# Haar multipliers, paraproducts and weighted inequalities

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**Abstract**

In this paper we present a brief survey on Haar multipliers, dyadic paraproducts, and recent results on their applications to deduce scalar and vector valued weighted inequalities. We present a new proof of the boundedness of a Haar multiplier in $L^p(\mathbf{R})$. The proof is based on a stopping time argument suggested by P. W. Jones for the case $p = 2$, that it is adapted to the case $1 < p < \infty$ using an new version of Cotlar's Lemma for $L^p$. We then prove some weighted inequalities for simple dyadic operators.

## 1 Introduction

A *Haar multiplier* is an operator of the form:

$$Tf(x) = \sum_{I\in\mathcal{D}} \omega_I(x)\langle f, h_I\rangle h_I(x),$$

where the sum runs over the dyadic intervals $\mathcal{D} = \{(k2^{-j}, (k+1)2^{-j}] : k, j \in \mathbf{Z}\}$; $h_I$ is the *Haar function* associated to $I$; $\langle .,. \rangle$ denotes the $L^2$ inner product; and finally the *symbol* $\omega_I(x)$ is a function of both the variables $x \in \mathbf{R}$ and $I \in \mathcal{D}$. These operators are formally similar to pseudodifferential operators, but the trigonometric functions have been replaced by step functions. When the symbol is a function independent of $x$, $\omega_I(x) = \omega_I$, the corresponding operators are the *constant Haar multipliers*, known to be bounded in $L^p$ if and only if the sequence $\{\omega_I\}_{I\in\mathcal{D}}$ is bounded. We will be concerned with symbols of the form

$$\omega_I^t(x) = \left(\frac{\omega(x)}{m_I\omega}\right)^t,$$

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where $t$ is a real number, $\omega$ is a weight, and $m_I\omega$ denotes the mean value of $\omega$ over $I$. The corresponding multipliers will be denoted $T^t_\omega$. We prove that under certain conditions on the weight these operators are bounded in $L^p(\mathbf{R})$.

The proof presented here is based on a stopping time argument suggested by P. W. Jones for the case $p = 2$, $t = 1$, that is adapted to the case $p \neq 2$ using a version of Cotlar's Lemma in $L^p$ which could be of interest in its own right.

The non constant Haar multipliers corresponding to $t = 1$ appeared for the first time in [P], in connection with the existence and boundedness of *the resolvent of the dyadic paraproduct*. The ones corresponding to $t = \pm 1/2$ appeared in the work of Treil and Volberg concerning matrix valued weighted inequalities for the Hilbert transform [TV].

The paraproducts are a class of operators that have made their way into mainstream harmonic analysis since their discovery in the late 70's. They were first introduced by Bony in the context of nonlinear differential equations [Bo]. After the proof of the $T(1)$ Theorem by David and Journé [DJ], it became clear that in the theory of singular integral operators the objects of study where of two very different natures: either similar to translation invariant operators, or to paraproducts, whose boundedness depended on a certain BMO/Carleson condition to be satisfied. The simplest version of the paraproduct is the dyadic one. Given a sequence of real numbers $\{b_I\}_{I\in\mathcal{D}}$, the paraproduct $\pi_b$, acts on locally integrable functions on the line by:

$$\pi_b f(x) = \sum_{I\in\mathcal{D}} m_I f \, b_I h_I(x),$$

where $m_I f$ denotes the mean value of $f$ over $I$. The operator $\pi_b$ is bounded in $L^p$ if and only if the sequence of $b_I^2$'s satisfies a Carleson condition, namely there exists a constant $C$ such that for all $I \in \mathcal{D}$:

$$\sum_{J\in\mathcal{D}(I)} b_J^2 \leq C|I|,$$

which is equivalent to saying that the function $b = \sum_{I\in\mathcal{D}} b_I h_I$ belongs to dyadic $BMO$, see [M], [Ch].

The boundedness of the paraproducts can be interpreted as an embedding theorem. This approach has been favored by the russian school, see [N], [TV1]. They refer to these results as *Carleson embedding theorems.*

After the Fourier transform the Hilbert transform is, perhaps, the most important operator in analysis. The Hilbert transform $H$, that acts on functions on the real line, is defined by:

$$Hf(x) = P.V. \int \frac{f(y)}{x-y} dy.$$

A necessary and sufficient condition for a single scalar weight $\omega$ so that $H$ maps

$L^2(\omega)$ into itself continuously, that is:

$$\int |Hf|^2\omega \leq C \int |f|^2\omega, \qquad \forall f \in L^2(\omega),$$

was first given (as part of a problem in prediction theory) by Helson and Szegö in 1960 using complex analysis, see [HS]. In the late 70's, Cotlar and Sadosky extended these results to two weights, and to matrix and operator valued contexts, see [CS1],[CS2].

In 1973, Hunt, Muckenhoupt and Wheeden, see [HMW], presented a new proof, where for the first time the $A_p$-condition for weights appeared as the necessary and sufficient condition for the boundedness of the Hilbert transform in $L^p(\omega)$:

$$\omega \in A_p \iff \left(\frac{1}{|I|}\int_I \omega\right)\left(\frac{1}{|I|}\int_I \omega^{\frac{-1}{p-1}}\right)^{p-1} < C;$$

for all intervals $I \subset \mathbf{R}$.

This was soon extended to a large class of singular integral operators, see [CF]. This work depended heavily on certain maximal functions defined only in the scalar case.

It was not understood how to make a sensible theory in the case of matrix (or operator) valued weights. In 1995 S. Treil and A. Volberg managed to prove that the natural matrix valued generalization of $A_2$ was the correct necessary and sufficient condition for the vector valued problem, with equal weights, $p = 2$, and *when the dimension of the Hilbert space is finite*, see [TV]. Their bounds depend on the dimension of the space. We suspect the bounds are not sharp although they should still depend on the dimension. Their proof depends on the existence of a weighted Carleson Embedding Theorem which can be interpreted as the boundedness of certain non constant *Haar multipliers*. They have since extended the results for all $1 < p < \infty$, for a classical approach see [V], for a novel one using Bellman functions see [NT].

The difficulties extending these results to the infinite dimensional setting become apparent by the recent discovery by Nazarov, Treil and Volberg [NTV1] of a counterexample for an infinite dimensional Carleson embedding theorem. They construct a sequence of matrix-valued Carleson measures indexed in the dimension $N$ of the underlying space, for which the embedding bounds are of the order $\sqrt{\log N}$. They believe those estimates are sharp. The best known upper bound, $\log N$, is due to Katz [K1], in his work on matrix valued paraproducts, and, independently, to Nazarov, Treil and Volberg [NTV1].

The scalar two weights problem for the Hilbert transform is still open. Sufficient conditions for the matrix and operator valued case were given in [KP]. The main tool used there is an operator valued Schur's Lemma to deduce, as Treil and Volberg did, the boundedness of the operator from the rate of decay of the off-diagonal coefficients of the standard matrix representation of the operator in the Haar basis. Parts of the operator are treated as if they were constant

Haar multipliers and others like paraproducts, very much in the spirit of the T(1) Theorem.

The techniques used, in particular paraproducts, stopping time arguments, Schur's Lemma, and Cotlar's Lemma had been exploited with success by the first author to solve a longstanding problem on maximal operators over arbitrary sets of directions, see [K2].

Recently Nazarov, Treil and Volberg solved the scalar two weights problem for a toy model of the Hilbert transform, namely the *signed constant Haar multipliers*:

$$T_\sigma f = \sum_{I\in\mathcal{D}} \pm\langle f, h_I\rangle h_I.$$

where $\sigma$ indicates a choice of signs. The result gives necessary and sufficient conditions on a pair of weights $(u, v)$ so that the $T_\sigma$'s are uniformly bounded from $L^2(u)$ into $L^2(v)$. It is not known what happens for $p \neq 2$, see [NTV2].

The paper is organized as follows. In the next section we present some notation and preliminaries. Then we define a decaying stopping time, we present the main Weight Lemma, and we deduce from it Gehring's Theorem and a dyadic relation between $RH_p^d$ weights and dyadic $BMO$ proved by R. Fefferman, C. Kenig and J. Pipher in the doubling case. We then prove a multilinear version of Schur's Lemma, and some lemmas in the $L^p$ theory for decaying stopping times, in particular one that can be thought as a version of Plancherel's Lemma in $L^p$, and the second being an $L^p$ version of Cotlar's Lemma. Then comes the stopping time proof of the boundedness of the Haar multiplier $T_\omega^1$ in $L^p$, as a corollary we deduce that $T_\omega^t$ is bounded in $L^p$, and that it is of weak type $(1,1)$. Finally we use those results to show weighted inequalities for constant Haar multipliers, dyadic paraproducts and the dyadic square function, this provides an alternative proof for the classical dyadic Littlewood-Paley Theory (some of these results appeared in [KP]).

# 2 Preliminaries

## 2.1 Dyadic intervals and Haar basis

Let us denote by $\mathcal{D}$ the family of all dyadic intervals in $\mathbf{R}$, i.e. intervals of the form $(j2^{-k}, (j+1)2^{-k}]$, $j, k \in \mathbf{Z}$. $\mathcal{D}_k$ denotes the $k^{th}$ generation of $\mathcal{D}$, consisting of those dyadic intervals of length $2^{-k}$. Given any interval $J$, $\mathcal{D}(J)$ denotes the family of dyadic subintervals of $J$; $\mathcal{D}_n(J) = \{I \in \mathcal{D}(J) : |I| = 2^{-n}|J|\}$, $|I|$ denotes the length of the interval $I$. Given an interval $J$ we will denote the right and left halves respectively by $J_r$ and $J_l$; they are the elements of $\mathcal{D}_1(J)$.

The *Haar function* associated to an interval $I$ is given by:

$$h_I(x) = \frac{1}{|I|^{1/2}}\big(\chi_{I_r}(x) - \chi_{I_l}(x)\big),$$

here $\chi_I$ denotes the characteristic function of the interval $I$. The Haar functions indexed on the dyadics, $\{h_I\}_{I\in\mathcal{D}}$, form a basis of $L^2(\mathbf{R})$, see [H].

### 2.2 Weights

A *weight* is a locally integrable function which is almost everywhere strictly positive. Let $1 < p < \infty$.

A weight $\omega$ is in *dyadic reverse Hölder p*, $RH_p^d$, if there exists a constant $C$ such that for every interval $I \in \mathcal{D}$

$$\frac{1}{|I|}\int_I \omega^p \leq C\left(\frac{1}{|I|}\int_I \omega\right)^p.$$

The infimum of such constants is called the $RH_p^d$ *constant of* $\omega$. Eg: $\omega(x) = |x|^\alpha$ for $\alpha > -1/p$.

A weight $\omega$ is in the *dyadic Muckenhoupt's class*, $A_p^d$, if there exists a constant $C$ such that for every interval $I \in \mathcal{D}$

$$\left(\frac{1}{|I|}\int_I \omega\right)\left(\frac{1}{|I|}\int_I \omega^{\frac{-1}{p-1}}\right)^{p-1} < C.$$

Eg: $\omega(x) = |x|^\alpha$ for $-1 < \alpha < p-1$.

We say that a weight $\omega$ is in $A_\infty^d$ if it belongs to $RH_p^d$ for some $1 < p < \infty$.

The main property of these classes of weights is Gehring's Theorem which states that if $\omega \in RH_p^d$ then $\omega \in RH_{p+\epsilon}^d$ for some $\epsilon > 0$, see [G].

This result is well known for the corresponding non dyadic classes, see [GC-RF]. We present a proof in the next section based in the Weight Lemma 3.

Corresponding properties for matrix valued weights are less clear. In particular an analogue of Gehring's Theorem would be very useful.

The following lemma will be used later. For non dyadic classes it is due to Strömberg and Wheeden [SW], see also [CU-N]. For dyadic classes it still holds. For completeness we present a proof.

**Lemma 1** *Assume $s > 1$. A weight $\omega \in RH_s^d$ if and only if $\omega^s \in A_\infty^d$.*

**Proof:** ($\Rightarrow$) Enough to show that there exists $1 < p < \infty$ such that $\omega^s \in RH_p^d$ by the definition of $A_\infty^d$.

Assuming $\omega \in RH_s^d$ then Gehring's Theorem implies that $\omega \in RH_{s+\epsilon}^d$ for some $\epsilon > 0$. Let $p = \frac{s+\epsilon}{s} > 1$ then,

$$\begin{aligned}\frac{1}{|I|}\int_I \omega^{sp} &= \frac{1}{|I|}\int_I \omega^{s+\epsilon} \leq C\left(\frac{1}{|I|}\int_I \omega\right)^{s+\epsilon} \\ &= C\left(\frac{1}{|I|}\int_I \omega\right)^{sp} \leq C\left(\frac{1}{|I|}\int_I \omega^s\right)^p\end{aligned}$$

where we have used the $RH^d_{s+\epsilon}$ in the first inequality and Hölder's inequality in the last step. This implies $\omega^s \in RH^d_p$.

($\Leftarrow$) We assume that there exists $1 < p < \infty$ such that $\omega^s \in RH^d_p$. Let $t = \frac{sp(s-1)}{sp-1}$ and $q = \frac{sp-1}{s-1} > p$, then $tq = sp$ and $(s-t)q' = 1$, where $\frac{1}{q} + \frac{1}{q'} = 1$. Then by Hölder's inequality

$$\begin{aligned}\frac{1}{|I|}\int_I \omega^s &= \frac{1}{|I|}\int_I \omega^t \omega^{s-t} \leq \left(\frac{1}{|I|}\int_I \omega^{tq}\right)^{\frac{1}{q}} \left(\frac{1}{|I|}\int_I \omega^{(s-t)q'}\right)^{\frac{1}{q'}} \\ &= \left(\frac{1}{|I|}\int_I \omega^{sp}\right)^{\frac{1}{q}} \left(\frac{1}{|I|}\int_I \omega\right)^{\frac{1}{q'}}.\end{aligned}$$

Since $\omega^s \in RH^d_p$ we get,

$$\left(\frac{1}{|I|}\int_I \omega^{sp}\right)^{\frac{1}{q}} \leq C\left(\frac{1}{|I|}\int_I \omega^s\right)^{\frac{p}{q}},$$

therefore

$$\left(\frac{1}{|I|}\int_I \omega^{s}\right)^{1-\frac{p}{q}} \leq C\left(\frac{1}{|I|}\int_I \omega\right)^{\frac{1}{q'}}.$$

But the choice of $t$ and $q$ forces $\frac{q}{q'} = s(q-p)$, hence

$$\frac{1}{|I|}\int_I \omega^s \leq C\left(\frac{1}{|I|}\int_I \omega\right)^{s},$$

and $\omega \in RH^d_s$. □

It is well known that any doubling $RH^d_q$ weight is in $A^d_p$ for some $p > 1$ and viceversa. Without the doubling condition half of it is still true, namely,

**Lemma 2** *If $\omega \in A^d_p$ then $\omega \in RH^d_q$ for some $q > 1$.*

**Proof:** This is a consequence of Gehring's Theorem and the following,

**Claim:** If $\omega \in A^d_p$ then $\omega^{\frac{1}{p}} \in RH^d_p$.

Assuming the claim, $\omega \in A^d_p$ implies $\omega^{\frac{1}{p}} \in RH^d_p$ implies $\omega^{\frac{1}{p}} \in RH^d_{p+\epsilon}$ which by the previous lemma implies $\omega^{1+\frac{\epsilon}{p}} \in A^d_\infty$ and by the same lemma it implies that $\omega \in RH^d_{1+\frac{\epsilon}{p}}$.

**Proof of the claim:** By Hölder's inequality twice and the hypothesis $\omega \in A^d_p$ we get,

$$1 \leq \left(\frac{1}{|I|}\int_I \omega^{\frac{1}{p}}\right)^{p} \left(\frac{1}{|I|}\int_I \omega^{\frac{-p'}{p^2}}\right)^{\frac{p^2}{p'}}$$

$$\leq C\left(\frac{1}{|I|}\int_I \omega^{\frac{1}{p}}\right)^p \left(\frac{1}{|I|}\int_I \omega^{\frac{-1}{p-1}}\right)^{p-1}$$
$$\leq C\left(\frac{1}{|I|}\int_I \omega^{\frac{1}{p}}\right)^p \left(\frac{1}{|I|}\int_I \omega\right)^{-1},$$

where $\frac{1}{p}+\frac{1}{p'}=1$ implies $\frac{p}{p'}=p-1$. We conclude that $\frac{1}{|I|}\int_I \omega \leq C\left(\frac{1}{|I|}\int_I \omega^{\frac{1}{p}}\right)^p$, that is $\omega^{\frac{1}{p}} \in RH_p^d$. □

If a given weight $\omega$ is bounded away from zero and from infinity, that is: $\lambda^{-1} < \omega(x) < \lambda$, for some $\lambda > 1$; then it is doubling and it belongs to any of the classes. In general belonging to $A_\infty^d$ does not guarantee being bounded or doubling (in the non dyadic setting $A_\infty$ implies doubling), but it says that the set where the weight is close to zero intersected with any interval where the mean is far from zero is small in a very precise way, to be described next.

# 3 Weight lemma and decaying stopping times

Let us first define a *stopping time* $\mathcal{J}$ for an interval $I$.

For a given interval $I$, let $\mathcal{J}(I)$ be the collection of dyadic intervals contained in $I$ which are maximal with respect to a given property. Let $\mathcal{F}(I)$ be the collection of dyadic intervals contained in $I$ but not contained in any interval $J \in \mathcal{J}(I)$.

We say the property is *admissible* if for all $J \in \mathcal{D}(I)$, $J \in \mathcal{F}(J)$ and hence $\mathcal{F}(J)$ is not empty.

Given an admissible property, let $\mathcal{J}^o(I) = \{I\}$. For $n > 0$ define now the collections $\mathcal{J}^n(I)$ and $\mathcal{F}^n(I)$ inductively. $\mathcal{J}^n(I)$ is the collection of intervals belonging to $\mathcal{J}(J)$ for some $J$ in $\mathcal{J}^{n-1}(I)$. Similarly, $\mathcal{F}^n(I)$ is the collection of intervals belonging to $\mathcal{F}(J)$ for some $J$ in $\mathcal{J}^{n-1}(I)$.

The family of collections of intervals $(\mathcal{J}^n, \mathcal{F}^n)$ is the stopping time $\mathcal{J}$ for the interval $I$ corresponding to the given admissible property. The intervals in $\mathcal{F}^n$ are "good", those in $\mathcal{J}^n$ are "bad" but not so bad because their parents are "good".

Clearly for each $n > 0$ the intervals in $\mathcal{J}^n(I)$ are pairwise disjoint. By definition the elements of $\mathcal{J}^n(I)$ are subintervals of the elements of $\mathcal{J}^{n-1}(I)$.

Also $\mathcal{D}(I) = \bigcup_{j=0}^{\infty} \mathcal{F}^j(I)$, and the $\mathcal{F}^j$'s are disjoint collections of dyadic subintervals.

We say that $\mathcal{J}$ is a *decaying stopping time* if there exists $0 < c < 1$ so that for every $I \in \mathcal{D}$, one has
$$\sum_{J \in \mathcal{J}(I)} |J| \leq c|I|.$$

Iterating this property we conclude that for decaying stopping times,

$$\sum_{J \in \mathcal{J}^k(I)} |J| \leq c^k |I|. \tag{1}$$

We now prove what will be the most fundamental lemma in the theory of weights. It should be viewed as an analogue of the John-Nirenberg theorem for functions in $BMO$.

Given a weight $\omega$, we define the stopping time $\mathcal{J}_\omega$ where $\mathcal{J}_\omega(I)$ denotes the set of pairwise disjoint dyadic subintervals $J$ of $I$ which are maximal with respect to the property that $m_J\omega \geq \lambda m_I\omega$ or $m_J\omega \leq \frac{1}{\lambda} m_I\omega$, where $\lambda \geq 1$ is to be specified in the proof of the following Lemma. It depends only on the $RH_p^d$ constant of $\omega$.

**Lemma 3 (Weight Lemma)** *Let $\omega \in RH_p^d$. Then, for $\lambda$ sufficiently large, $\mathcal{J}_\omega$ is a decaying stopping time.*

**Proof:** First let $\lambda > 3$. We may divide up $I$ into three disjoint subsets,

$$I = \bigcup_j I_j^\lambda \bigcup_j I_j^{\frac{1}{\lambda}} \bigcup G,$$

where the intervals $I_j^\lambda, I_j^{\frac{1}{\lambda}}$ are an enumeration of all the different elements of $\mathcal{J}_\omega(I)$ such that for each $j$

$$\lambda\, m_I\omega \leq m_{I_j^\lambda}\omega \leq 2\lambda\, m_I\omega, \tag{2}$$

$$m_{I_j^{\frac{1}{\lambda}}}\omega \leq \frac{1}{\lambda} m_I\omega,$$

and that $\omega(x) \leq \lambda\, m_I\omega$ a.e. on $G$. Notice that the second inequality in (2) is just a consequence of the maximality assumption in the definition of $\mathcal{J}_\omega(I)$.

Suppose the lemma is false. Then $G$ can be arbitrarily small. Suppose $|G| \leq \frac{|I|}{3\lambda}$. Thus

$$\int_G \omega \leq \frac{1}{3} \int_I \omega,$$

and since $\lambda > 3$,

$$\sum_j \int_{I_j^{\frac{1}{\lambda}}} \omega \leq \frac{1}{3} \int_I \omega,$$

so that

$$\sum_j \int_{I_j^\lambda} \omega \geq \frac{1}{3} \int_I \omega.$$

By the second inequality in (2), this implies that

$$\sum_j |I_j^\lambda| \geq \frac{1}{6\lambda}|I|. \qquad (3)$$

We will now use (3) and (2) to contradict $\omega \in RH_p^d$,

$$\int_I \omega^p \geq \sum_j \int_{I_j^\lambda} \omega^p \geq \sum_j \frac{1}{|I_j^\lambda|^{p-1}} \left( \int_{I_j^\lambda} \omega \right)^p = \sum_j |I_j^\lambda| (m_{I_j^\lambda} \omega)^p,$$

with the second inequality being just an application of Hölder's inequality. But

$$\sum_j |I_j^\lambda| (m_{I_j^\lambda} \omega)^p \geq \lambda^p \sum_j |I_j^\lambda| (m_I \omega)^p \geq \frac{1}{6} \lambda^{p-1} |I| (m_I \omega)^p.$$

This contradicts $RH_p^d$ provided we chose $\lambda \geq (6C)^{\frac{1}{p-1}}$, where $C$ is the $RH_p^d$ constant of $\omega$. If this is the case, then $|G| \geq \frac{1}{3\lambda}|I|$ and hence, we have proved the lemma with $c = 1 - \frac{1}{3\lambda}$. □

We will use this lemma to show the classical Gehring's Theorem (see Section 2.2). It can also be used to prove the fact that for doubling weights $\omega$, $\log \omega$ is in dyadic $BMO$ whenever $\omega \in A_\infty^d$, see [K3]. First we prove a more "dyadic" relation between $RH_p^d$ weights and dyadic $BMO$ first discovered by Fefferman, Kenig, and Pipher [FKP].

A function $b$ is in dyadic $BMO$ if and only if the sequence $b_I = \langle b, h_I \rangle$ satisfies a Carleson condition, namely,

$$\sum_{J \in \mathcal{D}(I)} b_J^2 \leq C|I|, \quad \forall I \in \mathcal{D}.$$

**Corollary 1** *Let $\omega \in RH_p^d$ for some $1 < p < \infty$. We define the function*

$$b(x) = \sum_{I \in \mathcal{D}} \frac{\langle \omega, h_I \rangle}{m_I \omega} h_I.$$

*Then $b(x)$ is in dyadic $BMO$.*

**Proof:** It suffices to show that there exist a constant $C > 0$, so that for every $I \in \mathcal{D}$,

$$\sum_{J \in \mathcal{D}(I)} \left| \frac{\langle \omega, h_J \rangle}{m_J \omega} \right|^2 \leq C|I|.$$

We define $\mathcal{F}_\omega^j(I)$ to be the set of dyadic intervals contained in some interval of $\mathcal{J}_\omega^{j-1}(I)$ but not contained in any interval of $\mathcal{J}_\omega^j(I)$. This is the decaying stopping time $\mathcal{J}_\omega$. For any interval $J$, we define the function $\omega_J(x)$ supported on $J$ to be equal to $\omega(x)$ when $x$ is not contained in any interval of $\mathcal{J}_\omega(J)$ and to be equal to $m_K\omega$ when $x \in K \in \mathcal{J}_\omega(J)$. Then

$$|J| m_J^2\omega \ + \sum_{K\in\mathcal{F}_\omega(J)} |\langle \omega, h_K\rangle|^2 = \int_J \omega_J^2(x).$$

However, by definition of $\mathcal{J}_\omega(J)$, in particular by (2),

$$\sum_{K\in\mathcal{F}_\omega(J)} |\langle \omega, h_K\rangle|^2 \le \int_J \omega_J^2(x) \le 2\lambda m_J\omega \int_J \omega_J(x) = 2\lambda |J| m_J^2\omega.$$

Now for any $K \in \mathcal{F}_\omega^j(J)$, there is a unique interval $J$ of $\mathcal{J}_\omega^{j-1}(I)$ containing $K$. Then $K \in \mathcal{F}_\omega(J)$ and $m_K\omega > \frac{1}{\lambda} m_J\omega$. Thus

$$\begin{aligned}
\sum_{K\in\mathcal{F}_\omega^j(I)} \left| \frac{\langle w, h_K\rangle}{m_K\omega} \right|^2 &\le \lambda^2 \sum_{J\in\mathcal{J}_\omega^{j-1}(I)} \sum_{K\in\mathcal{F}_\omega(J)} \left| \frac{\langle \omega, h_K\rangle}{m_J\omega} \right|^2 \\
&\le \lambda^2 \sum_{J\in\mathcal{J}_\omega^{j-1}(I)} \frac{1}{m_J^2\omega} 2\lambda |J| m_J^2\omega \\
&\le 2\lambda^3 \sum_{J\in\mathcal{J}_\omega^{j-1}(I)} |J| \ \le \ 2\lambda^3 c^{j-1}|I|,
\end{aligned}$$

where we have used (1) to get the last inequality. Thus

$$\sum_{J\in\mathcal{D}(I)} \left| \frac{\langle \omega, h_J\rangle}{m_J\omega} \right|^2 \le 2\lambda^3(1 + c + c^2 + \ldots)|I| \le \frac{2\lambda^3|I|}{1-c},$$

which was to be shown. □

**Lemma 4 (Gehring's Theorem)** *Suppose $\omega \in RH_p^d$ for some $1 < p < \infty$. Then there is $\epsilon > 0$ depending only on $p$ and the $RH_p^d$ constant of $\omega$ so that $\omega \in RH_{p+\epsilon}^d$.*

**Proof:** For any interval $J$, define

$$\mathcal{G}_1(I) = \left(\bigcup \mathcal{J}_\omega(I)\right)^c,$$

where we are denoting $\bigcup \mathcal{J}_\omega(I) = \bigcup_{J\in\mathcal{J}_\omega(I)} J$. Observe that for almost every $x \in \mathcal{G}_1(I)$, one has $\frac{1}{\lambda} m_I\omega \le \omega(x)$, and one has, for $0 < c < 1$,

$$|\mathcal{G}_1(I)| \ge (1-c)|I|,$$

by Lemma 3. Thus

$$\int_{\mathcal{G}_1(I)} \omega^p \geq (1-c)|I|(\frac{1}{\lambda})^p (m_I \omega)^p.$$

Since $\omega \in RH_p^d$ this means there exists $a > 0$ depending only on $p$ and the $RH_p^d$ constant for $\omega$ (since $\lambda$ and $c$ depend only on these) so that for every $I$,

$$\int_{\mathcal{G}_1(I)} \omega^p \geq a \int_I \omega^p. \tag{4}$$

We define

$$\mathcal{G}_j(I) = \bigcup \mathcal{J}_\omega^{j-1}(I) \setminus \bigcup \mathcal{J}_\omega^j(I).$$

Clearly for $x \in \mathcal{G}_j(I)$ we have that

$$\omega(x) \leq (2\lambda)^j m_I \omega. \tag{5}$$

Further, we claim that

$$\int_{\mathcal{G}_j(I)} \omega^p \leq (1-a)^{j-1} \int_I \omega^p. \tag{6}$$

We will prove (6) by induction.

To prove (6), it suffices to show that

$$\int_{\bigcup \mathcal{J}_\omega^j(I)} \omega^p \leq (1-a) \int_{\bigcup \mathcal{J}_\omega^{j-1}(I)} w^p,$$

for every $j$, since

$$\int_{\mathcal{G}_j(I)} \omega^p \leq \int_{\bigcup \mathcal{J}_\omega^{j-1}(I)} \omega^p.$$

But by (4)

$$\begin{aligned} \int_{\bigcup \mathcal{J}_\omega^j(I)} \omega^p &= \sum_{J \in \mathcal{J}_\omega^{j-1}(I)} (\int_J \omega^p - \int_{\mathcal{G}_1(J)} \omega^p) \\ &\leq (1-a) \int_{\bigcup \mathcal{J}_\omega^{j-1}(I)} \omega^p. \end{aligned}$$

Thus (6) is shown.

Now we estimate

$$\begin{aligned} \int_I \omega^{p+\epsilon} &\leq \sum_{j=1}^{\infty} \int_{\mathcal{G}_j(I)} \omega^{p+\epsilon} \\ &\leq (m_I \omega)^\epsilon \sum_{j=1}^{\infty} (2\lambda)^{j\epsilon} \int_{\mathcal{G}_j(I)} \omega^p \\ &\leq (m_I \omega)^\epsilon \sum_{j=1}^{\infty} (2\lambda)^{j\epsilon} (1-a)^{j-1} \int_I \omega^p. \end{aligned} \tag{7}$$

Here we have obtained the second inequality using (5) and the third using (6). Now we choose $\epsilon$ sufficiently small so that $(1-a)(2\lambda)^{\epsilon} < 1$, and the sum in (7) converges. There is a $C > 0$ so that for every $I$,

$$\frac{1}{|I|}\int_I \omega^{p+\epsilon} \le C(m_I\omega)^{\epsilon}\frac{1}{|I|}\int_I \omega^p.$$

Now applying the $RH_p^d$ condition shows there exists $C > 0$ so that

$$\frac{1}{|I|}\int_I \omega^{p+\epsilon} \le C(m_I\omega)^{p+\epsilon},$$

which was to be shown. $\Box$

The following result will be used later. For a proof see [M].

**Lemma 5 (Carleson's Lemma)** *Assume the sequence of positive numbers $\{a_I\}_{I\in\mathcal{D}}$ satisfies the following Carleson condition:*

$$\sum_{J\in\mathcal{D}(I)} a_J \le C|I|, \quad \forall I \in \mathcal{D}.$$

*Let $\{\lambda_I\}_{I\in\mathcal{D}}$ be any sequence of positive numbers, then*

$$\sum_{I\in\mathcal{D}} \lambda_I a_I \le C\int \sup_{I\ni x} \lambda_I\, dx.$$

# 4 $L^p$ Lemmas for decaying stopping times

## 4.1 $L^p$ Plancherel Lemma

First we state explicitly a multilinear version of Schur's Lemma that we will need.

**Lemma 6** *Suppose $C(j_1, j_2, \ldots, j_{n-1})$ are given positive numbers where the $j$'s range over $\mathbf{Z}$ and suppose that for some $C \ge 0$*

$$\sum_{j_1=-\infty}^{\infty}\cdots\sum_{j_{n-1}=-\infty}^{\infty} C(j_1, j_2, \ldots, j_{n-1}) \le C,$$

*then for any sequence $\{x_j\}_{j\in\mathbf{Z}}$ in $l^n$, one has*

$$\sum_{j_1=-\infty}^{\infty}\cdots\sum_{j_n=-\infty}^{\infty} C(j_2-j_1, j_3-j_2, \ldots, j_n-j_{n-1})x_{j_1}\ldots x_{j_n} \le C^n\|x\|_n^n.$$

**Proof:** We simply use Hölder's inequality,

$$\sum_{j_1=-\infty}^{\infty} \cdots \sum_{j_n=-\infty}^{\infty} C(j_2-j_1, j_3-j_2, \ldots, j_n-j_{n-1}) x_{j_1} \ldots x_{j_n}$$
$$\leq \Big( \sum_{j_1, j_2, \ldots, j_n} C(j_2-j_1, \ldots, j_n-j_{n-1}) x_{j_1}^n \Big)^{\frac{1}{n}} \ldots$$
$$\Big( \sum_{j_1, j_2, \ldots, j_n} C(j_2-j_1, \ldots, j_n-j_{n-1}) x_{j_n}^n \Big)^{\frac{1}{n}} \leq C^n \|x\|_n^n. \qquad \square$$

We now prove a small result in the theory of decaying stopping times, which can be viewed as an $L^p$ Plancherel type lemma.

Let $\mathcal{J}$ be a decaying stopping time. Let us denote $\mathcal{F}^j = \mathcal{F}^j([0,1])$, similarly $\mathcal{J}^j$. We define the operator

$$\Delta_{\mathcal{F}^j} f = \sum_{I \in \mathcal{F}^j} \langle f, h_I \rangle h_I,$$

which is just the projection onto the closed subspace spanned by $\{h_I\}_{\mathcal{F}_j}$. This is a constant Haar multiplier with bounded symbol $c_I^j = \chi_{\mathcal{F}^j}(I)$, hence it is bounded on $L^p$ for $1 < p < \infty$ with constant depending only on $p$, by Lemma 10 in the next section.

**Lemma 7** *Let $\mathcal{J}$ be a decaying stopping time and the $\mathcal{F}^j$'s be as above. For any $1 < p < \infty$ there is a constant $C(p) > 0$ so that for all $f \in L^p(\mathbf{R})$,*

$$\sum_{j=1}^{\infty} \|\Delta_{\mathcal{F}^j} f\|_p^p \leq C(p) \|f\|_p^p \tag{8}$$

**Proof** Consider the sequence $\Delta_{\mathcal{F}} = \{\Delta_{\mathcal{F}^j}\}$ as a single operator from $L^p(\mathbf{R})$ to $L^p(l^p)$. Then (8) asserts the boundedness of this operator. Let $X$ be the subspace of $L^p(\mathbf{R})$ of functions supported on $[0,1]$ and having mean zero on [0,1]. Let $E_X$ be the $L^2$ projection into this space. $E_X$ is a constant Haar multiplier, $E_X f = \sum_{I \in \mathcal{D}} c_I \langle f, h_I \rangle h_I$ where $c_I = \chi_{\mathcal{D}([0,1])}(I) \in l^\infty$ hence, by the classical Littlewood-Paley theory $E_X$ is bounded on $L^p$, see Lemma 10. Now, it is easy to see that $\Delta_{\mathcal{F}} E_X = \Delta_{\mathcal{F}}$ so it suffices to show (8) for $f$ in $X$. Let $\Delta_{\mathcal{F}}^*$ be the dual mapping acting on sequences $L^q(l^q)$ for $\frac{1}{p} + \frac{1}{q} = 1$, it can be seen that,

$$\Delta_{\mathcal{F}}^*(\{f_j\}) = \sum_j \Delta_{\mathcal{F}_j} f_j.$$

Therefore.

$$\Delta_{\mathcal{F}}^* \Delta_{\mathcal{F}} = E_X,$$

Thus showing the reverse inequality to (8), for $f \in X$ and all $1 < p < \infty$, that is,

$$\|f\|_p^p \leq C(p) \sum_{j=1}^{\infty} \|\Delta_{\mathcal{F}^j} f\|_p^p, \tag{9}$$

implies that $\Delta_{\mathcal{F}}^*$ is bounded from $L^p(l^p)$ to $L^p(\mathbf{R})$ for all $1 < p < \infty$ which, by duality, implies the boundedness of $\Delta_{\mathcal{F}}$.

Now we assert that there exists $C(p) > 0$ and $0 < c_1 < 1$ so that for any $j, k$ and any $f \in L^p(\mathbf{R})$, one has the inequality

$$\int |\Delta_{\mathcal{F}^j} f|^{\frac{p}{2}} |\Delta_{\mathcal{F}^k} f|^{\frac{p}{2}} \leq C(p) c_1^{|k-j|} \|\Delta_{\mathcal{F}^j} f\|_p^{\frac{p}{2}} \|\Delta_{\mathcal{F}^k} f\|_p^{\frac{p}{2}}. \tag{10}$$

To show (10), it suffices to let $k > j$. We define $f_j = \Delta_{\mathcal{F}^j} f$ and $f_k = \Delta_{\mathcal{F}^k} f$. We observe that on any $J \in \mathcal{J}^j([0,1])$, the function $f_j$ is constant, and we denote by $f_j(J)$ its value there. We compute

$$\begin{aligned}
\int |f_j|^{\frac{p}{2}} |f_k|^{\frac{p}{2}} &= \int_{\bigcup \mathcal{J}^{k-1}([0,1])} |f_j|^{\frac{p}{2}} |f_k|^{\frac{p}{2}} \\
&\leq \left( \int_{\bigcup \mathcal{J}^{k-1}([0,1])} |f_j|^p \right)^{\frac{1}{2}} \left( \int_{\bigcup \mathcal{J}^{k-1}([0,1])} |f_k|^p \right)^{\frac{1}{2}} \\
&\leq \|f_k\|_p^{\frac{p}{2}} \left( \sum_{J \in \mathcal{J}^j([0,1])} \int_{\bigcup \mathcal{J}^{k-j-1}(J)} |f_j(J)|^p \right)^{\frac{1}{2}} \\
&\leq c^{\frac{k-j-1}{2}} \|f_k\|_p^{\frac{p}{2}} \left( \sum_{J \in \mathcal{J}^j([0,1])} |J| \, |f_j(J)|^p \right)^{\frac{1}{2}} \leq c^{\frac{k-j-1}{2}} \|f_k\|_p^{\frac{p}{2}} \|f_j\|_p^{\frac{p}{2}}.
\end{aligned}$$

As in (1), we iterated the fact that $\mathcal{J}$ is a decaying stopping time to conclude,

$$\sum_{I \in \mathcal{J}^{k-j-1}(J)} |I| \leq c^{k-j-1} |J|.$$

This implies (10). Fix $n > p$. Let $f \in X$. First, by Hölder's inequality and (10) for the same $C(p) > 0$ and $0 < c_1 < 1$, for $j_1 < j_2 < \dots j_n$, one has

$$\begin{aligned}
\int |f_{j_1}|^{\frac{p}{n}} \dots |f_{j_n}|^{\frac{p}{n}} &\leq \left( \int |f_{j_1}|^{\frac{p}{2}} |f_{j_2}|^{\frac{p}{2}} \right)^{\frac{1}{n}} \dots \left( \int |f_{j_n}|^{\frac{p}{2}} |f_{j_1}|^{\frac{p}{2}} \right)^{\frac{1}{n}} \\
&\leq C(p) c_1^{\frac{j_2-j_1}{n}} \dots c_1^{\frac{j_n-j_{n-1}}{n}} c_1^{\frac{j_n-j_1}{n}} \|f_{j_1}\|^{\frac{p}{n}} \dots \|f_{j_n}\|^{\frac{p}{n}} \\
&\leq C(p) c_1^{j_n-j_1} \|f_{j_1}\|^{\frac{p}{n}} \dots \|f_{j_n}\|^{\frac{p}{n}}.
\end{aligned} \tag{11}$$

Furthermore $f = \sum_{j=1}^{\infty} f_j$. Thus by Jensen's inequality for $\frac{p}{n} < 1$

$$\int |f|^p \leq \sum_{j_1, j_2, \ldots, j_n} \int |f_{j_1}|^{\frac{p}{n}} \ldots |f_{j_n}|^{\frac{p}{n}}. \tag{12}$$

But (11) together with (12) and Lemma 6 for the choice $x_j = \|f_j\|^{\frac{p}{n}}$, $C(j_1, \ldots, c_{n-1}) = c^{j_1 + \cdots j_{n-1}}$, prove (9) and with it the lemma. □

## 4.2 $L^p$ version of Cotlar's Lemma

Much effort goes into the search for good $L^p$ substitutes to Cotlar's lemma, see [C]. The proof we just did to prove the $L^p$ Plancherel Lemma was the source of inspiration to state and prove the following version of Cotlar's Lemma for $L^p$.

**Lemma 8** *Let $\mathcal{J}$ be a decaying stopping time, and $\Delta_{\mathcal{F}^j}$ be as above. Let $T$ be a linear operator on functions on $\mathbf{R}$ and write $T_j = T\Delta_{\mathcal{F}^j}$. Suppose $T = \sum_{j=1}^{\infty} T_j$. Let $1 < p < \infty$. Suppose there are $C > 0$ and $0 < c < 1$ so that for every $j, k$ (possibly equal), one has*

$$\int |T_j f|^{\frac{p}{2}} |T_k f|^{\frac{p}{2}} \leq C c^{|k-j|} \|f_j\|_p^{\frac{p}{2}} \|f_k\|_p^{\frac{p}{2}}. \tag{13}$$

*Then $T$ is bounded on $L^p(\mathbf{R})$ with constant depending only on $p$, $C$, $c$, and the rate of decay of $\mathcal{J}$.*

**Proof:** Let $n$ be an integer so that $2n > p$. We wish to estimate

$$\int |Tf|^p = \int (|Tf|^{\frac{p}{2n}})^{2n}.$$

Now we observe that by Jensen's inequality,

$$|Tf|^{\frac{p}{2n}} \leq \sum_j |T_j f|^{\frac{p}{2n}}.$$

Thus we observe that

$$\int |Tf|^p \leq C(p) \sum_{j_1 \leq j_2 \leq \cdots \leq j_{2n}} \int |Tf_{j_1}|^{\frac{p}{2n}} \ldots |Tf_{j_{2n}}|^{\frac{p}{2n}}. \tag{14}$$

Now applying Hölder's inequality and (13), as we did in (11), it is readily apparent that

$$\int |Tf_{j_1}|^{\frac{p}{2n}} \ldots |Tf_{j_{2n}}|^{\frac{p}{2n}} \leq A c^{\frac{j_{2n} - j_1}{r}} \|f_{j_1}\|_p^{\frac{p}{2n}} \ldots \|f_{j_{2n}}\|_p^{\frac{p}{2n}}. \tag{15}$$

The reader may easily verify that (14), (15) and Lemma 6 imply that there is a constant $C$ depending only on $A$, $c$, and $p$ so that

$$\int |Tf|^p \leq C \sum_k \|f_k\|_p^p = C \sum_k \|\Delta_{\mathcal{F}^k} f\|_p^p, \tag{16}$$

we apply now the Littlewood-Paley Lemma to prove the lemma. □

# 5 Boundedness of $T^t_\omega$

## 5.1 Boundedness of $T_\omega$

We are interested in the boundedness of Haar multipliers. Let us discuss the simplest possible case. Which is of course well known, see [Ch], where the proof uses the classical Littlewood-Paley Theory. Our proof will use the following lemma, which also holds for $p \neq 2$, $1 < p < \infty$.

**Lemma 9** *Let $T$ be a linear or sublinear operator which is of strong type $(2,2)$. Suppose that for every dyadic interval $I$, the function $Th_I$ is supported only on $I$. Then $T$ is of weak type $(1,1)$.*

**Proof:** Suppose we have $f \in L^1(\mathbf{R})$. We pick $\lambda > 0$. We now apply the Calderón-Zygmund decomposition. We write $f = g + b$ with $||g||_\infty \leq 2\lambda$, $||g||_1 \leq 2||f||_1$, and $b$ supported on a disjoint sequence of dyadic intervals $\{I_j\}$, having mean zero on those intervals, and so that

$$\sum_j |I_j| \leq \frac{||f||_1}{\lambda}.$$

Now observe that

$$|\{x : |(Tf)(x)| \geq \lambda\}| \leq |\{x : |(Tg)(x)| \geq \frac{\lambda}{2}\}| + |\{x : |(Tb)(x)| \geq \frac{\lambda}{2}\}|. \tag{17}$$

For the first term on the right, we apply the fact that $T$ is of strong type (2,2) and hence of weak type (2,2) and that by Hölder's inequality

$$\|g\|_2 \leq \|g\|_1^{\frac{1}{2}} \|g\|_\infty^{\frac{1}{2}} \leq \sqrt{2\lambda \|f\|_1}.$$

Thus,

$$|\{x : |(Tg)(x)| \geq \frac{\lambda}{2}\}| \leq \frac{4C\|g\|_2^2}{\lambda^2} \leq \frac{8C\|f\|_1}{\lambda}. \tag{18}$$

On the other hand $\langle b, h_J \rangle = 0$ for $J$ dyadic unless $J \subset I_j$ for some $j$. Thus by assumption,

$$|\{x : |(Tb)(x)| \geq \frac{\lambda}{2}\}| \leq |\bigcup_j I_j| \leq \frac{\|f\|_1}{\lambda}. \tag{19}$$

Now applying (18) and (19) to (17) shows that $T$ is of weak type (1,1) which was to be shown. □

**Lemma 10** *The constant Haar multiplier $Tf = \sum_{I \in \mathcal{D}} c_I \langle f, h_I \rangle h_I$ is bounded in $L^p$ if and only if the sequence $\{c_I\}_{I \in \mathcal{D}}$ is bounded.*

**Proof:** The necessity follows immediately from applying $T$ to the Haar functions.

The sufficiency is a consequence of the previous lemma. Clearly, the multiplier $T$ satisfies the hypotheses of Lemma 9. Thus $T$ is of weak type (1,1). Thus by the Marcinkiewicz Interpolation Theorem $T$ is bounded on $L^p(\mathbf{R})$ for all $p$ with $1 < p \leq 2$. But $T$ is selfadjoint. Thus $T$ is bounded on $L^p(\mathbf{R})$ for all $1 < p < \infty$. $\square$

We will now use Cotlar's Lemma in $L^p$ to prove the main theorem of this section. See [P] for a different argument.

**Theorem 1** *Let $\omega$ be a weight. Define the operator*

$$T_\omega f(x) = \sum_{I \in \mathcal{D}} \frac{\omega(x)}{m_I \omega} \langle f, h_I \rangle h_I(x) = w(x)(M_\omega f)(x),$$

*where $M_\omega$ is the (possibly unbounded) Haar multiplier with coefficients $\frac{1}{m_I \omega}$. Then $T_\omega$ is bounded on $L^p$ if and only if $\omega \in RH_p^d$.*

**Remark:** If $\frac{1}{\lambda} < \omega(x) < \lambda$ for a.e. $x$, then there is a trivial bound, far from being sharp. Just observe,

$$\|T_\omega f\|_p^p = \int \omega^p |M_\omega f|^p \leq \lambda^p \|M_\omega f\|_p^p \leq C \lambda^{2p} \|f\|_p^p,$$

where the last inequality uses the fact that $M_\omega$ is a constant Haar multiplier with symbol bounded by $\lambda$.

**Proof:** The necessity of $\omega \in RH_p^d$ follows immediately from applying $T_w$ to the Haar functions.

For the proof of sufficiency, it suffices to prove the theorem for

$$Tf(x) = \sum_{I \in \mathcal{D}([0,1])} \frac{\omega(x)}{m_I \omega} \langle f, h_I \rangle h_I(x).$$

If $\omega \in RH_p^d$, by the Weight Lemma, the stopping time $\mathcal{J} = \mathcal{J}_\omega([0,1])$ is decaying. We will abuse our notation denoting by $\bigcup \mathcal{J}^n(I)$ the set $\bigcup_{J \in \mathcal{J}^n(I)} J$, and $\mathcal{F}^j = \mathcal{F}^j([0,1])$, similarly for $\mathcal{J}^j$.

We define

$$T_j = T \Delta_{\mathcal{F}^j} = \omega M_j,$$

where

$$M_j f(x) = \sum_{J \in \mathcal{F}^j} \frac{1}{m_J \omega} \langle f, h_J \rangle h_J(x).$$

We define for every dyadic $I$, the multiplier

$$M_I f(x) = \sum_{J \in \mathcal{F}_\omega(I)} \frac{1}{m_J \omega} \langle f, h_J \rangle h_J(x).$$

Then

$$M_j = \sum_{I \in \mathcal{J}^{j-1}} M_I.$$

Each $M_I$ is a bounded constant Haar multiplier since by the definition of the stopping time, $(m_J\omega)^{-1} \leq \lambda(m_I\omega)^{-1}$ for all $I \in \mathcal{D}$ and for all $J \in \mathcal{F}_\omega(I)$. Therefore, for any $f \in L^p$ and for each $I$, there is $C(p) > 0$ depending only on $p$ such that

$$\|M_I f\|_p \leq C(p) \frac{\lambda}{m_I \omega} \|f\|_p.$$

Here $\lambda$ is fixed, depending just on $p$, so that $\mathcal{J}$ is decaying. In fact, defining

$$\Delta_I f = \sum_{K \in \mathcal{F}_\omega(I)} \langle f, h_K \rangle h_K,$$

we have $M_I \Delta_I f = M_I f$ hence,

$$\|M_I f\|_p \leq C(p) \frac{\lambda}{m_I \omega} \|\Delta_I f\|_p. \tag{20}$$

Also notice that $f_j = \Delta_{\mathcal{F}^j} f = \sum_{I \in \mathcal{J}^{j-1}([0,1])} \Delta_I f$.

We shall prove that the $T_j$'s satisfy the condition (13) in Cotlar's Lemma. We will begin by proving that each is bounded on $L^p$. We write

$$\int |T_j f|^p = \int_{\bigcup \mathcal{J}^{j-1}([0,1]) \setminus \bigcup \mathcal{J}^j([0,1])} |T_j f|^p + \int_{\bigcup \mathcal{J}^j([0,1])} |T_j f|^p, \tag{21}$$

and we will estimate each term separately.

Observe that for every $I$, on $I \setminus \bigcup \mathcal{J}(I)$ one has almost everywhere that $\omega \leq \lambda m_I \omega$. Thus by the remark right before this proof and (20) we conclude that,

$$\begin{aligned} \int_{\bigcup \mathcal{J}^{j-1}([0,1]) \setminus \bigcup \mathcal{J}^j([0,1])} |T_j f|^p &= \sum_{J \in \mathcal{J}^{j-1}([0,1])} \int_{J \setminus \bigcup \mathcal{J}(J)} |T_j f|^p \\ &\leq \sum_{J \in \mathcal{J}^{j-1}([0,1])} (\lambda m_J \omega)^p \int_{J \setminus \bigcup \mathcal{J}(J)} |M_J f|^p \\ &\leq \sum_{J \in \mathcal{J}^{j-1}([0,1])} C(p) \lambda^{2p} \|\Delta_J f\|_p^p \ \leq \ C(p) \lambda^{2p} \|f_j\|_p^p. \end{aligned}$$

Here the last line comes from the disjointness of the elements of $\mathcal{J}^{j-1}$. We have completed our estimate of the first term of (21).

Observe that for every $I$, we have that $M_I f$ is constant on all $J \in \mathcal{J}(I)$. We denote its value on $J$ by $(M_I f)(J)$. Also observe that for such $J$, we have by the definition of $\mathcal{J}$ that $m_J\omega \leq 2\lambda m_I\omega$. Now we estimate using the $RH_p^d$ condition, and (20),

$$\begin{aligned}
\int_{\bigcup \mathcal{J}^j([0,1])} |T_j f|^p &= \sum_{J\in\mathcal{J}^{j-1}([0,1])} \int_{\bigcup \mathcal{J}(J)} |T_j f|^p \\
&= \sum_{J\in\mathcal{J}^{j-1}([0,1])} \sum_{K\in\mathcal{J}(J)} \big(M_J f(K)\big)^p \int_K \omega^p \\
&\leq C \sum_{J\in\mathcal{J}^{j-1}([0,1])} \sum_{K\in\mathcal{J}(J)} \big(M_J f(K)\big)^p |K| m_K^p\omega \\
&\leq C \sum_{J\in\mathcal{J}^{j-1}([0,1])} 2^p\lambda^p m_J^p\omega \int_J |M_J f|^p \;\leq\; C2^p\lambda^{2p}\|f_j\|_p^p.
\end{aligned}$$

Thus we have shown that there exists $C(p)$ so that

$$\int |T_j f|^p \leq C(p)\|f_j\|^p.$$

We claim further that there exists $0 < c < 1$ so that for any $k > j$, one has that

$$\int_{\bigcup \mathcal{J}^{k-1}([0,1])} |T_j f|^p \leq C(p)c^{k-j}\|f_j\|_p^p. \tag{22}$$

We will use Hölder's inequality, the decaying of $\mathcal{J}$ and the fact that $\omega \in RH_{p+\epsilon}^d$ to compute

$$\begin{aligned}
\int_{\bigcup \mathcal{J}^{k-1}([0,1])} |T_j f|^p &= \sum_{J\in\mathcal{J}^{j}([0,1])} \int_{\bigcup \mathcal{J}^{k-j-1}(J)} |T_j f|^p \\
&= \sum_{J\in\mathcal{J}^{j}([0,1])} |(M_j f)(J)|^p \int_{\bigcup \mathcal{J}^{k-j-1}(J)} \omega^p \\
&\leq \sum_{J\in\mathcal{J}^{j}([0,1])} c^{\frac{\epsilon(k-j-1)}{p+\epsilon}} |(M_j f)(J)|^p \left(\int_{\bigcup \mathcal{J}^{k-j-1}(J)} \omega^{p+\epsilon}\right)^{\frac{p}{p+\epsilon}} |J|^{\frac{\epsilon}{p+\epsilon}} \\
&\leq \sum_{J\in\mathcal{J}^{j}([0,1])} c^{\frac{\epsilon(k-j-1)}{p+\epsilon}} |(M_j f)(J)|^p |J| \left(\frac{1}{|J|}\int_J \omega^{p+\epsilon}\right)^{\frac{p}{p+\epsilon}} \\
&\leq C \sum_{J\in\mathcal{J}^{j}([0,1])} c^{\frac{\epsilon(k-j-1)}{p+\epsilon}} |J| \left(\frac{1}{|J|}\int_J w\,(M_j f)(J)\right)^p \\
&\leq \sum_{J\in\mathcal{J}^{j}([0,1])} Cc^{\frac{\epsilon(k-j-1)}{p+\epsilon}} \int_J \omega^p |M_j f|^p \\
&= Cc^{\frac{\epsilon(k-j-1)}{p+\epsilon}} \int |T_j f|^p \;\leq\; C(p)c^{\frac{\epsilon(k-j-1)}{p+\epsilon}}\|f_j\|_p^p.
\end{aligned}$$

This proves (22). But (22) implies (13) because $T_k f$ is supported on $\bigcup \mathcal{J}^{k-1}([0,1])$. Thus the theorem is proven. $\square$

**Remark:** Notice that $T_\omega h_J = \frac{\omega(x)}{m_J\omega} h_J(x)$ is supported on $J$, and the previous theorem for $p=2$ will give the strong type (2,2) for the operator (for this result ordinary Cotlar's Lemma is all that is needed, not the $L^p$ version), hence by Lemma 9, $T_\omega$ is of weak type (1,1). Thus by interpolation it is of strong type $(p,p)$ for $1<p<2$. It is not true that the adjoint $T_\omega^*$ is localized when acting on Haar functions, hence we cannot repeat the argument and then use duality to get $2<p<\infty$.

## 5.2 Some corollaries

We can consider the following one parameter family of Haar multipliers:

$$T_\omega^t f(x) = \sum_{I\in\mathcal{D}} \left(\frac{\omega(x)}{m_I\omega}\right)^t \langle f, h_I\rangle h_I(x) = \omega^t M_\omega^t f(x),$$

where $t \in \mathbf{R}$, and $M_\omega^t$ is the constant Haar multiplier with symbol $(m_I\omega)^{-t}$.

Checking the action on the Haar functions one obtains a necessary condition for the boundedness of $T_\omega^t$ in $L^p(\mathbf{R})$, namely *Condition* $C_{tp}$:

$$m_I\omega^{tp} \le (m_I\omega)^{tp}, \quad \forall I \in \mathcal{D}.$$

Clearly condition $C_s$ coincides with $RH_s^d$ when $s>1$, with $A_{1-1/s}^d$ when $s<0$, and it always holds when $0\le s\le 1$.

The following lemma will allow us to deduce the boundedness of $T_\omega^t$ as a simple corollary of Theorem 1.

**Lemma 11** *If $\omega \in C_{tp}\cap A_\infty^d$ then (i) $\omega^t \in RH_p^d$ and (ii) $m_I\omega^t \le C(m_I\omega)^t$ for all $I\in\mathcal{D}$.*

**Proof:** The proof of (i) follows from Lemma 1. Enough to show that $\omega^{tp}\in A_\infty^d$ since that would imply that $\omega^t \in RH_p^d$.

The cases $tp=0$ and $tp=1$ are trivial. For $0<tp<1$ then $\omega\in A_\infty^d$ implies that $\omega^{tp}\in RH_{1/tp}^d \subset A_\infty^d$. For $1<tp$ then $\omega \in RH_{tp}^d$ if and only if $\omega^{tp}\in A_\infty^d$. For $tp<0$ then $\omega\in A_{1-1/tp}^d$ if and only if $\omega^{tp}\in A_{1-tp}^d$ but Lemma 2 implies that $\omega^{tp}\in RH_q^d$ for some $q>1$ therefore $\omega^{tp}\in A_\infty^d$.

The proof of (ii) is a simple application of Hölder's inequality for $1<p<\infty$ and the use of the $C_{tp}$ condition:

$$(m_I\omega^t)^p \le m_I(\omega^{tp}) \le C(m_I\omega)^{tp}. \quad \square$$

**Corollary 2** *Assume $\omega\in A_\infty^d$, $t\in\mathbf{R}$, and $1<p<\infty$. Then $T_\omega^t$ is bounded in $L^p(\mathbf{R})$ if and only if $\omega\in C_{tp}$.*

**Proof:** We already pointed out the necessity of the $C_{tp}$ condition.

As for the sufficiency, notice that we can factorize $T^t_\omega$ as a composition of two bounded operators. In fact:

$$T^t_\omega = T_{\omega^t} S_{\omega,t},$$

where $S_{\omega,t}$ is the constant Haar multiplier with symbol $c_I = \frac{m_I \omega^t}{(m_I\omega)^t}$ which is a bounded sequence by Lemma 11, therefore $S_{\omega,t}$ is a bounded operator in $L^p$. Since $\omega \in A^d_\infty \cap C_{tp}$ also by Lemma 11, $\omega^t \in RH^d_p$, therefore by the Theorem 1, $T_{\omega^t}$ is bounded in $L^p$. □

**Corollary 3** *Assume $\omega \in A^d_\infty \cap C_{tp}$ for some $1 < p < A_\infty$ then $T^t_\omega$ is of weak type $(1,1)$, for all $t \in \mathbf{R}$.*

**Proof:** By Corollary 2 $T^t_\omega$ is of strong type $(p,p)$, and it clearly satisfies the hypothesis of Lemma 9. Therefore it is of weak type $(1,1)$. □

# 6 Haar multipliers and weighted inequalities

As an example on how to prove weighted inequalities with the aid of the Haar multipliers $T^t_\omega$, we show that $A^d_p$ is a sufficient condition for the boundedness in $L^p(\omega)$ of constant Haar multipliers, the dyadic square function $Sf$, and dyadic paraproducts.

Recall that, for $\omega$ some weight, and for $t$ any real number, we denote by $M^t_\omega$, the multiplier given by

$$M^t_\omega f = \sum_{I\in\mathcal{D}} \left(\frac{1}{m_I\omega}\right)^t \langle f, h_I\rangle h_I.$$

**Corollary 4** *Let $\omega \in A^d_p$. Then for any $1 < p < \infty$, the operators $\omega^{\frac{1}{p}} M^{\frac{1}{p}}_\omega$ and $M^{-\frac{1}{p}}_\omega \omega^{-\frac{1}{p}}$ are bounded on $L^p(\mathbf{R})$.*

**Proof:** Since, by Lemma 2, $A^d_p \subset A^d_\infty$, setting $t = 1/p$, $t = -1/p$ in Corollary 2 we get, respectively that $T^{\frac{1}{p}}_\omega = \omega^{\frac{1}{p}} M^{\frac{1}{p}}_\omega$ is bounded in $L^p$ and $\omega^{-\frac{1}{p}} M^{-\frac{1}{p}}_\omega$ is bounded on $L^{\frac{p}{p-1}}(\mathbf{R})$. Since $\frac{p}{p-1}$ is the dual index of $p$ and $M^{-\frac{1}{p}}_\omega \omega^{-\frac{1}{p}}$ is the dual of $\omega^{-\frac{1}{p}} M^{-\frac{1}{p}}_\omega$, we have shown that $\omega^{\frac{1}{p}} M^{\frac{1}{p}}_\omega$ is bounded in $L^p$. □

This allows us to interchange the weight $\omega$ and the multiplier $M^{-1}_\omega$ when proving weighted norm inequalities. We will give some easy applications.

**Corollary 5** *Let $\{\alpha_I\}_{I\in\mathcal{D}}$ be a bounded set of numbers and $T_\alpha$ be the associated Haar multiplier, i.e*

$$T_\alpha f = \sum_{I\in\mathcal{D}} \alpha_I \langle f, h_I\rangle h_I.$$

*Let $1 < p < \infty$ be given and let $\omega \in A_p^d$. Then $T_\alpha$ is bounded from $L^p(\omega)$ to $L^p(\omega)$.*

**Proof:** It suffices to show that $\omega^{\frac{1}{p}} T_\alpha \omega^{-\frac{1}{p}}$ is bounded on $L^p(\mathbf{R})$. Since Haar multipliers commute,

$$T_\alpha = M_\omega^{-\frac{1}{p}} T_\alpha M_\omega^{\frac{1}{p}},$$

and is bounded on $L^p(\mathbf{R})$. Then

$$\omega^{\frac{1}{p}} T_\alpha \omega^{-\frac{1}{p}} = (\omega^{\frac{1}{p}} M_\omega^{\frac{1}{p}})(M_\omega^{-\frac{1}{p}} T_\alpha M_\omega^{\frac{1}{p}})(M_\omega^{-\frac{1}{p}} \omega^{-\frac{1}{p}}).$$

Since all three factor of the right hand side are bounded, then so is the left hand side. □

**Corollary 6** *Let $\omega \in A_p^d$ and $b$ in dyadic $BMO$. Then $\pi_b$, the dyadic paraproduct defined by*

$$\pi_b f = \sum_{I \in \mathcal{D}} b_I m_I f \, h_I,$$

*is bounded on $L^p(\omega)$.*

**Proof:** We will prove the result for $p = 2$ and then an extrapolation argument proves it for $p \neq 2$ , see [GC-RF].

It suffices to bound $\omega^{\frac{1}{2}} \pi_b \omega^{-\frac{1}{2}}$ on $L^2$. We may write

$$\omega^{\frac{1}{2}} \pi_b \omega^{-\frac{1}{2}} = (\omega^{\frac{1}{2}} M_\omega^{\frac{1}{2}})(M_\omega^{-\frac{1}{2}} \pi_b \omega^{-\frac{1}{2}}).$$

Thus it suffices to bound $M_\omega^{-\frac{1}{2}} \pi_b \omega^{-\frac{1}{2}}$ on $L^2$. However for any $f \in L^2$,

$$\|M_\omega^{-\frac{1}{2}} \pi_b \omega^{-\frac{1}{2}} f\|_2^2 = \sum_{I \in \mathcal{D}} (m_I \omega) b_I^2 (m_I(\omega^{-\frac{1}{2}} f))^2.$$

We use the fact that $\omega$, $\omega^{-1} \in A_2^d$, which implies that $\omega^{-1} \in RH^d_{\frac{2+\epsilon}{2}}$ for some $\epsilon > 0$, Carleson's Lemma 5, boundedness of the maximal function on $L^{\frac{2+2\epsilon}{2+\epsilon}}$, and Hölder's inequality to estimate

$$\begin{aligned} \|M_\omega^{-\frac{1}{2}} \pi_b \omega^{-\frac{1}{2}} f\|_2^2 &\leq \sum_I (m_I \omega) b_I^2 (m_I \omega^{-\frac{2+\epsilon}{2}})^{\frac{2}{2+\epsilon}} (m_I f^{\frac{2+\epsilon}{1+\epsilon}})^{\frac{2+2\epsilon}{2+\epsilon}} \\ &\leq C \sum_I (m_I \omega) b_I^2 (m_I(\omega^{-1}))(m_I f^{\frac{2+\epsilon}{1+\epsilon}})^{\frac{2+2\epsilon}{2+\epsilon}} \leq C \sum_I b_I^2 (m_I f^{\frac{2+\epsilon}{1+\epsilon}})^{\frac{2+2\epsilon}{2+\epsilon}} \\ &\leq C \int \big( \sup_{I \ni x} m_I f^{\frac{2+\epsilon}{1+\epsilon}} \big)^{\frac{2+2\epsilon}{2+\epsilon}} dx \leq C \|f^{\frac{2+\epsilon}{1+\epsilon}}\|_{\frac{2+2\epsilon}{2+\epsilon}}^{\frac{2+2\epsilon}{2+\epsilon}} = C\|f\|_2^2. \end{aligned}$$

Which was to be shown. □

We can also prove weighted inequalities for dyadic square functions. For an alternative proof see [B]. In particular the classical dyadic Littlewood-Paley Theory can be deduced setting $\omega = 1$. The next result can be viewed as weighted Littlewood-Paley Theory.

**Corollary 7** *Let $\omega \in A_p^d$ then the dyadic square function*

$$Sf(x) = \big(\sum_j |\Delta_j f|^2\big)^{\frac{1}{2}}, \quad \Delta_j f = \sum_{I \in \mathcal{D}_j} \langle f, h_I \rangle h_I,$$

*is bounded in $L^p(\omega)$.*

**Proof:** We prove it for $p = 2$ and use extrapolation for $p \neq 2$ as Steve Buckley did in his proof of the same fact, see [B].

It suffices to show that $\omega^{\frac{1}{2}} S \omega^{-\frac{1}{2}}$ is bounded on $L^2$.

Computing we get

$$\begin{aligned} \|\omega^{\frac{1}{2}} S \omega^{-\frac{1}{2}} f\|_2^2 &= \sum_j \int \omega(x) |\Delta_j (f w^{-\frac{1}{2}})|^2 \\ &= \sum_{I \in \mathcal{D}} |\langle f\omega^{-\frac{1}{2}}, h_I \rangle|^2 m_I \omega = \|M_\omega^{-\frac{1}{2}} w^{-\frac{1}{2}} f\|_2^2 \leq C \|f\|_2^2. \end{aligned}$$

The last inequality by Corollary 4. □

For operators like the Hilbert transform it is more complicated than these, see for example [TV1], [KP]. But the same ideas are used. The following tautology holds: *an operator $T$ is bounded from $L^p(u)$ into $L^p(v)$ if and only if the operator $v^{\frac{1}{p}} T u^{-\frac{1}{p}}$ is bounded from $L^p(\mathbf{R})$ into itself.* As we mentioned before, and we hope has been highlighted by the examples, the Haar multipliers will allow us to replace multiplication on space side by multiplication on frequency side, whenever it is convenient. For example to show the boundedness of the Hilbert Transforming from $L^2(\omega)$ into itself it is enough to check that $S = M_\omega^{\frac{1}{2}} H M_\omega^{\frac{-1}{2}}$ is bounded in $L^2(\mathbf{R})$. In this case the estimates are more laborious to obtain than in the cases presented here. The strategy followed in [TV1], [KP] follows a method introduced by Coifman and Semmes, widely used in Wavelet Theory. One studies the decay of the matrix $t_{IJ} = \langle Sh_I, h_J \rangle$ using, for example, Schur's Lemma [CJS]. Some pieces will be analized like if they were constant Haar multipliers, others as if they were paraproducts, very much in the spirit of the $T(1)$ Theorem of Journé and David [DJ].

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