jk} = [k/2^j, (k+1)/2^j)$. Then $f = \sum_{jk} c_{jk}\psi_{jk} \in M^\lambda(\mathbf{R})$, $0 \le \lambda < n$ if and only if there is a constant $C$ such that, for all intervals $I, |I| \le 1$, one has $\sum_{Q_{jk}\subset I} |c_{jk}|^2 \le C|I|^\lambda$.*

The same method of proof also shows that $f = \sum_{jk} c_{jk}\psi_{jk} \in M^{\lambda,1}(\mathbf{R})$ if and only if there is a constant $C$ such that $\frac{1}{|I|^\lambda}\sum_{Q_{jk}\subset I} 2^{2j}|c_{jk}|^2 \le C$. In view of this fact we have

DEFINITION 5. *For $\alpha \ge 0$ and $0 \le \lambda < n$, we say that $f = \sum_{jk} c_{jk}\psi_{jk} \in M^{\lambda,\alpha}(\mathbf{R})$ if and only if*

$$\sup_{I:|I|<1} \frac{1}{|I|^\lambda} \sum_{Q_{jk}\subset I} 2^{2j\alpha}|c_{jk}|^2 < \infty.$$

Actually the wavelets must have sufficient regularity in order that the definition makes sense. With this caveat in mind, it is a fact that $M^{\lambda,\alpha}(\mathbf{R})$ is a Banach space whose norm is the square root of the supremum of $\frac{1}{|I|^\lambda}\sum_{Q_{jk}\subset I} 2^{2j\alpha}|c_{jk}|^2$ taken over all intervals. When $\alpha = 0$ or $\alpha = 1$ this norm is equivalent to the standard Morrey norm. At this stage we will just state the results about extending Morrey functions and about which Morrey norms are adapted to products.

THEOREM 5.2. *The even extension mapping $f \mapsto f^e$ is continuous from $M^{\lambda,\alpha}(\mathbf{R}^n_+)$ to $M^{\lambda,\alpha}(\mathbf{R}^n)$ whenever $0 \le \alpha \le 1, 0 \le \lambda \le n$.*

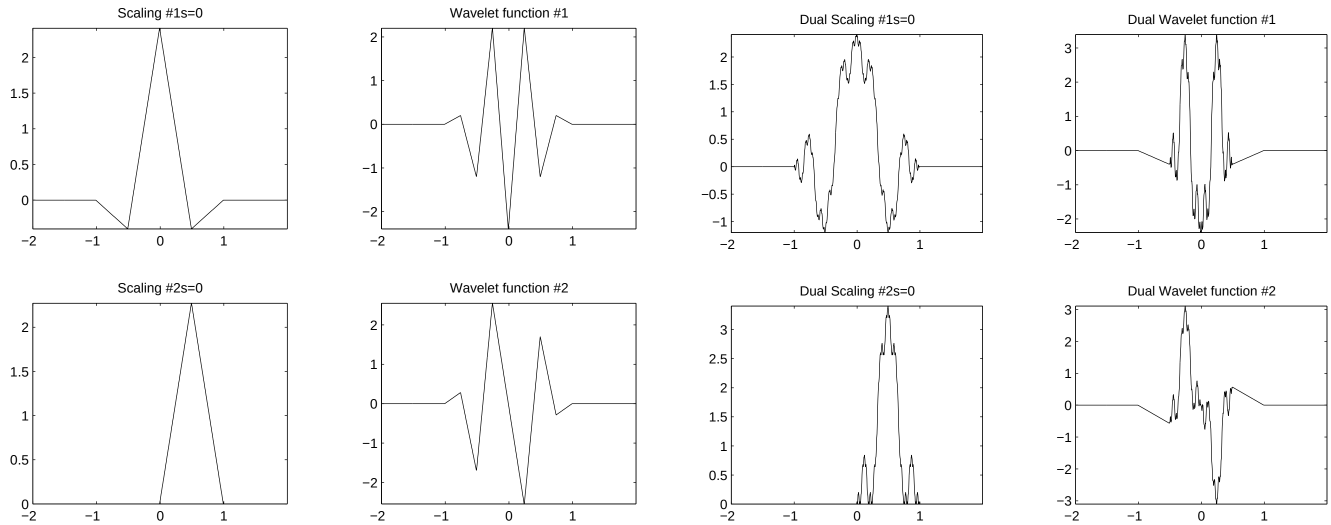
COROLLARY 5.3. *If $f \in L^\partial_2(\mathbf{R}_+) \cap M^{\lambda,\alpha}(\mathbf{R}_+)$ $(0 \le \lambda < 1, 0 \le \alpha \le 1)$ then there is a constant $C$ such that the frame expansion $f = c_0\phi^e_{10}\chi_{[0,\infty)} + \sum_{j=1}^\infty c_j\psi^e_{j0}\chi_{[0,\infty)}$ satisfies $\frac{1}{|I|^\lambda}\sum_{Q_{jk}\subset I} 2^{2j\alpha}|c_{jk}|^2 \le C$ for all intervals $I \subset \mathbf{R}_+, |I| \le 1$. That is, this frame expansion converges to $f$ in $M^{\lambda,\alpha}(\mathbf{R}_+)$.*

Of course, a similar analogue of Corollary 3.3 also holds. Finally, the analogue of Theorem 4.2 is

THEOREM 5.4. *Let $Q_j$ denote the projection onto the $j$-th wavelet space of a multiresolution analysis of $L^2(\mathbf{R})$ where the wavelets are orthogonal and Lipschitz with compact support. Then for $0 \le \lambda < 1, 0 \le \alpha \le 1$ and $j \ge 1$,*

$$\|Q_j(fg)\|_{M^{\lambda,\alpha}} \le C2^{j(\frac{1-\lambda}{2}-\alpha)}\|f\|_{M^{\lambda,\alpha}}\|g\|_{M^{\lambda,\alpha}}$$

**Figure 1.** Plots of HM scaling functions and wavelets and their duals for $s = 0$





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