A note on a maximal function over arbitrary sets of directions*

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Abstract

Let $\mathcal{M}_S$ be the universal maximal operator over unit vectors of arbitrary directions. This operator is not bounded in $L^2(\mathbf{R}^2)$. We consider a sequence of operators over sets of finite equidistributed directions converging to $\mathcal{M}_S$. We provide a new proof of N. Katz's bound for such operators. As a corollary we deduce that $\mathcal{M}_S$ is bounded from some subsets of L^2 to L^2. These subsets are composed of positive functions whose Fourier transforms have a logarithmic decay or are supported on a disc.

Let Σ be a set of unit vectors on $\mathbf{R}^2$. Define the following maximal operator for functions f in the Schwartz class $\mathcal{S}(\mathbf{R}^2)$:

$$\mathcal{M}_\Sigma f(x) = \sup_{v\in\Sigma} \frac{1}{2}\int_{|t|\leq 1} |f(x-tv)|\, dt.$$

This operator was introduced by S. Wainger in [W]. He showed that bounds of the order $(1+\log N)$ hold for any set of N directions. The best bounds known to date for arbitrary sets of N directions were found recently by N. Katz (see [K1]), he showed that:

Given any set Σ_N of N directions, there is a constant $C>0$ such that

$$\|M_{\Sigma_N} f\|_2 \leq \mathcal{C}\sqrt{1+\log N}\,\|f\|_2, \qquad \forall f\in L^2(\mathbf{R}^2). \tag{1}$$

The second author was the first to observe that this type of bounds hold for the set of directions whose slopes belong to $\mathcal{C}_n$, a truncation of $\mathcal{C}$ the 1/3 Cantor set, see [V]. Her proof is more restrictive than Katz's proof but it implies the boundedness of $\mathcal{M}_\mathcal{C}$ on particular subsets of $L^2(\mathbf{R}^2)$. One of the subsets corresponds to functions band-limited to a *strip*, and the other to a sort of *Sobolev space with logarithmic derivatives.*

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In this note we observe that Vargas' proof holds for equidistributed sets of directions, therefore similar conclusions can be drawn on the boundedness of the maximal function on the limiting set of directions of arbitrarily large equidistributed sets, that is, the unit circle S. This means that those estimates hold for $\mathcal{M}_S$ the *universal maximal operator* and therefore are independent of the chosen set of directions. More precisely we show

Theorem A: *Let* $S = \{v \in \mathbf{R}^2 : |v| = 1\}$.

(a) *There is a constant* $C > 0$ *such that, for every* $q \geq 1$ *and* $f \in L^2(\mathbf{R}^2)$, $f \geq 0$, *satisfying supp*$\hat{f} \subseteq \{\xi \in \mathbf{R}^2 : |\xi| \leq 2^q\}$, *we have*

$$\|\mathcal{M}_S f\|_2 \leq C\sqrt{q}\,\|f\|_2.$$

(b) *For all* $\epsilon > 0$, *there is a constant* C_ϵ *such that, for every* $f \in L^2(\mathbf{R}^2)$, $f \geq 0$ *and* $\|f\|^2_{Wlog} := \int |\hat{f}(\xi)|^2 (1 + (\log^+ |\xi|)(\log^+ \log^+ |\xi|)^{1+\epsilon})\, d\xi_1\, d\xi_2 < \infty$ *then*

$$\|\mathcal{M}_S f\|_2 \leq C_\epsilon \|f\|_{Wlog}$$

Katz also showed in [K2] that for truncated Cantor sets of directions the bounds cannot be better than $\sqrt{\log\log N}$, similar examples show that for equidistributed sets the bound $\sqrt{\log N}$ is sharp. These implies that $\mathcal{M}_S$ cannot be bounded in $L^2(\mathbf{R}^2)$. Notice that positive results for truncated Cantor sets follow as a corollary from positive results for equidistributed sets.

Let $\mathcal{A}_n$ denote the set of 2^n equidistributed points on $[0,1]$, namely:

$$\mathcal{A}_n = \left\{\frac{1}{2^n}, \frac{2}{2^n}, \dots, \frac{2^n - 1}{2^n}, 1\right\}.$$

Let Ω_n denote the set of directions whose slopes are in $\mathcal{A}_n$, more precisely:

$$\Omega_n = \left\{v = (v_1, v_2) \in \mathbf{R}^2 : |v| = 1, \;\; v_1 > 0, \;\; \frac{v_2}{v_1} \in \mathcal{A}_n\right\}.$$

We also give a proof of a particular case of Katz's Theorem,

$$\|\mathcal{M}_{\Omega_n} f\|_2 \leq C\sqrt{1+n}\,\|f\|_2. \tag{2}$$

Notice $\cup\Omega_n$ is dense in $S \cap \{v = (v_1, v_2) : \; 0 < v_2 \leq v_1\} = \Omega$, and the sequence $\mathcal{M}_{\Omega_n} f(x)$ is increasing for every $x \in \mathbf{R}^2$ and f in the Schwartz class $\mathcal{S}(\mathbf{R}^2)$. Hence for such f and x, $\mathcal{M}_{\Omega_n} f(x)$ converges to $\mathcal{M}_\Omega f(x)$. By simmetry, it is enough to prove Theorem A with Ω replacing $\mathcal{S}$. Actually in this case we can prove a stronger statement:

(a') *There is a constant* $C > 0$ *such that, for every* $q \geq 1$ *and* $f \in L^2(\mathbf{R}^2)$, $f \geq 0$, *satisfying supp*$\hat{f} \subseteq \{\xi = (\xi_1, \xi_2) \in \mathbf{R}^2 : |\xi_2| \leq 2^q\}$, *we have*

$$\|\mathcal{M}_\Omega f\|_2 \leq C\sqrt{q}\,\|f\|_2.$$

(b') *For all* $\epsilon > 0$, *there is a constant* C_ϵ *such that, for every* $f \in L^2(\mathbf{R}^2)$, $f \geq 0$ *and* $\|f\|^2_{Wlog_2} := \int |\hat{f}(\xi)|^2(1 + (\log^+ |\xi_2|)(\log^+ \log^+ |\xi_2|)^{1+\epsilon})\, d\xi_1\, d\xi_2 < \infty$ *then*

$$\|\mathcal{M}_\Omega f\|_2 \leq C_\epsilon \|f\|_{Wlog_2}$$

This will be proved by showing similar estimates, uniformly on n, for $\mathcal{M}_{\Omega_n} f$.

We will only indicate the adjustments that must be done on Vargas' proof for it to work for equidistributed sets with cardinality 2^n, very much in the spirit of S. Wainger's original paper. A similar proof works for equidistributed sets with cardinality $N \sim 2^n$, hence for all equidistributed sets. This provides an alternative proof to Katz's for equidistributed sets of directions.

Proof of Theorem A and (2):

We can assume $f \geq 0$, $f \in \mathcal{S}(\mathbf{R}^2)$.

Let $p_k : \mathbf{R}^2 \to \mathcal{A}_k$ be a measurable function. Associated to such mappings there is a smooth linearization of the maximal function, given by:

$$\mathcal{L}_k f(x_1, x_2) = \mathcal{L}_{p_k(x_1,x_2)} f(x_1, x_2) = \int \phi(t) f(x_1 - t, x_2 - tp_k(x_1, x_2))\, dt,$$

where ϕ is a positive function in $\mathcal{S}(\mathbf{R})$ and $\hat{\phi} \in C_0^\infty([-1/2, 1/2])$. It is enough to bound $\mathcal{L}_n$ independently of the choice of p_n with a bound of the order $\sqrt{1+n}$ to show that $\mathcal{M}_{\Omega_n}$ is bounded in $L^2(\mathbf{R}^2)$ with a bound of the same order. Similarly, in order to prove Theorem A, we will have to prove bounds for $\mathcal{L}_n f$, uniform on n, for $f \in Wlog_2$ or for $supp \hat{f} \subset \{|\xi_2| \leq 2^q\}$.

We split the function f as $f = \sum_{l=0}^\infty f_l$, where $\hat{f}_l(\xi_1, \xi_2) = \beta_l(\xi_1, \xi_2)\hat{f}(\xi_1, \xi_2)$, and $\beta_l \in \mathcal{S}(\mathbf{R}^2)$, $0 \leq \beta_l \leq 1$, $\sum_{l=0}^\infty \beta_l = 1$, $supp\, \beta_0 \subseteq \{|\xi_2| \leq 4\}$ and $supp\, \beta_l \subseteq \{2^l < |\xi_2| \leq 2^{l+2}\}$ for $l = 1, 2, \ldots$ Define $\hat{f^n} = (1 - \sum_{l=0}^{n-1} \beta_l)\hat{f}$.

(a'), (b') and (2) are straightforward consequences of the following estimates:

$$\|\mathcal{L}_n f_l\|_2 \leq C(\phi)\|f_l\|_2, \quad n \geq 1, \; l = 0, 1, \cdots, n, \tag{3}$$
$$\|\mathcal{L}_n f^n\|_2 \leq C(\phi)\|f^n\|_2, \quad n \geq 0. \tag{4}$$

To show (3) it will be enough to control $\sum_{k=l+1}^n \|\mathcal{L}_k f_l - \mathcal{L}_{k-1} f_l\|_2 + \|\mathcal{L}_l f_l\|_2$. Where given $p_n : \mathbf{R}^2 \to \mathcal{A}_n$, we define $p_k : \mathbf{R}^2 \to \mathcal{A}_k$ for $k < n$ recursively by: $p_{k-1}(x) = p_k(x)$ if $p_k(x) \in \mathcal{A}_{k-1}$, otherwise $p_{k-1} = p_k(x) + \frac{1}{2^k}$ which will be in $\mathcal{A}_{k-1}$.

Estimates (3) and (4) boil down to the study of the following square functions:

$$\psi^l(\xi_1, \xi_2) = \sum_{p \in \mathcal{A}_l} |\hat{\phi}(\xi_1 + p\xi_2)|^2,$$
$$\psi_k(\xi_1, \xi_2) = \sum_{p \in \mathcal{A}_k \setminus \mathcal{A}_{k-1}} |\hat{\phi}(\xi_1 + p\xi_2) - \hat{\phi}(\xi_1 + (p + \frac{1}{2^k})\xi_2)|^2.$$

To explain this claim, define $M_p f(x_1, x_2) = \int \phi(t) f(x_1 - t, x_2 - tp)\, dt$. Then, we majorize

$$\|\mathcal{L}_k f_l - \mathcal{L}_{k-1} f_l\|_2 \le \|(\sum_{p \in \mathcal{A}_k \setminus \mathcal{A}_{k-1}} |M_p f_l - M_{p+\frac{1}{2^k}} f_l|^2)^{1/2}\|_2.$$

By Plancherel's theorem, this is equal to

$$\|(\psi_k)^{1/2} \widehat{f_l}\|_2.$$

A similar argument can be used to bound

$$\|\mathcal{L}_l f_l\|_2 \le \|(\psi^l)^{1/2} \widehat{f_l}\|_2$$

and

$$\|\mathcal{L}_n f^n\|_2 \le \|(\psi^n)^{1/2} \widehat{f^n}\|_2.$$

Hence, it is enough to show

Lemma: *There exists a constant C such that*

(i) $\|\psi^l\|_{L^\infty(supp \sum_{k \ge l} \beta_k)} \le C \|\hat{\phi}\|_\infty^2$,

(ii) $\|\psi_k\|_{L^\infty(supp\, \beta_l)} \le C \|\hat{\phi}'\|_\infty^2 2^{l-k}, \qquad \forall k > l.$

Proof: The proof of (i) is similar to that on [V]. The main observation is that in this setting it is also true that, for $l \ge 1$, if $(\xi_1, \xi_2) \in supp \sum_{k \ge l} \beta_k$ then there is a unique $p \in \mathcal{A}_l$ such that $\xi_1 + p\xi_2 \in supp\, \hat{\phi} \subseteq [-1/2, 1/2]$. (If $l = 0$ then $\mathcal{A}_0$ has only one element).

To show (ii), notice that $supp\, \hat{\phi}'(\xi_1 + \cdot\, \xi_2) \subseteq I(\xi_1, \xi_2) = [-\frac{1}{2|\xi_2|} - \frac{\xi_1}{\xi_2}, \frac{1}{2|\xi_2|} - \frac{\xi_1}{\xi_2}]$. The proof of (ii) depends on a careful counting of the number m_k of points $p \in \mathcal{A}_k \setminus \mathcal{A}_{k-1}$ such that $[p, p + \frac{1}{2^k}] \cap I(\xi_1, \xi_2) \ne \emptyset$ for $(\xi_1, \xi_2) \in supp\, \beta_l$. For such (ξ_1, ξ_2) one can estimate:

$$\psi_k(\xi_1, \xi_2) \le m_k \sup_{p \in \mathcal{A}_k \setminus \mathcal{A}_{k-1}} \left| \int_p^{p+\frac{1}{2^k}} \xi_2 \hat{\phi}'(\xi_1 + t\xi_2)\, dt \right|^2.$$

Moreover, $|I(\xi_1, \xi_2)| = \frac{1}{|\xi_2|} \le 2^{-l}$, hence there are at most 2^{k-l} points $p \in \mathcal{A}_k \setminus \mathcal{A}_{k-1}$ such that the intersection of $[p, p + \frac{1}{2^k}]$ and $I(\xi_1, \xi_2)$ is not empty. We conclude that $m_k \le C 2^{k-l}$, and therefore

$$\begin{aligned} \psi_k(\xi_1, \xi_2) &\le C 2^{k-l} |\xi_2|^2 \|\hat{\phi}'\|_\infty^2 2^{-2k} \\ &\le C 2^{k-l} 2^{2l+4} 2^{-2k} \|\hat{\phi}'\|_\infty^2 \quad \le \quad C \|\hat{\phi}'\|_\infty^2 2^{l-k} \end{aligned}$$

This finishes the proof of the lemma. □

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