# HAAR BASES ON QUASI-METRIC MEASURE SPACES, AND DYADIC STRUCTURE THEOREMS FOR FUNCTION SPACES ON PRODUCT SPACES OF HOMOGENEOUS TYPE

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Abstract. We give an explicit construction of Haar functions associated to a system of dyadic cubes in a geometrically doubling quasi-metric space equipped with a positive Borel measure, and show that these Haar functions form a basis for $L^p$. Next we focus on spaces $X$ of homogeneous type in the sense of Coifman and Weiss, where we use these Haar functions to define a discrete square function, and hence to define dyadic versions of the function spaces $H^1(X)$ and $\mathrm{BMO}(X)$. In the setting of product spaces $\widetilde{X} = X_1 \times \cdots \times X_n$ of homogeneous type, we show that the space $\mathrm{BMO}(\widetilde{X})$ of functions of bounded mean oscillation on $\widetilde{X}$ can be written as the intersection of finitely many dyadic BMO spaces on $\widetilde{X}$, and similarly for $A_p(\widetilde{X})$, reverse-Hölder weights on $\widetilde{X}$, and doubling weights on $\widetilde{X}$. We also establish that the Hardy space $H^1(\widetilde{X})$ is a sum of finitely many dyadic Hardy spaces on $\widetilde{X}$, and that the strong maximal function on $\widetilde{X}$ is pointwise comparable to the sum of finitely many dyadic strong maximal functions. These dyadic structure theorems generalize, to product spaces of homogeneous type, the earlier Euclidean analogues for BMO and $H^1$ due to Mei and to Li, Pipher and Ward.

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## 1. Introduction

This paper has three main components: (1) the explicit construction of a Haar basis associated to a system of dyadic cubes on a geometrically doubling quasi-metric space equipped with a positive Borel measure, (2) definitions of dyadic product function spaces, by means of this Haar basis, on product spaces of homogeneous type in the sense of Coifman and Weiss, and (3) dyadic structure theorems relating the continuous and dyadic versions of these function spaces on product spaces of homogeneous type. We describe these components in more detail below.

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The function spaces we deal with in this paper are the Hardy space $H^1$, the space BMO of functions of bounded mean oscillation, and the classes of Muckenhoupt $A_p$ weights, reverse-Hölder weights, and doubling weights. Much of the theory of these function spaces can be found in [JoNi], [FS], [GCRF] and [Ste]. Here we are concerned with generalising the dyadic structure theorems established for Euclidean underlying spaces in [LPW] and [Mei].

A central theme in modern harmonic analysis has been the drive to extend the Calderón–Zygmund theory from the Euclidean setting, namely where the underlying space is $\mathbb{R}^n$ with the Euclidean metric and Lebesgue measure, to more general settings. To this end Coifman and Weiss formulated the concept of spaces of homogeneous type, in [CW71]. Specifically, by a *space* $(X, \rho, \mu)$ *of homogeneous type* in the sense of Coifman and Weiss, we mean a set $X$ equipped with a quasi-metric $\rho$ and a Borel measure $\mu$ that is doubling. There is a large literature devoted to spaces of homogeneous type, in both their one-parameter and more recently multiparameter forms. See for example [CW71, CW77, MS, DJS, H94, HS, H98, ABI07, DH, HLL, CLW] and [AH]. Some non-Euclidean examples of spaces of homogeneous type are given by the Carnot–Carathéodory spaces whose theory is developed by Nagel, Stein and others in [NS04], [NS06] and related papers; there the quasi-metric is defined in terms of vector fields satisfying the Hörmander condition on an underlying manifold.

Meyer noted in his preface to [DH] that *"One is amazed by the dramatic changes that occurred in analysis during the twentieth century. In the 1930s complex methods and Fourier series played a seminal role. After many improvements, mostly achieved by the Calderón–Zygmund school, the action takes place today on spaces of homogeneous type. No group structure is available, the Fourier transform is missing, but a version of harmonic analysis is still present. Indeed the geometry is conducting the analysis."*

When we go beyond the Euclidean world and attempt to prove our dyadic structure theorems in the setting of spaces of homogeneous type, we immediately encounter the following two obstacles. First, the Euclidean proofs rely on the so-called one-third trick, which says in effect that each ball is contained in some cube whose measure is (uniformly) comparable to that of the ball, and which belongs either to the usual dyadic lattice or to a fixed translate of the dyadic lattice. However, in a space of homogeneous type there is no notion of translation. Second, in the Euclidean proofs one uses a definition of dyadic function spaces in terms of the Haar coefficients. However, the theory of Haar bases on spaces of homogeneous type is not fully developed in the literature. In the present paper, we overcome the first obstacle by means of the adjacent systems of dyadic cubes constructed by the first author and Hytönen in [HK], and the second obstacle by our full development of an explicit construction of Haar bases for $L^p$ in a setting somewhat more general than that of spaces of homogeneous type.

We now describe in more detail the three main components of this paper, listed at the start of the introduction.

(1) We include in this paper a detailed construction of a Haar basis associated to a system of dyadic cubes on a geometrically doubling quasi-metric space equipped with a positive Borel measure. The geometric-doubling condition says that each ball can be covered by a uniformly bounded number of balls of half the radius of the original ball. This setting is somewhat more general than that of spaces of homogeneous type in the sense of Coifman and Weiss. Although such bases have been discussed in the existing literature (see below), to our knowledge a full construction has not appeared before. We believe that such bases will also be useful for other purposes, beyond the definitions of dyadic function spaces for which they are used in this paper.

Haar-type bases for $L^2(X, \mu)$ have been constructed in general metric spaces, and the construction is well known to experts. Haar-type wavelets associated to nested partitions in abstract measure spaces are provided by Girardi and Sweldens [GS]. Such Haar functions are also used in [NRV], in geometrically doubling metric spaces. For the case of spaces of homogeneous type, we refer to the papers by Aimar et al. [Aim, ABI07, AG]; see also [ABI05, ABN11a, ABN11b] for related results. In the present paper the context is a slightly more general geometrically doubling quasi-metric space $(X, \rho)$, with a positive Borel measure $\mu$. Our approach differs from that of the mentioned prior results, and it uses results from general martingale theory. We provide the details, noting that essentially the same construction can be found, for example, in [Hyt, Chapter 4] in the case of Euclidean space with the usual dyadic cubes and a non-doubling measure.

(2) It is necessary to develop in the setting of product spaces $\widetilde{X}$ of homogeneous type careful definitions of the continuous and dyadic versions of the function spaces we consider. In the Euclidean setting the theory of product BMO and $H^1$ was developed by Chang and R. Fefferman in the continuous case [Cha, Fef, CF], and by Bernard in the dyadic case [Ber].

We recall that $(X, \rho, \mu)$ is a *space of homogeneous type* in the sense of Coifman and Weiss if $\rho$ is a quasi-metric and $\mu$ is a nonzero measure satisfying the doubling condition. A *quasi-metric* $\rho : X \times X \longrightarrow [0, \infty)$ satisfies (i) $\rho(x, y) = \rho(y, x) \geq 0$ for all $x$, $y \in X$; (ii) $\rho(x, y) = 0$ if and only if $x = y$; and (iii) the *quasi-triangle inequality*: there is a constant $A_0 \in [1, \infty)$ such that for all $x$, $y$, $z \in X$,

$$\rho(x, y) \leq A_0[\rho(x, z) + \rho(z, y)]. \tag{1.1}$$

In contrast to a metric, the quasi-metric may not be Hölder regular and quasi-metric balls may not be open; see for example [HK, p.5]. A nonzero measure $\mu$ satisfies the *doubling condition* if there is a constant $C_\mu$ such that for all $x \in X$ and all $r > 0$,

$$\mu(B(x, 2r)) \leq C_\mu \mu(B(x, r)) < \infty. \tag{1.2}$$

We do not impose any assumptions about regularity of the quasi-metric, nor any additional properties of the measure.

As shown in [CW71], spaces of homogeneous type are geometrically doubling. Thus our construction of the Haar basis is valid on $(X, \rho, \mu)$.

We denote the product of such spaces by $\widetilde{X} = X_1 \times \cdots \times X_n$, with the product quasi-metric $\rho = \rho_1 \times \cdots \times \rho_n$ and the product measure $\mu = \mu_1 \times \cdots \times \mu_n$.

The task of defining our function spaces on the product space $\widetilde{X}$ is straightforward for $A_p$, $RH_p$ and doubling weights. As in the Euclidean case, the product weights on $\widetilde{X}$ are simply those weights that have the one-parameter $A_p$, $RH_p$ or doubling property in each factor, and Lebesgue measure is replaced by the Borel measure $\mu$. The situation is more delicate for $H^1$ and BMO. For the continuous versions, we use the definition from [HLW] of $H^1(\widetilde{X})$ via a square function that makes use of the orthonormal wavelet bases developed in [AH] for spaces of homogeneous type in the sense of Coifman and Weiss. We define BMO$(\widetilde{X})$ in terms of summation conditions on the [AH] wavelet coefficients. As is shown in [HLW], these definitions yield the expected Hardy space theory and the duality relation between $H^1(\widetilde{X})$ and BMO$(\widetilde{X})$.

For the dyadic versions of $H^1$ and BMO on $\widetilde{X}$, we replace the [AH] wavelets in these definitions by a basis of Haar wavelets, constructed in the present paper, associated to a given system of dyadic cubes as constructed in [HK].

(3) In this paper we show that on product spaces $\widetilde{X}$ of homogeneous type, in the sense of Coifman and Weiss, the space BMO of functions of bounded mean oscillation coincides with the intersection of finitely many dyadic BMO spaces. The product Euclidean version of this result appears in [LPW], the one-parameter Euclidean version in [Mei], and the one-parameter version for spaces $X$ of homogeneous type in [HK]. In addition, generalizing the Euclidean results of [LPW], we establish the analogous intersection results for Muckenhoupt's $A_p$ weights $A_p(\widetilde{X})$, $1 \leq p \leq \infty$, for the reverse-Hölder weights $RH_p(\widetilde{X})$, $1 \leq p \leq \infty$, and for the class of doubling weights on $\widetilde{X}$; the result that the Hardy space $H^1(\widetilde{X})$ is the sum of

finitely many dyadic Hardy spaces on $\widetilde{X}$; the result that the strong maximal function on $\widetilde{X}$ is comparable to the sum of finitely many dyadic maximal functions on $\widetilde{X}$.

We note that our methods should suffice to establish the generalizations to spaces of homogeneous type of the weighted dyadic structure results in [LPW], namely Theorem 8.1 for product $H^1$ weighted by an $A_\infty$ weight and Theorem 8.2 for the strong maximal function weighted by a doubling weight. We leave it to the interested reader to pursue this direction.

We note that in a different direction, not pursued in the present paper, a connection between continuous and dyadic function spaces via averaging is developed in the papers [GJ, War, PW, Tre, PWX], and [CLW], for both Euclidean spaces and spaces of homogeneous type. Specifically, a procedure of translation-averaging (for BMO) or geometric-arithmetic averaging (for $A_p$, $RH_p$, and doubling weights) converts a suitable family of functions in the dyadic version of a function space into a single function that belongs to the continuous version of that function space. We do not discuss the averaging approach further in the present paper.

This paper is organized as follows. In Section 2, we first recall the definition of systems of dyadic cubes and the related properties of those cubes. Next we recall the result of Hytönen and the first author (see Theorem 2.7) on the existence of a collection of adjacent systems of dyadic cubes, which provides a substitute in the setting of spaces of homogeneous type for the one-third trick in the Euclidean setting. In Section 3 we prove our dyadic structure results for the $A_p$, $RH_p$ and doubling weights, as well as for maximal functions. In Section 4, we construct the Haar functions on a geometrically doubling quasi-metric space $(X, \rho)$ equipped with a positive Borel measure $\mu$ (Theorem 4.8). Then we prove that the Haar wavelet expansion holds on $L^p(X)$ for all $p \in (1, \infty)$ (Theorem 4.9). In Section 5 we recall the definitions of the continuous product Hardy and BMO spaces from [HLW], and provide the definitions of the dyadic product Hardy and BMO spaces by means of the Haar wavelets we have constructed. In Section 6, we establish the dyadic structure theorems for the continuous and dyadic product Hardy and BMO spaces (Theorems 6.1 and 6.2), by proving the dyadic atomic decomposition for the dyadic product Hardy spaces (Theorem 6.5).

## 2. Systems of dyadic cubes

The set-up for Section 2 is a *geometrically doubling quasi-metric space*: a quasi-metric space $(X, \rho)$ that satisfies the *geometric doubling property* that there exists a positive integer $A_1 \in \mathbb{N}$ such that any open ball $B(x, r) := \{y \in X \colon \rho(x, y) < r\}$ of radius $r > 0$ can be covered by at most $A_1$ balls $B(x_i, r/2)$ of radius $r/2$; by a quasi-metric we mean a mapping $\rho\colon X \times X \to [0, \infty)$ that satisfies the axioms of a metric except for the triangle inequality which is assumed in the weaker form

$$\rho(x, y) \le A_0(\rho(x, z) + \rho(z, y)) \quad \text{for all } x, y, z \in X$$

with a constant $A_0 \ge 1$.

A subset $\Omega \subseteq X$ is *open* (in the topology induced by $\rho$) if for every $x \in \Omega$ there exists $\varepsilon > 0$ such that $B(x, \varepsilon) \subseteq \Omega$. A subset $F \subseteq X$ is *closed* if its complement $X \setminus F$ is open. The usual proof of the fact that $F \subseteq X$ is closed, if and only if it contains its limit points, carries over to the quasi-metric spaces. However, some open balls $B(x, r)$ may fail to be open sets, see [HK, Sec 2.1].

Constants that depend only on $A_0$ (the quasi-metric constant) and $A_1$ (the geometric doubling constant), are referred to as *geometric constants.*

### 2.1. A system of dyadic cubes.

In a geometrically doubling quasi-metric space $(X, \rho)$, a countable family

$$\mathscr{D} = \bigcup_{k \in \mathbb{Z}} \mathscr{D}_k, \quad \mathscr{D}_k = \{Q^k_\alpha \colon \alpha \in \mathscr{A}_k\},$$

of Borel sets $Q^k_\alpha \subseteq X$ is called *a system of dyadic cubes with parameters* $\delta \in (0,1)$ and $0 < c_1 \leq C_1 < \infty$ if it has the following properties:

$$X = \bigcup_{\alpha \in \mathscr{A}_k} Q^k_\alpha \quad \text{(disjoint union) for all } k \in \mathbb{Z}; \tag{2.1}$$

$$\text{if } \ell \geq k, \text{ then either } Q^\ell_\beta \subseteq Q^k_\alpha \text{ or } Q^k_\alpha \cap Q^\ell_\beta = \emptyset; \tag{2.2}$$

$$\text{for each } (k,\alpha) \text{ and each } \ell \leq k, \text{ there exists a unique } \beta \text{ such that } Q^k_\alpha \subseteq Q^\ell_\beta; \tag{2.3}$$

(2.4)
for each $(k,\alpha)$ there exist between 1 and $M$ (a fixed geometric constant) $\beta$ such that
$$Q^{k+1}_\beta \subseteq Q^k_\alpha, \text{ and } Q^k_\alpha = \bigcup_{\substack{Q \in \mathscr{D}_{k+1} \\ Q \subseteq Q^k_\alpha}} Q;$$

$$B(x^k_\alpha, c_1\delta^k) \subseteq Q^k_\alpha \subseteq B(x^k_\alpha, C_1\delta^k) =: B(Q^k_\alpha); \tag{2.5}$$

$$\text{if } \ell \geq k \text{ and } Q^\ell_\beta \subseteq Q^k_\alpha, \text{ then } B(Q^\ell_\beta) \subseteq B(Q^k_\alpha). \tag{2.6}$$

The set $Q^k_\alpha$ is called a *dyadic cube of generation* $k$ with center point $x^k_\alpha \in Q^k_\alpha$ and side length $\delta^k$. The interior and closure of $Q^k_\alpha$ are denoted by $\widetilde{Q}^k_\alpha$ and $\bar{Q}^k_\alpha$, respectively.

We recall from [HK] the following construction, which is a slight elaboration of seminal work by M. Christ [Chr], as well as Sawyer–Wheeden [SW].

**Theorem 2.1.** *Let $(X,\rho)$ be a geometrically doubling quasi-metric space. Then there exists a system of dyadic cubes with parameters $0 < \delta \leq (12A_0^3)^{-1}$ and $c_1 = (3A_0^2)^{-1}, C_1 = 2A_0$. The construction only depends on some fixed set of countably many center points $x^k_\alpha$, having the properties that*

$$\rho(x^k_\alpha, x^k_\beta) \geq \delta^k \quad (\alpha \neq \beta), \qquad \min_\alpha \rho(x, x^k_\alpha) < \delta^k \quad \textit{for all } x \in X,$$

*and a certain partial order $\leq$ among their index pairs $(k,\alpha)$. In fact, this system can be constructed in such a way that*

$$\overline{Q}^k_\alpha = \overline{\{x^\ell_\beta : (\ell,\beta) \leq (k,\alpha)\}} \tag{2.7}$$

*and*

$$\widetilde{Q}^k_\alpha = \operatorname{int} \overline{Q}^k_\alpha = \Big(\bigcup_{\gamma \neq \alpha} \overline{Q}^k_\gamma\Big)^c, \tag{2.8}$$

*and*

$$\widetilde{Q}^k_\alpha \subseteq Q^k_\alpha \subseteq \overline{Q}^k_\alpha, \tag{2.9}$$

*where $Q^k_\alpha$ are obtained from the closed sets $\overline{Q}^k_\alpha$ and the open sets $\widetilde{Q}^k_\alpha$ by finitely many set operations.*

**Remark 2.2.** The proof in [HK] shows that the first and the second inclusion in (2.5) hold with $Q^k_\alpha$ replaced by $\widetilde{Q}^k_\alpha$ and $\overline{Q}^k_\alpha$, respectively. We mention that, for any $Q \in \mathscr{D}$, the number $M$ of dyadic sub-cubes as in (2.4) is bounded by $M \leq A_1^2(A_0/\delta)^{\log_2 A_1}$.

2.2. **Further properties of dyadic cubes.** The following additional properties follow directly from the properties listed in (2.1)–(2.6):

$$X \text{ is bounded if and only if there exists } Q \in \mathscr{D} \text{ such that } X = Q; \tag{2.10}$$

$$\begin{aligned} &\text{for every } x \in X \text{ and } k \in \mathbb{Z}, \text{ there exists a unique } Q \in \mathscr{D}_k \text{ such that} \\ &x \in Q =: Q^k(x); \end{aligned} \tag{2.11}$$

$$\begin{aligned} &x \in X \text{ is an isolated point if and only if there exists } k \in \mathbb{Z} \text{ such that } \{x\} = Q^\ell(x) \\ &\text{for all } \ell \geq k. \end{aligned} \tag{2.12}$$

2.3. **Dyadic system with a distinguished center point.** The construction of dyadic cubes requires their center points and an associated partial order be fixed *a priori*. However, if either the center points or the partial order is not given, their existence already follows from the assumptions; any given system of points and partial order can be used as a starting point. Moreover, if we are allowed to choose the center points for the cubes, the collection can be chosen to satisfy the additional property that a fixed point becomes a center point at *all levels*:

$$\begin{aligned} &\text{given a fixed point } x_0 \in X, \text{ for every } k \in \mathbb{Z}, \text{ there exists } \alpha \text{ such that} \\ &x_0 = x_\alpha^k, \text{ the center point of } Q_\alpha^k \in \mathscr{D}_k. \end{aligned} \tag{2.13}$$

This property is crucial in some applications and has useful implications, such as the following.

**Lemma 2.3.** *Given $x, y \in X$, there exists $k \in \mathbb{Z}$ such that $y \in Q^k(x)$. Moreover, if $\rho(x,y) \geq \delta^k$, then $y \notin Q^{k+1}(x)$. In particular, if $\rho(x,y) > 0$, there do not exist arbitrarily large indices $k$ such that $y \in Q^k(x)$.*

*Proof.* Pick $k \in \mathbb{Z}$ such that $x, y \in B(x_0, c_1\delta^k)$. The first assertion follows from (2.5) for the $Q_\alpha^k$ that has $x_0$ as a center point. For the second assertion, suppose $\rho(x,y) \geq \delta^k$. Denote by $x_\alpha^{k+1}$ the center point of $Q^{k+1}(x)$. Then

$$\rho(y, x_\alpha^{k+1}) \geq A_0^{-1}\rho(x,y) - \rho(x, x_\alpha^{k+1}) \geq A_0^{-1}\delta^k - C_1\delta^{k+1} > C_1\delta^{k+1}$$

since $12A_0^3\delta \leq 1$ and $C_1 = 2A_0$, showing that $y \notin Q^{k+1}(x)$. $\square$

**Lemma 2.4.** *Suppose $\sigma$ and $\omega$ are non-trivial positive Borel measures on $X$, and $A \subseteq X$ is a measurable set with $\omega(A) > 0$. Then there exists a dyadic cube $Q \in \mathscr{D}$ such that $\sigma(Q) > 0$ and $\omega(A \cap Q) > 0$. In particular, if $(X, \rho, \mu)$ is a quasi-metric measure space, $E \subseteq X$ is a set with $\mu(E) > 0$, and $x \in X$, then there exists a dyadic cube $Q$ such that $x \in Q$ and $\mu(E \cap Q) > 0$.*

*Proof.* For $k \in \mathbb{Z}$, consider the sets $B_k := B(x_0, c_1\delta^{-k})$ and $A_k := A \cap B_k$. First observe that $\sigma(B_k) > 0$ for $k > k_0$ and $\omega(A_k) > 0$ for $k > k_1$. Indeed, $X = \cup_{k=1}^\infty B_k$ and $B_1 \subseteq B_2 \subseteq \ldots$, so that $0 < \sigma(X) = \lim_{k\to\infty} \sigma(B_k)$. Similarly, $A = \cup_{k=1}^\infty A_k$ and $A_1 \subseteq A_2 \subseteq \ldots$, so that $0 < \omega(A) = \lim_{k\to\infty} \omega(A_k)$. Set $k = \max\{k_0, k_1\}$ and let $Q \in \mathscr{D}$ be the dyadic cube of generation $-k$ centred at $x_0$. Then $B_k \subseteq Q$ by (2.5), and it follows that $\sigma(Q) \geq \sigma(B_k) > 0$ and $\omega(A \cap Q) \geq \omega(A \cap B_k) = \omega(A_k) > 0$. $\square$

We will need the following consequence of (2.13).

**Lemma 2.5.** *For any $x \in X$, $Q^k(x) \to X$ as $k \to -\infty$.*

*Proof.* Given $x \in X$, pick $k \in \mathbb{Z}$ such that $x \in B(x_0, c_1\delta^k)$ where $x_0 \in X$ is as in (2.13). Then $x \in Q^k$ where $Q^k$ is the dyadic cube in $\mathscr{D}_k$ with center point $x_0$. The assertion follows by

$$Q^k \supseteq B(x_0, c_1\delta^k) \to X \text{ as } k \to -\infty.$$

$\square$

**Remark 2.6.** Note that the usual Euclidean dyadic cubes of the form $2^{-k}([0,1)^n+m), k\in\mathbb{Z}, m\in\mathbb{Z}^n$, in $\mathbb{R}^n$ do not have the property (2.13), and that Lemmata 2.3, 2.4 and 2.5 depend on this property. A simple example of a system with a distinguished center point, using triadic rather than dyadic intervals, is as follows. In the real line divide each interval $[n,n+1)$, $n\in\mathbb{Z}$, into three triadic intervals of equal length, divide each of these intervals in three, and so on. Now require that the parent interval of $[0,1)$ is $[-1,2)$, so that $[0,1)$ is the middle third of its parent interval. Similarly require that the parent interval of $[-1,2)$ is $[-4,5)$, and so on. Then the point $x_0=1/2$ is a distinguished center point for this system.

2.4. **Adjacent systems of dyadic cubes.** In a geometrically doubling quasi-metric space $(X,\rho)$, a finite collection $\{\mathscr{D}^t\colon t=1,2,\dots,T\}$ of families $\mathscr{D}^t$ is called a *collection of adjacent systems of dyadic cubes with parameters* $\delta\in(0,1), 0<c_1\le C_1<\infty$ and $1\le C<\infty$ if it has the following properties: individually, each $\mathscr{D}^t$ is a system of dyadic cubes with parameters $\delta\in(0,1)$ and $0<c_1\le C_1<\infty$; collectively, for each ball $B(x,r)\subseteq X$ with $\delta^{k+3}<r\le\delta^{k+2}, k\in\mathbb{Z}$, there exist $t\in\{1,2,\dots,T\}$ and $Q\in\mathscr{D}^t$ of generation $k$ and with center point ${}^tx^k_\alpha$ such that $\rho(x,{}^tx^k_\alpha)<2A_0\delta^k$ and

$$B(x,r)\subseteq Q\subseteq B(x,Cr). \tag{2.14}$$

We recall from [HK] the following construction.

**Theorem 2.7.** *Let $(X,\rho)$ be a geometrically doubling quasi-metric space. Then there exists a collection $\{\mathscr{D}^t\colon t=1,2,\dots,T\}$ of adjacent systems of dyadic cubes with parameters $\delta\in(0,(96A_0^6)^{-1}), c_1=(12A_0^4)^{-1}, C_1=4A_0^2$ and $C=8A_0^3\delta^{-3}$. The center points ${}^tx^k_\alpha$ of the cubes $Q\in\mathscr{D}^t_k$ have, for each $t\in\{1,2,\dots,T\}$, the two properties*

$$\rho({}^tx^k_\alpha,{}^tx^k_\beta)\ge(4A_0^2)^{-1}\delta^k\quad(\alpha\ne\beta),\qquad \min_\alpha\rho(x,{}^tx^k_\alpha)<2A_0\delta^k\quad\textit{for all } x\in X.$$

*Moreover, these adjacent systems can be constructed in such a way that each $\mathscr{D}^t$ satisfies the distinguished center point property* (2.13).

**Remark 2.8.** For $T$ (the number of the adjacent systems of dyadic cubes), we have the estimate

$$T=T(A_0,A_1,\delta)\le A_1^6(A_0^4/\delta)^{\log_2 A_1}. \tag{2.15}$$

Note that in the Euclidean space $\mathbb{R}^n$ with the usual structure we have $A_0=1, A_1\ge 2^n$ and $\delta=\frac{1}{2}$, so that (2.15) yields an upper bound of order $2^{7n}$. We mention that T. Mei [Mei] has shown that in $\mathbb{R}^n$ the conclusion (2.14) can be obtained with just $n+1$ cleverly chosen systems $\mathscr{D}^t$.

Further, we have the following result on the smallness of the boundary.

**Proposition 2.9.** *Suppose that $144A_0^8\delta\le 1$. Let $\mu$ be a positive $\sigma$-finite measure on $X$. Then the collection $\{\mathscr{D}^t\colon t=1,2,\dots,T\}$ may be chosen to have the additional property that*

$$\mu(\partial Q)=0\quad\text{for all}\quad Q\in\bigcup_{t=1}^T\mathscr{D}^t.$$

## 3. Dyadic structure theorems for maximal functions and for weights

In this section we establish that in the setting of product spaces of homogeneous type, the strong maximal function is pointwise comparable to a sum of finitely many dyadic maximal functions, and that the strong $A_p$ class is the intersection of finitely many dyadic $A_p$ classes, and similarly for reverse-Hölder weights $RH_p$ and for doubling measures.

3.1. **Maximal functions.** Let $(X,\rho,\mu)$ be a space of homogenenous type, and let $\{\mathscr{D}^t : t = 1,\ldots,T\}$ be a collection of adjacent systems of dyadic cubes in $X$ as in Theorem 2.1.

Similarly, for each $j \in \{1,\ldots,k\}$ let $(X_j,\rho_j,\mu_j)$ be a space of homogeneous type, with an associated collection of adjacent systems of dyadic cubes $\{\mathscr{D}^{t_j} : t_j = 1,\ldots,T_j\}$. Let $\widetilde{X} := X_1 \times \cdots \times X_k$ with the product quasi-metric $\rho_1 \times \cdots \times \rho_k$ and the product measure $\mu_1 \times \cdots \times \mu_k$.

**Theorem 3.1.** *The maximal function $M$ (or strong maximal function $M_s$ in the product case) controls each dyadic maximal function pointwise, and is itself controlled pointwise by a sum of dyadic maximal functions, as follows.*

(i) *Let $(X,\rho,\mu)$ be a space of homogeneous type. Then there is a constant $C > 0$ such that for each $f \in L^1_{\mathrm{loc}}(X)$, and for all $x \in X$, we have the pointwise estimates*
$$M^t_d f(x) \le C Mf(x) \qquad \text{for each } t \in \{1,\ldots,T\}, \text{ and}$$
$$Mf(x) \le C \sum_{t=1}^{T} M^t_d f(x).$$

(ii) *Let $(X_j,\rho_j,\mu_j)$, $j \in \{1,\ldots,k\}$, be spaces of homogeneous type. Then there is a constant $C > 0$ such that for each $f \in L^1_{\mathrm{loc}}(\widetilde{X})$,*
$$M_d^{t_1,\ldots,t_k} f(x) \le C M_s f(x) \qquad \text{for each } t_j \in \{1,\ldots,T_j\},\ j \in \{1,\ldots,k\}, \text{ and}$$
$$M_s f(x) \le C \sum_{t_1=1}^{T_1} \cdots \sum_{t_k=1}^{T_k} M_d^{t_1,\ldots,t_k} f(x).$$

These maximal operators are defined as follows. For $f \in L^1_{\mathrm{loc}}(X,\mu)$ let $Mf$ denote the Hardy–Littlewood maximal function, given by
$$Mf(x) := \sup_{B \ni x} \frac{1}{\mu(B)} \int_B |f(y)|\, d\mu(y),$$
where the supremum is taken over all balls $B \subset X$ that contain $x$. For each $t \in \{1,\ldots,T\}$, denote by $M^t_d f$ the dyadic maximal function with respect to the system $\mathscr{D}^t$ of dyadic cubes in $X$; here the supremum is taken over only those dyadic cubes $Q \in \mathscr{D}^t$ that contain $x$.

In the multiparameter case, instead of the Hardy–Littlewood maximal function we consider the strong maximal function $M_s f$, defined as follows. Take $x = (x_1,\ldots,x_k) \in \widetilde{X}$ and $f \in L^1_{\mathrm{loc}}(\widetilde{X}, \mu_1 \times \cdots \times \mu_k)$. Let
$$M_s f(x) := \sup_{B_1 \times \cdots \times B_k \ni x} \frac{1}{\prod_{j=1}^k \mu_j(B_j)} \int_{B_1 \times \cdots \times B_k} |f(y)|\, d\mu_1(y_1) \times \cdots \times d\mu_k(y_k), \tag{3.1}$$
where $y = (y_1,\ldots,y_k)$ and the supremum is taken over all products $B_1 \times \cdots \times B_k$ of balls $B_j \subset X_j$, $B_j \ni x_j$, for $j \in \{1,\ldots,k\}$.

Next, for each choice of $t_j \in \{1,\ldots,T_j\}$, for $j \in \{1,\ldots,k\}$, let $M_d^{t_1,\ldots,t_k} f$ denote the associated dyadic strong maximal function, defined by
$$M_d^{t_1,\ldots,t_k} f(x) := \sup_{Q_1 \times \cdots \times Q_k \ni x} \frac{1}{\prod_{j=1}^k \mu_j(Q_j)} \int_{Q_1 \times \cdots \times Q_k} |f(y)|\, d\mu_1(y_1) \times \cdots \times d\mu_k(y_k),$$
restricting the supremum in formula (3.1) to dyadic rectangles $Q_1 \times \cdots \times Q_k \in \mathscr{D}^{t_1} \times \cdots \times \mathscr{D}^{t_k}$ that contain $x$.

*Proof of Theorem 3.1.* (i) Fix $x \in X$. Fix $t \in \{1,\ldots,T\}$ and suppose $Q \ni x$, $Q \in \mathscr{D}^t$. Then $Q = Q^{t,k}_\alpha$ for some $k$ and $\alpha$, and by condition (2.5), there is a ball $B = B(x^k_\alpha, C_1\delta^k)$ that contains $Q$. Since $\mu$ is doubling, we see that
$$\frac{1}{\mu(Q)} \int_Q |f(y)|\, d\mu(y) \le \frac{C^*}{\mu(B)} \int_B |f(y)|\, d\mu(y),$$

where $C^* = C_{dbl}^{\log_2(C_1/c_1)+1}$ and $C_1, c_1$ are the constants in (2.5), see Proposition 3.4(i). It follows that $M_d^t f(x) \lesssim Mf(x)$.

For the second inequality, fix $x_0 \in X$ and a ball $B(x,r) \ni x_0$. By (2.14) there exist $t \in \{1,\dots,T\}$ and $Q \in \mathscr{D}^t$ such that $B(x,r) \subset Q \subset B(x,Cr)$. Since $\mu$ is doubling, $\mu(B(x,Cr)) \lesssim \mu(B(x,r)) \lesssim \mu(Q)$. It follows that $Mf(x) \lesssim \sum_{t=1}^T M_d^t f(x)$.

(ii) Fix $x \in X$, and fix $(t_1,\dots,t_k) \in \{1,\dots,T_1\} \times \cdots \times \{1,\dots,T_k\}$. Then suppose $Q_1 \times \cdots \times Q_k \ni x$, $Q_1 \times \cdots \times Q_k \in \mathscr{D}^{t_1} \times \cdots \times \mathscr{D}^{t_k}$. Iteration of the argument in (i), using in each factor the condition (2.5) and the assumption that each $\mu_j$ is doubling, establishes the first inequality.

Similarly, iteration of the argument for the second inequality in (i), using in each factor the condition (2.14) and the assumption that $\mu_j$ is doubling, establishes that $M_s f(x) \lesssim \sum_{t_1=1}^{T_1} \cdots \sum_{t_k=1}^{T_k} M_d^{t_1,\dots,t_k} f(x)$. □

3.2. **Doubling, $A_p$, and $RH_p$ weights.** Like in the previous section, for each $j \in \{1,\dots,k\}$ let $(X_j,\rho_j,\mu_j)$ be a space of homogeneous type, with an associated collection of adjacent systems of dyadic cubes $\{\mathscr{D}^{t_j} : t_j = 1,\dots,T_j\}$. Let $\widetilde{X} := X_1 \times \cdots \times X_k$ with the product quasi-metric $\rho_1 \times \cdots \times \rho_k$ and the product measure $\mu_1 \times \cdots \times \mu_k$.

By a *weight* on $(X,\rho,\mu)$, a space of homogeneous type, we mean a nonnegative locally integrable function $\omega : X \to [0,\infty]$. We begin with the main result of this subsection; definitions of doubling weights, $A_p$ weights and $RH_p$ weights are discussed below.

**Theorem 3.2.** *Fix $k \in \mathbb{N}$. For each $j \in \{1,2,\dots,k\}$, let $(X_j,\rho_j,\mu_j)$ be a space of homogeneous type, and as in Theorem 2.7, let $\{\mathscr{D}^{t_j} : t_j = 1,\dots,T_j\}$ be a collection of adjacent systems of dyadic cubes for $X_j$. Then the following assertions hold.*

(a) *A weight $\omega(x_1,\dots,x_k)$ is a product doubling weight if and only if $\omega$ is dyadic doubling with respect to each of the $T_1T_2\cdots T_k$ product dyadic systems $\mathscr{D}_1^{t_1} \times \cdots \times \mathscr{D}_k^{t_k}$, where $t_j$ runs over $\{1,\dots,T_j\}$ for each $j \in \{1,2,\dots,k\}$, with comparable constants.*

(b) *For each $p$ with $1 \le p \le \infty$, $k$-parameter $A_p(X_1 \times \cdots \times X_k)$ is the intersection of $T_1T_2\cdots T_k$ $k$-parameter dyadic $A_p$ spaces, as follows:*

$$A_p(X_1 \times \cdots \times X_k) = \bigcap_{t_1=1}^{T_1} \bigcap_{t_2=1}^{T_2} \cdots \bigcap_{t_k=1}^{T_k} A_{p,d}^{t_1,t_2,\dots,t_k}(X_1 \times \cdots \times X_k),$$

*with comparable constants.*

(c) *For each $p$ with $1 \le p \le \infty$, $k$-parameter $RH_p(X_1 \times \cdots \times X_k)$ is the intersection of $T_1T_2\cdots T_k$ $k$-parameter dyadic $RH_p$ spaces, as follows:*

$$RH_p(X_1 \times \cdots \times X_k) = \bigcap_{t_1=1}^{T_1} \bigcap_{t_2=1}^{T_2} \cdots \bigcap_{t_k=1}^{T_k} RH_{p,d}^{t_1,t_2,\dots,t_k}(X_1 \times \cdots \times X_k),$$

*with comparable constants.*

*The constant $A_p(\omega)$ depends only on the constants $A_{p,d}^{t_1,t_2,\dots,t_k}(\omega)$ for $1 \le t_j \le T_j$, $1 \le j \le k$, and vice versa, and similarly for the other classes.*

We note that one difference from the Euclidean setting is that on $X$ or $\widetilde{X}$, it is not immediate that a continuous function space is a subset of its dyadic counterpart, since in general the dyadic cubes are not balls. We address this question in the proof of Theorem 3.2, below.

We begin with some definitions and observations. For brevity, we include only the one-parameter versions on $X$. The product definitions on $\widetilde{X}$ follow the pattern described, for the Euclidean case, in [LPW, Section 7.2]. In particular the product $A_p$ weights are those which are uniformly $A_p$ in each variable separately, and so on.

**Definition 3.3.** (i) A weight $\omega$ on a homogeneous space $(X, \rho, \mu)$ is *doubling* if there is a constant $C_{\mathrm{dbl}}$ such that for all $x \in X$ and all $r > 0$,

$$0 < \omega(B(x, 2r)) \leq C_{\mathrm{dbl}}\, \omega(B(x, r)) < \infty.$$

As usual, $\omega(E) := \int_E \omega\, d\mu$ for $E \subset X$.

(ii) A weight $\omega$ on a homogeneous space $(X, \rho, \mu)$ equipped with a system $\mathscr{D}$ of dyadic cubes is *dyadic doubling* if there is a constant $C_{\mathrm{dydbl}}$ such that for every dyadic cube $Q \in \mathscr{D}$ and for each child $Q'$ of $Q$,

$$0 < \omega(Q) \leq C_{\mathrm{dydbl}}\, \omega(Q') < \infty.$$

**Proposition 3.4.** (i) *Let $(X, \rho, \mu)$ be a space of homogeneous type. Then for all $x \in X$, $r > 0$, and $\lambda > 0$, we have*

$$\mu(B(x, \lambda r)) \leq (C_{\mathrm{dbl}})^{1+\log_2 \lambda}\, \mu(B(x, r)). \tag{3.1}$$

(ii) *The condition in Definition 3.3* (ii) *of dyadic doubling weights, above, is equivalent to the following condition: there is a constant $C$ such that for each cube $Q \in \mathscr{D}$, for all children (subcubes) $Q'$ and $Q''$ of $Q$,*

$$\frac{1}{C}\omega(Q'') \leq \omega(Q') \leq C\omega(Q''). \tag{3.2}$$

*Proof.* The proofs are elementary; (ii) is an immediate consequence of the fact that each cube $Q$ in $\mathscr{D}$ is the disjoint union of its children (condition (2.4)). □

We note that in the Euclidean setting, the theory of product weights was developed by K.-C. Lin in his thesis [Lin], and the dyadic theory was developed in Buckley's paper [Buc]. The definitions of $A_p$ and $RH_p$, and their dyadic versions, on $X$ and $\widetilde{X}$, are obtained by the natural modifications of the Euclidean definitions, which are summarised in, for example, [PWX] for $A_p$, $1 \leq p \leq \infty$, and $RH_p$, $1 < p \leq \infty$, and in [LPW] for $RH_1$. For brevity, we omit the definitions for $A_p$ and $A^t_{p,d}$, and include only those for the reverse-Hölder classes $RH_p$ and $RH^t_{p,d}$.

**Definition 3.5.** Let $\omega(x)$ be a nonnegative locally integrable function on $(X, \rho, \mu)$. For $p$ with $1 < p < \infty$, we say $\omega$ is a *reverse-Hölder-$p$ weight*, written $\omega \in RH_p$, if

$$RH_p(\omega) := \sup_B \left( ⨍_B \omega^p \right)^{1/p} \left( ⨍_B \omega \right)^{-1} < \infty.$$

For $p = 1$, we say $\omega$ is a *reverse-Hölder-1 weight*, written $\omega \in RH_1$ or $\omega \in B_1$, if

$$RH_1(\omega) := \sup_B ⨍_B \left( \frac{\omega}{⨍_B \omega} \log \frac{\omega}{⨍_B \omega} \right) < \infty.$$

For $p = \infty$, we say $\omega$ is a *reverse-Hölder-infinity weight*, written $\omega \in RH_\infty$ or $\omega \in B_\infty$, if

$$RH_\infty(\omega) := \sup_B \left( \operatorname*{ess\,sup}_{x \in B} \omega \right) \left( ⨍_B \omega \right)^{-1} < \infty.$$

Here the suprema are taken over all quasi-metric balls $B \subset X$, and $⨍_B$ denotes $\frac{1}{\mu(B)} \int_B d\mu$. The quantity $RH_p(\omega)$ is called the *$RH_p$ constant* of $\omega$.

For $p$ with $1 \leq p \leq \infty$, and $t \in \{1, 2, \ldots, T\}$, we say $\omega$ is a *dyadic reverse-Hölder-$p$ weight related to the dyadic system $\mathcal{D}^t$*, written $\omega \in RH^t_{p,d}$, if

(i) the analogous condition $RH^t_{p,d}(\omega) < \infty$ holds with the supremum being taken over only the dyadic cubes $Q \subset \mathcal{D}^t$, and

(ii) in addition $\omega$ is a dyadic doubling weight.

We define the *$RH^t_{p,d}$ constant* $RH^t_{p,d}(\omega)$ of $\omega$ to be the larger of this dyadic supremum and the dyadic doubling constant.

Note that the $A_p$ inequality (or the $RH_p$ inequality) implies that the weight $\omega$ is doubling, and the dyadic $A_p$ inequality implies that $\omega$ is dyadic doubling. However, the dyadic $RH_p$ inequality does not imply that $\omega$ is dyadic doubling, which is why the dyadic doubling assumption is needed in the definition of $RH^t_{p,d}$.

It is shown in [AHT] that the self-improving property of reverse-Hölder weights does not hold on some spaces of homogeneous type, but does hold on doubling metric measure spaces with some additional geometric properties such as the $\alpha$-annular decay property; see also [Maa]. Our work here does not involve the self-improving property.

*Proof of Theorem 3.2.* The multiparameter case follows from the one-parameter case by a straightforward iteration argument, as in the Euclidean setting (see [LPW, Theorem 7.3]).

The one-parameter proof that each continuous function class contains the intersection of its dyadic counterparts follows that of Theorem 7.1 in [LPW], replacing Lebesgue measure by $\mu$, and replacing the two dyadic grids related through translation by a collection of adjacent systems of dyadic grids. We give the details for the cases of doubling weights and $RH_p$ weights (parts (a) and (c)), and omit them for $A_p$ weights (part (b)).

(a) We show that doubling weights $\omega$ on $X$ are dyadic doubling with respect to each of the systems $\mathscr{D}^t$ of dyadic cubes, $t = 1, \ldots, T$, given by Theorem 2.7; these systems have parameters $\delta$, $c_1 = 1/(12A_0^4)$, $C_1 = 4A_0^2$ and $C = 8A_0^3/\delta^3$. Let $\omega$ be a doubling weight on $X$. Fix $t \in \{1, \ldots, T\}$. Fix a dyadic cube $Q^k_\alpha \in \mathscr{D}^t$ and a child $Q^{k+1}_\beta$ of $Q^k_\alpha$. Then $B(Q^k_\alpha) := B(x^k_\alpha, C_1\delta^k) \subset B(x^{k+1}_\beta, 2A_0C_1\delta^k)$, since for $x \in B(Q^k_\alpha)$,

$$\rho(x, x^{k+1}_\beta) \leq A_0[\rho(x, x^k_\alpha) + \rho(x^k_\alpha, x^{k+1}_\beta))] \leq 2A_0C_1\delta^k.$$

By Proposition 3.4 (i), it follows that

$$\begin{aligned}\omega(Q^k_\alpha) &\leq \omega(B(x^k_\alpha, C_1\delta^k)) \leq \omega(B(x^{k+1}_\beta, 2A_0C_1\delta^k))\\ &= \omega(B(x^{k+1}_\beta, 2A_0C_1/(c_1\delta)\, c_1\delta^{k+1}))\\ &\leq (C_{\mathrm{dbl}})^{1+\log_2[2A_0C_1/(c_1\delta)]}\, \omega(B(x^{k+1}_\beta, c_1\delta^{k+1}))\\ &\leq (C_{\mathrm{dbl}})^{1+\log_2[2A_0C_1/(c_1\delta)]}\, \omega(Q^{k+1}_\beta).\end{aligned}$$

Therefore $\omega$ is dyadic doubling, with constant $(C_{\mathrm{dbl}})^{1+\log_2[2A_0C_1/(c_1\delta)]}$, with respect to each system $\mathscr{D}^t$, for $t = 1, \ldots, T$.

For the other inclusion, let $\omega$ be a weight that is dyadic doubling with constant $C_{\mathrm{dydbl}}$ with respect to each $\mathscr{D}^t$, for $t = 1, \ldots, T$. Fix $x \in X$ and $r > 0$. Pick $k \in \mathbb{Z}$ such that $\delta^{k+3} < 2r \leq \delta^{k+2}$. By property (2.14) applied to $B(x, 2r)$, there is some $t \in \{1, \ldots, T\}$ and some $Q = Q^k_\alpha \in \mathscr{D}^t$ of generation $k$ such that

$$B(x, 2r) \subset Q = Q^k_\alpha \subset B(x, 2Cr). \tag{3.3}$$

We claim there is an integer $N$ independent of $Q$ and $t$ such that there is a descendant $Q' = Q^\ell_\beta$ of $Q$ at most $N$ generations below $Q$, with $Q' \subset Q$, $Q' \in \mathscr{D}^t$, and $Q' \subset B(x, r)$. If so, then

$$\omega(B(x, r)) \leq \omega(Q) \leq C^N_{\mathrm{dydbl}}\, \omega(Q') \leq C^N_{\mathrm{dydbl}}\, \omega(B(x, r)),$$

and so $\omega$ is a doubling weight.

It remains to establish the claim. First, choose $\ell \in \mathbb{Z}$ such that

$$\delta^\ell \leq (6A_0^3)^{-1} r < \delta^{\ell-1}.$$

Second, by the choice of the center points for the adjacent systems of dyadic cubes as in Theorem 2.7, there is a $\beta$ such that $\rho(x, x^\ell_\beta) < 2A_0\delta^\ell$. It follows that $Q' := Q^\ell_\beta \subset B(x, r)$. For given $y \in Q^\ell_\beta$, we have

$$\begin{aligned}\rho(y, x) &\leq A_0[\rho(y, x^\ell_\beta) + \rho(x^\ell_\beta, x)]\\ &< A_0[C_1\delta^\ell + 2A_0\delta^\ell]\end{aligned}$$

$$\leq 6A_0^3\delta^\ell$$
$$\leq r,$$

so $y \in B(x,r)$. Hence $Q_\beta^\ell \subset B(x,r)$.

Now we have

$$\left(\frac{1}{\delta}\right)^{\ell-k} = \frac{\delta^k}{\delta^\ell} \leq \frac{2r/\delta^3}{\delta r/(8A_0^3)} = \frac{16A_0^3}{\delta^4}.$$

It follows that

$$\ell - k \leq \frac{\log[16A_0^3/\delta^4]}{\log[1/\delta]} < \infty.$$

Thus it suffices to choose $N := \lceil \log[16A_0^3/\delta^4])/(\log[1/\delta])\rceil$.

We note that with the parameter choices (as in Theorem 2.7) of $c_1 = 1/(12A_0^4)$, $C_1 = 4A_0^2$, $C = 8A_0^3/\delta^3$, and choosing $\delta = 1/(96A_0^6)$ at the upper end of the range in Theorem 2.7, we find that $16A_0^3/\delta^4 = 16 \cdot 96^4 A_0^{27} \gg 1$ while $1/\delta = 96A_0^2$. Thus with these parameter choices we have $1 < N < \infty$.

We also note that the parameter choices in Theorem 2.7 are consistent with those in Theorem 2.1. This completes the proof of part (a).

(b) The proof for $A_p$ weights is similar to part (c) below, and we omit the details.

(c) Now we turn to the reverse-Hölder weights. It suffices to prove the one-parameter case. Suppose $1 < p < \infty$. We show first that $RH_p \subset RH_{p,d}^t$ for each $t \in \{1, 2, \dots, T\}$. Fix such a $t$ and fix $\omega \in RH_p$. Consider the quantity

$$V := \left(\fint_Q \omega^p\right)^{1/p} \left(\fint_Q \omega\right)^{-1}, \tag{3.4}$$

where $Q$ is any fixed dyadic cube in $\mathcal{D}^t$. For that dyadic cube $Q$, we denote by $B_{c_1}$ and $B_{C_1}$ the two balls from property (2.5).

Then, since $Q \subseteq B_{C_1}$, we have that

$$V = \left(\frac{1}{\mu(Q)}\int_Q \omega^p\, d\mu\right)^{1/p} \frac{\mu(Q)}{\omega(Q)} \leq \left(\frac{1}{\mu(Q)}\int_{B_{C_1}} \omega^p\, d\mu\right)^{1/p} \frac{\mu(B_{C_1})}{\omega(Q)}.$$

Next, since $B_{c_1} \subseteq Q$, we have $\mu(Q) \geq \mu(B_{c_1}) \geq \widetilde{C}\mu(B_{C_1})$, where the last inequality follows from the doubling property of $\mu$, and the constant $\widetilde{C} = \big(C_\mu^{1+\log_2\frac{C_1}{c_1}}\big)^{-1}$. Similarly, we have $\omega(Q) \geq \omega(B_{c_1}) \geq \widetilde{\widetilde{C}}\,\omega(B_{C_1})$, where the last inequality follows from the doubling property of $\omega$, and the constant $\widetilde{\widetilde{C}} = \big(C_{\rm dbl}^{1+\log_2\frac{C_1}{c_1}}\big)^{-1}$. Hence

$$V \leq \frac{1}{\widetilde{C}^{\frac1p}\widetilde{\widetilde{C}}}\left(\frac{1}{\mu(B_{C_1})}\int_{B_{C_1}} \omega^p\, d\mu\right)^{1/p} \frac{\mu(B_{C_1})}{\omega(B_{C_1})} \leq \frac{1}{\widetilde{C}^{\frac1p}\widetilde{\widetilde{C}}}\sup_B \left(\fint_B \omega^p\right)^{1/p}\left(\fint_B \omega\right)^{-1} = \frac{RH_p(\omega)}{\widetilde{C}^{\frac1p}\widetilde{\widetilde{C}}},$$

which implies that $\omega \in RH_{p,d}^t$.

Next we prove that $\bigcap_{t=1}^T RH_{p,d}^t \subset RH_p$. Suppose $\omega \in \bigcap_{t=1}^T RH_{p,d}^t$. Then for each fixed quasi-metric ball $B(x,r) \subset X$, by property (2.14), there exist an integer $k$ with $\delta^{k+3} < r \leq \delta^{k+2}$, a number $t \in \{1, \dots, T\}$, and a cube $Q \in \mathscr{D}^t$ of generation $k$ and with center point $x_\alpha^k$ such that $\rho(x, x_\alpha^k) < 2A_0\delta^k$ and $B(x,r) \subseteq Q \subseteq B(x, Cr)$. Here each $\mathscr{D}^t$ is a system of dyadic cubes with parameters $\delta \in (0,1)$ and $0 < c_1 \leq C_1 < \infty$. Hence, writing $B := B(x,r)$, we have

$$\left(\fint_B \omega^p\right)^{1/p}\left(\fint_B \omega\right)^{-1} = \left(\frac{1}{\mu(B)}\int_B \omega^p\, d\mu\right)^{1/p}\frac{\mu(B)}{\omega(B)} \leq \left(\frac{1}{\mu(B)}\int_Q \omega^p\, d\mu\right)^{1/p}\frac{\mu(Q)}{\omega(B)}.$$

Also, from the doubling property of $\mu$, we have $\mu(B) \geq \overline{C}\mu(B(x,Cr)) \geq \overline{C}\mu(Q)$, where $\overline{C} = \big(C_\mu^{1+\log_2 C}\big)^{-1}$. Similarly, $\omega(B) \geq \overline{\overline{C}}\omega(B(x,Cr)) \geq \overline{\overline{C}}\omega(Q)$, where $\overline{\overline{C}} = \big(C_{\rm dbl}^{1+\log_2 C}\big)^{-1}$. As a consequence, we get

$$\left(\fint_B \omega^p\right)^{1/p}\left(\fint_B \omega\right)^{-1} \leq \frac{1}{\overline{C}^{\frac{1}{p}}\overline{\overline{C}}}\left(\frac{1}{\mu(Q)}\int_Q \omega^p\, d\mu\right)^{1/p}\frac{\mu(Q)}{\omega(Q)} \leq \frac{1}{\overline{C}^{\frac{1}{p}}\overline{\overline{C}}} RH^t_{p,d}(\omega),$$

which implies that $\omega \in RH_p$.

Similar arguments apply to the cases $p = 1$ and $p = \infty$. □

## 4. Explicit construction of Haar functions, and completeness

This section is devoted to our construction of a Haar basis $\{h^Q_u \colon Q \in \mathscr{D}, u = 1, \ldots, M_Q - 1\}$ for $L^p(X,\mu)$, $1 < p < \infty$, associated to the dyadic cubes $Q \in \mathscr{D}$, with the properties listed in Theorems 4.1 and 4.2 below. Here $M_Q := \#\operatorname{Ch}(Q) = \#\{R \in \mathscr{D}_{k+1} \colon R \subseteq Q\}$ denotes the number of dyadic sub-cubes ("children") the cube $Q \in \mathscr{D}_k$ has.

**Theorem 4.1.** *Let $(X,\rho)$ be a geometrically doubling quasi-metric space and suppose $\mu$ is a positive Borel measure on $X$ with the property that $\mu(B) < \infty$ for all balls $B \subseteq X$. For $1 < p < \infty$, for each $f \in L^p(X,\mu)$, we have*

$$f(x) = m_X(f) + \sum_{Q \in \mathscr{D}} \sum_{u=1}^{M_Q - 1} \langle f, h^Q_u\rangle h^Q_u(x),$$

*where the sum converges (unconditionally) both in the $L^p(X,\mu)$-norm and pointwise $\mu$-almost everywhere, and*

$$m_X(f) := \begin{cases} \frac{1}{\mu(X)}\int_X f d\mu, & \text{if } \mu(X) < \infty,\\ 0, & \text{if } \mu(X) = \infty.\end{cases}$$

The following theorem collects several basic properties of the functions $h^Q_u$.

**Theorem 4.2.** *The Haar functions $h^Q_u$, $Q \in \mathscr{D}$, $u = 1, \ldots, M_Q - 1$, have the following properties:*

(i) *$h^Q_u$ is a simple Borel-measurable real function on $X$;*
(ii) *$h^Q_u$ is supported on $Q$;*
(iii) *$h^Q_u$ is constant on each $R \in \operatorname{Ch}(Q)$;*
(iv) *$\int h^Q_u\, d\mu = 0$ (cancellation);*
(v) *$\langle h^Q_u, h^Q_{u'}\rangle = 0$ for $u \neq u'$, $u$, $u' \in \{1, \ldots, M_Q - 1\}$;*
(vi) *the collection*

$$\big\{\mu(Q)^{-1/2}1_Q\big\} \cup \{h^Q_u : u = 1, \ldots, M_Q - 1\}$$

*is an orthogonal basis for the vector space $V(Q)$ of all functions on $Q$ that are constant on each sub-cube $R \in \operatorname{Ch}(Q)$;*

(vii) *if $h^Q_u \not\equiv 0$ then*

$$\|h^Q_u\|_{L^p(X,\mu)} \simeq \mu(Q_u)^{\frac{1}{p}-\frac{1}{2}} \quad \text{for } 1 \leq p \leq \infty;$$

*and*

(viii) $$\|h^Q_u\|_{L^1(X,\mu)} \cdot \|h^Q_u\|_{L^\infty(X,\mu)} \simeq 1.$$

In the remainder of this section, we develop the Haar functions and their properties and establish Theorems 4.1 and 4.2. The section is organized as follows: Section 4.1 describes our assumptions on the underlying space $(X,\rho)$ and measure $\mu$, Section 4.2 presents the dyadic $\sigma$-algebras and conditional expectations we use, Section 4.3 provides our indexing of the sub-cubes of a given dyadic cube, and Section 4.4 develops the martingale difference decomposition that leads to our explicit definition of the Haar functions. In Section 4.4

we also deal with the situation when a dyadic cube has only one child, and sum up our Haar function definitions and results in Theorems 4.8 and 4.9, which complete the proof of Theorems 4.1 and 4.2.

4.1. **Set-up.** The set-up for Section 4 is a geometrically doubling quasi-metric space $(X, \rho)$ equipped with a positive Borel measure $\mu$. We assume that the $\sigma$-algebra of measurable sets $\mathscr{F}$ contains all balls $B \subseteq X$ with $\mu(B) < \infty$. This implies that $\mu$ is $\sigma$-finite, in other words, the sub-collection

$$\mathscr{F}^0 := \{F \in \mathscr{F} : \mu(F) < \infty\}$$

contains a countable cover: there is a collection of at most countably many sets $F_1, F_2, \ldots$ in $\mathscr{F}^0$ such that

$$X = \bigcup_{i=1}^{\infty} F_i.$$

The space $(X, \rho, \mu)$ is called a *geometrically doubling quasi-metric measure space.*

We emphasize that in this section (Section 4) we do not assume that the measure $\mu$ is doubling. That assumption is only needed in Sections 5 and 6 below.

4.2. **Dyadic $\sigma$-algebras and conditional expectations.** Let $\mathscr{D} = \cup_{k\in\mathbb{Z}} \mathscr{D}_k$ be a fixed system of dyadic cubes with the additional distinguished center point property (2.13). Let $\mathscr{F}_k := \sigma(\mathscr{D}_k)$ be the (dyadic) $\sigma$-algebra generated by the countable partition $\mathscr{D}_k$. It is an easy exercise to check that

$$\mathscr{F}_k = \left\{ \bigcup_{\alpha\in I} Q^k_\alpha : I \subseteq \mathscr{A}_k \right\},$$

the collection of all unions.

**Lemma 4.3** (Properties of the filtration $(\mathscr{F}_k)$)**.** *The family $(\mathscr{F}_k)$ of $\sigma$-algebras is a filtration, that is, $\mathscr{F}_i \subseteq \mathscr{F}_j \subseteq \mathscr{F}$ for all $i < j$. Each $(X, \mathscr{F}_k, \mu)$ is $\sigma$-finite and*

$$\sigma\Big( \bigcup_{k\in\mathbb{Z}} \mathscr{F}_k \Big) = \mathscr{F}.$$

*Proof.* The first assertion is clear since every dyadic cube is a finite union of smaller dyadic cubes. The collection $\mathscr{D}_k \subseteq \mathscr{F}_k$ forms a countable cover, and the second assertion follows immediately from the assumption $\mu(B) < \infty$ imposed on balls. Recall that dyadic cubes are Borel sets. Thus, $\cup_{k\in\mathbb{Z}} \mathscr{F}_k \subseteq \mathscr{F}$ and consequently, $\sigma(\cup_{k\in\mathbb{Z}} \mathscr{F}_k) \subseteq \mathscr{F}$. Then suppose that $\Omega \in \mathscr{F}$. We may assume that $\Omega$ is an open set. For each $x \in \Omega$, there is $Q = Q_x \in \mathscr{D}$ such that $x \in Q \subseteq \Omega$. Consequently, $\Omega$ is a (countable) union of dyadic cubes, and hence, $\Omega \in \sigma(\cup_{k\in\mathbb{Z}} \mathscr{F}_k)$. □

We now recall Doob's martingale convergence theorem, the proof of which can be found in [Wil] (see also the lecture notes [Hyt]).

**Theorem 4.4** (Doob's martingale convergence theorem)**.** *Let $(X, \rho, \mu)$ be a geometrically doubling quasi-metric measure space. Suppose that $(\mathscr{F}_k)_{k\in\mathbb{Z}}$ is any filtration such that the spaces $(X, \mathscr{F}_k, \mu)$ are $\sigma$-finite and*

$$\sigma\left( \bigcup_{k\in\mathbb{Z}} \mathscr{F}_k \right) = \mathscr{F}.$$

*Then for every $f \in L^p(\mathscr{F}, \mu), 1 < p < \infty$, there holds*

$$\mathbb{E}[f|\mathscr{F}_k] \to f \text{ as } k \to \infty.$$

*The convergence takes place both in the $L^p(X, \mu)$-norm and pointwise $\mu$-a.e.*

We consider the filtration $(\mathscr{F}_k)_{k\in\mathbb{Z}}$ generated by the dyadic cubes, the associated conditional expectation operators, and the corresponding martingale differences. For $k \in \mathbb{Z}$ and $Q \in \mathscr{D}_k$, the conditional expectation and its local version are defined for $f \in L^1_{\mathrm{loc}}(X, \mathscr{F})$ by

$$\mathbb{E}_k f := \mathbb{E}[f|\mathscr{F}_k] \quad \text{and} \quad \mathbb{E}_Q f := 1_Q \mathbb{E}_k f,$$

and they admit the explicit representation

$$\mathbb{E}_k f = \sum_{Q\in\mathscr{D}_k} \frac{1_Q}{\mu(Q)} \int_Q f d\mu = \sum_{Q\in\mathscr{D}_k} \mathbb{E}_Q f.$$

If $\mu(Q) = 0$ for some cube, then the corresponding term in the above series is taken to be 0. We use the shorthand notation

$$\langle f\rangle_Q := \frac{1}{\mu(Q)} \int_Q f\, d\mu$$

for the integral average of $f$ on $Q$. With this notation, $\mathbb{E}_Q f = 1_Q\langle f\rangle_Q$.

The martingale difference operators and their local versions are defined by

$$\mathbb{D}_k f := \mathbb{E}_{k+1} f - \mathbb{E}_k f, \quad \mathbb{D}_Q f := 1_Q \mathbb{D}_k f.$$

By martingale convergence, for each $m \in \mathbb{Z}$,

$$f = \sum_{k\geq m} \mathbb{D}_k f + \mathbb{E}_m f = \sum_{k\geq m} \sum_{Q\in\mathscr{D}_k} \mathbb{D}_Q f + \sum_{Q\in\mathscr{D}_m} \mathbb{E}_Q f. \tag{4.5}$$

4.3. **The indexing of the sub-cubes.** Recall that a classical $L^2$-normalized Haar function $h_Q$ in $\mathbb{R}^n$, associated to a standard Euclidean dyadic cube $Q$, satisfies the size conditions $\|h_Q\|_{L^1} = |Q|^{1/2}$ and $\|h_Q\|_{L^\infty} = |Q|^{-1/2}$. In the present context, for the cancellative Haar functions there is no upper bound for the norm $\|h_u^Q\|_{L^\infty}$ in terms of the measure $\mu(Q)$. However, this will be compensated for by the smallness of the norm $\|h_u^Q\|_{L^1}$, in the sense that the following estimate still holds:

$$\|h_u^Q\|_{L^1}\|h_u^Q\|_{L^\infty} \lesssim 1.$$

(This type of estimate is established in [LSMP] for Haar functions on $\mathbb{R}$.) To obtain this control, we introduce the following ordering of the sub-cubes of $Q$.

Let the index sets $\mathscr{A}_k$, indexing the cubes $Q^k_\alpha$ of generation $k$, be initial intervals (finite or infinite) in $\mathbb{N}$. Recall from (2.4) that for $Q \in \mathscr{D}_k$, the cardinality of the set of dyadic sub-cubes $\mathrm{Ch}(Q) := \{R \in \mathscr{D}_{k+1} : R \subseteq Q\}$ is bounded by

$$\#\,\mathrm{Ch}(Q) =: M_Q \in [1, M]; \quad M = M(A_0, A_1, \delta) < \infty.$$

The number $M_Q$ depends, of course, on $Q$ but we may omit this dependence in the notation whenever it is clear from the context. For the usual dyadic cubes in the Euclidean space $\mathbb{R}^n$ we have $M_Q = 2^n$ for every $Q$. We order the sub-cubes from the "smallest" to the "largest". More precisely, we have the following result.

**Lemma 4.5.** *Given $Q \in \mathscr{D}$, there is an indexing of the sub-cubes $Q_j \in \mathrm{Ch}(Q)$ such that*

$$\sum_{j=u}^{M_Q} \mu(Q_j) \geq [1 - (u-1)M_Q^{-1}]\mu(Q) \tag{4.6}$$

*for every $u = 1, 2, \ldots, M_Q$.*

*Proof.* The case $u = 1$ is clear for any ordering of the sub-cubes. For $u > 1$, fix some indexing. If $M_Q = 1$, we are done. Otherwise, we proceed as follows. Suppose we have an indexing of the sub-cubes $Q_1, \dots, Q_m, m \geq 1$, such that (4.6) holds for all $u = 1, \dots, m < M_Q$. In particular,

$$[1-(m-1)M_Q^{-1}]\mu(Q) \leq \sum_{j=m}^{M_Q} \mu(Q_j) = \frac{1}{M_Q - m} \sum_{j=m}^{M_Q} \sum_{\substack{\ell=m \\ \ell \neq j}}^{M_Q} \mu(Q_\ell)$$

$$\leq \frac{M_Q-(m-1)}{M_Q-m} \max_{j\in\{m,\dots,M_Q\}} \mu\Big(\big(\bigcup_{\ell=m}^{M_Q} \mu(Q_\ell)\big) \setminus Q_j\Big).$$

Thus, we may choose $n \in \{m, \dots, M_Q\}$ such that

$$\mu\Big(\big(\bigcup_{\ell=m}^{M_Q} \mu(Q_\ell)\big) \setminus Q_n\Big) \geq \frac{M_Q-m}{M_Q-(m-1)} \cdot [1-(m-1)M_Q^{-1}]\mu(Q) = (1-mM_Q^{-1})\mu(Q).$$

We reorder the cubes $Q_j$ with $j \geq m$ by setting $m = n$, obtaining an indexing such that inequality (4.6) holds for all $u = 1, \dots, m+1$. If $M_Q > m+1$, we proceed as before. The claim follows. □

We observe that one way to obtain an indexing satisfying inequality (4.6) of Lemma 4.5 is to order the cubes so that $\mu(Q_1) \leq \mu(Q_2) \leq \dots \leq \mu(Q_{M_Q})$, as a short calculation shows.

From now on, let the indexing of the sub-cubes $Q_j$ of a given cube $Q \in \mathscr{D}$ be one provided by Lemma 4.5. For each $u \in \{1, \dots, M_Q\}$, set

$$E_u = E_u(Q) := \bigcup_{j=u}^{M_Q} Q_j. \tag{4.7}$$

Then, in particular, $E_1 = Q, E_{M_Q} = Q_{M_Q}$, and $E_u = Q_u \cup E_{u+1}$ where the union is disjoint.

**Lemma 4.6.** *For every $u \in \{1, \dots, M_Q - 1\}$,*

$$\frac{\mu(Q)}{M} \leq \mu(E_u) \leq \mu(Q) \tag{4.8}$$

*and*

$$\frac{1}{M} \leq \frac{\mu(E_{u+1})}{\mu(E_u)} \leq \frac{M}{2} \tag{4.9}$$

*where $M$ is the uniform bound for the maximal number of sub-cubes. In particular, $\mu(E_u) \simeq \mu(Q)$ and $\mu(E_{u+1})/\mu(E_u) \simeq 1$ for all $u$.*

*Proof.* The first estimate in (4.8) follows directly from Lemma 4.6, and the second estimate is immediate. For (4.9), by Lemma 4.6,

$$\frac{1}{M} \leq \frac{1}{M_Q} \leq \frac{M_Q-u}{M_Q} = \frac{[1-uM_Q^{-1}]\mu(Q)}{\mu(Q)} \leq \mu(E_u)^{-1} \sum_{j=u+1}^{M_Q} \mu(Q_j) = \frac{\mu(E_{u+1})}{\mu(E_u)}$$

$$\leq \left(\sum_{j=u}^{M_Q} \mu(Q_j)\right)^{-1} \mu(Q) \leq \frac{1}{[1-(u-1)M_Q^{-1}]} = \frac{M_Q}{M_Q-(u-1)} \leq \frac{M_Q}{2} \leq \frac{M}{2}. \qquad \square$$

4.4. **Martingale difference decomposition.** We will decompose the operators $\mathbb{E}_k$ and $\mathbb{D}_k$ by representing the projections $\mathbb{E}_Q$ and $\mathbb{D}_Q$ in terms of Haar functions as

$$\mathbb{E}_Q f = \langle f, h_0^Q \rangle h_0^Q, \quad \mathbb{D}_Q f = \sum_{u=1}^{M_Q-1} \langle f, h_u^Q \rangle h_u^Q. \tag{4.10}$$

Here $h_0^Q := \mu(Q)^{-1/2}1_Q$ is a non-cancellative Haar function and $\{h_u^Q : u = 1, \dots, M_Q - 1\}$ are cancellative ones. To this end, given $Q \in \mathscr{D}_k$, let the indexing of the sub-cubes $\{Q_j : j = 1, \dots, M_Q\}$ be the one provided by Lemma 4.5. Generalize the notation $\mathbb{E}_Q$ by denoting

$$\mathbb{E}_A f := \frac{1_A}{\mu(A)} \int_A f \, d\mu$$

for any measurable set $A$ with $\mu(A) > 0$. With this notation we obtain the splitting of the martingale difference $\mathbb{D}_Q$ as

$$\begin{aligned}
\mathbb{D}_Q = (\mathbb{E}_{k+1} - \mathbb{E}_k)1_Q &= \sum_{u=1}^{M} \mathbb{E}_{Q_u} - \mathbb{E}_Q = \sum_{u=1}^{M-1} \mathbb{E}_{Q_u} + (\mathbb{E}_{Q_M} - \mathbb{E}_Q) \\
&= \sum_{u=1}^{M-1} \mathbb{E}_{Q_u} + \sum_{u=1}^{M-1} (\mathbb{E}_{E_{u+1}} - \mathbb{E}_{E_u}) = \sum_{u=1}^{M-1} \big(\mathbb{E}_{Q_u} + \mathbb{E}_{E_{u+1}} - \mathbb{E}_{E_u}\big) =: \sum_{u=1}^{M-1} \mathbb{D}_u^Q;
\end{aligned}$$

here $E_u$ is the set defined in (4.7) with $E_1 = Q$ and $E_M = Q_M$.

Take a closer look at the operator $\mathbb{D}_u^Q$. If $\mu(Q_u) = 0$, then $\mathbb{D}_u^Q f = 0$, and we define the corresponding Haar function to be $h_u^Q \equiv 0$. For $\mu(Q_u) > 0$, we write (recall that $E_u = Q_u \cup E_{u+1}$ with a disjoint union)

$$\begin{aligned}
\mathbb{D}_u^Q f &= \big(\mathbb{E}_{Q_u} + \mathbb{E}_{E_{u+1}} - \mathbb{E}_{E_u}\big) f \\
&= \frac{1_{Q_u}}{\mu(Q_u)} \int_{Q_u} f d\mu + \frac{1_{E_{u+1}}}{\mu(E_{u+1})} \int_{E_{u+1}} f d\mu - \frac{1_{E_u}}{\mu(E_u)} \int_{E_u} f d\mu \\
&= \frac{1_{Q_u}}{\mu(Q_u)} \int_{Q_u} f d\mu + \frac{1_{E_{u+1}}}{\mu(E_{u+1})} \int_{E_{u+1}} f d\mu - \frac{1_{Q_u} + 1_{E_{u+1}}}{\mu(E_u)} \left( \int_{Q_u} f d\mu + \int_{E_{u+1}} f d\mu \right) \\
&= \frac{\mu(E_{u+1})\mu(Q_u)}{\mu(E_u)} \left[ \frac{1_{Q_u}}{\mu(Q_u)} \int \frac{1_{Q_u}}{\mu(Q_u)} f d\mu + \frac{1_{E_{u+1}}}{\mu(E_{u+1})} \int \frac{1_{E_{u+1}}}{\mu(E_{u+1})} f d\mu \right. \\
&\qquad \left. - \frac{1_{Q_u}}{\mu(Q_u)} \int \frac{1_{E_{u+1}}}{\mu(E_{u+1})} f d\mu - \frac{1_{E_{u+1}}}{\mu(E_{u+1})} \int \frac{1_{Q_u}}{\mu(Q_u)} f d\mu \right] \\
&= \frac{\mu(E_{u+1})\mu(Q_u)}{\mu(E_u)} \left( \frac{1_{Q_u}}{\mu(Q_u)} - \frac{1_{E_{u+1}}}{\mu(E_{u+1})} \right) \int \left( \frac{1_{Q_u}}{\mu(Q_u)} - \frac{1_{E_{u+1}}}{\mu(E_{u+1})} \right) f d\mu \\
&=: h_u^Q \int h_u^Q f d\mu = \langle f, h_u^Q \rangle h_u^Q,
\end{aligned}$$

where

$$h_u^Q := a_u 1_{Q_u} - b_u 1_{E_{u+1}}; \quad a_u := \frac{\mu(E_{u+1})^{1/2}}{\mu(Q_u)^{1/2}\mu(E_u)^{1/2}}, b_u := \frac{\mu(Q_u)^{1/2}}{\mu(E_u)^{1/2}\mu(E_{u+1})^{1/2}}. \tag{4.11}$$

**Remark 4.7.** We note that it may happen that a given cube $Q$ has only one child $R$, so that $R = Q$ as sets. In this case the formula (4.11) is not meaningful but also not relevant, as we do not need to add a cancellative Haar function corresponding to an "only child" $R$. We examine this situation more closely. A cube can be its own only child for finitely many generations, or even forever. In the first case, it means there will not be cancellative Haar functions associated to the cube until it truly subdivides. In the second case, it turns out that the cube $Q$ must be a single point, and the point must be an isolated point. In a geometrically doubling metric space there are at most countably many isolated points, and the only ones that contribute something are the ones that have positive (finite) measure (that is, point-masses).

Given a cube that is a point-mass, there are two scenarios: either the cube is its own parent forever, or else the cube is a proper child of its parent. In the first case, X is a point mass with finite measure, and in that case we need to include the function $1_X/\mu(X)^{1/2}$ to get a basis; in fact that is the only element in the basis (it is an orthonormal basis, although the orthogonality holds by default because there are no pairs to be checked). In the second case, we don't need to add any functions to the basis. In fact, one may then wonder whether the characteristic function of the point-mass set can be represented with the Haar functions we already have. The answer is yes: we can always represent the characteristic function of any given cube using the Haar functions associated to ancestors of the given cube, much as on $\mathbb{R}$ one can represent the function $1_{[0,1)}$ in terms of the Haar functions corresponding to the intervals $[0, 2^n)$ for $n \geq 1$; see Project 9.7 in [PerW]. For all this and more, see [Wei].

**Theorem 4.8.** *Let $(X, \rho)$ be a geometrically doubling quasi-metric space and suppose $\mu$ is a positive Borel measure on $X$ with the property that $\mu(B) < \infty$ for all balls $B \subseteq X$. For each $Q \in \mathscr{D}$, let*

$$h_0^Q := \mu(Q)^{-1/2} 1_Q$$

*and for $u = 1, 2, \ldots, M_Q - 1$ let*

$$h_u^Q := \begin{cases} 0, & \text{if } \mu(Q_u) = 0; \\ a_u 1_{Q_u} - b_u 1_{E_{u+1}}, & \text{if } \mu(Q_u) > 0, \end{cases}$$

*where*

$$a_u := \frac{\mu(E_{u+1})^{1/2}}{\mu(Q_u)^{1/2}\mu(E_u)^{1/2}}, \qquad b_u := \frac{\mu(Q_u)^{1/2}}{\mu(E_u)^{1/2}\mu(E_{u+1})^{1/2}}.$$

*The Haar functions $h_u^Q$, $Q \in \mathscr{D}$, $u = 0, 1, \ldots, M_Q - 1$, have the following properties:*

(i) *each $h_u^Q$ is a simple Borel-measurable real function on $X$;*
(ii) *each $h_u^Q$ is supported on $Q$;*
(iii) *each $h_u^Q$ is constant on each $R \in \mathrm{Ch}(Q)$;*
(iv) *$\int h_u^Q \, d\mu = 0$ for $u = 1, 2, \ldots, M_Q - 1$ (cancellation);*
(v) *$\langle h_u^Q, h_{u'}^Q \rangle = 0$ if $u \neq u'$, $u$, $u' \in \{0, 1, \ldots, M_Q - 1\}$;*
(vi) *the collection*
$$\{h_u^Q : u = 0, 1, \ldots, M_Q - 1\}$$
*is an orthogonal basis for the vector space $V(Q)$ of all functions on $Q$ that are constant on each sub-cube $R \in \mathrm{Ch}(Q)$;*
(vii) *for $u = 1, 2, \ldots, M_Q - 1$, if $h_u^Q \not\equiv 0$ then*
$$\|h_u^Q\|_{L^p(X,\mu)} \simeq \mu(Q_u)^{\frac{1}{p} - \frac{1}{2}} \quad \text{for } 1 \leq p \leq \infty;$$

*and*

(viii) *for $u = 0, 1, \ldots, M_Q - 1$, we have*
$$\|h_u^Q\|_{L^1(X,\mu)} \cdot \|h_u^Q\|_{L^\infty(X,\mu)} \simeq 1.$$

Note that Theorem 4.8 completes the proof of Theorem 4.2.

*Proof of Theorem 4.8.* Properties (i)–(v) are clear from the definition of the Haar functions. For property (vi), observe that $Q$ has $M_Q$ children, and $V(Q)$ is a finite-dimensional vector space with dimension $\dim(V(Q)) = M_Q$. The functions $\{h_u^Q\}_{u=0}^{M_Q-1}$ are orthogonal, and there are $M_Q$ of them, so they span $V(Q)$.

Property (vii): Take $h_u^Q$ with $u \in \{1, 2, \ldots, M_Q - 1\}$. For $1 \leq p < \infty$, we have

$$\begin{aligned} \|h_u^Q\|_{L^p(X,\mu)}^p &= \int_{Q_u} |h_u^Q(x)|^p \, d\mu(x) + \int_{E_{u+1}} |h_u^Q(x)|^p \, d\mu(x) \\ &= \int_{Q_u} (a_u)^p \, d\mu(x) + \int_{E_{u+1}} (b_u)^p \, d\mu(x) \\ &= \underbrace{\left(\frac{\mu(E_{u+1})}{\mu(E_u)}\right)^{p/2} \frac{\mu(Q_u)}{\mu(Q_u)^{p/2}}}_{\text{(I)}} + \underbrace{\frac{\mu(Q_u)^{p/2}}{\mu(E_u)^{p/2}\mu(E_{u+1})^{p/2}} \, \mu(E_{u+1})}_{\text{(II)}}. \end{aligned}$$

By Lemma 4.6,

$$\text{(I)} \simeq \mu(Q_u)^{1-p/2}$$

and

$$\text{(II)} \simeq \mu(Q_u)^{p/2}\frac{\mu(E_{u+1})}{\mu(E_{u+1})^p} \simeq \frac{\mu(Q_u)^{p/2}}{\mu(Q)^{p-1}} \lesssim \frac{\mu(Q_u)^{p/2}}{\mu(Q_u)^{p-1}} = \mu(Q_u)^{1-p/2}.$$

Since also (II) $\geq 0$, it follows that

$$\mu(Q_u)^{1-p/2} \lesssim \text{(I)} \leq \text{(I)} + \text{(II)} = \|h_u^Q\|^p_{L^p(X,\mu)} \lesssim \mu(Q_u)^{1-p/2},$$

and so

$$\|h_u^Q\|_{L^p(X,\mu)} \simeq \mu(Q_u)^{\frac{1}{p}-\frac{1}{2}}$$

as required.

Now suppose $p = \infty$. Note that by Lemma 4.6, $a_u \simeq \mu(Q_u)^{-1/2}$ and

$$b_u \simeq \frac{\mu(Q_u)^{1/2}}{\mu(E_u)} \simeq \frac{\mu(Q_u)^{1/2}}{\mu(Q)} \leq \frac{\mu(Q_u)^{1/2}}{\mu(Q_u)} = \mu(Q_u)^{-1/2},$$

as $0 \leq \mu(Q_u) \leq \mu(Q)$. Therefore, for all $x$ we have

$$|h_u^Q(x)| \leq a_u 1_{Q_u}(x) + b_u 1_{E_{u+1}}(x) \lesssim \mu(Q_u)^{-1/2}.$$

It follows that $\|h_u^Q\|_{L^\infty(X,\mu)} \lesssim \mu(Q_u)^{-1/2}$. Also, if $h_u^Q \not\equiv 0$, then for $x \in Q_u$ we have $|h_u^Q(x)| = a_u \simeq \mu(Q_u)^{-1/2}$, while for $x \in E_{u+1}$ we have $|h_u^Q(x)| = b_u \lesssim \mu(Q_u)^{-1/2}$. Then $\|h_u^Q\|_{L^\infty(X,\mu)} \gtrsim \mu(Q_u)^{-1/2}$. Thus for $h_u^Q \not\equiv 0$ we find that

$$\|h\|_{L^\infty(X,\mu)} \sim \mu(Q_u)^{-1/2},$$

as required.

As an aside, we note that although $|h_u^Q(x)| \lesssim \mu(Q_u)^{-1/2}$ for all $x$, the reverse inequality need not hold, since if the measure $\mu$ is not doubling, for $x \in E_{u+1}$ the quantity $|h_u^Q(x)| = b_u \sim \mu(Q_u)^{1/2}/\mu(Q)$ may be arbitrarily small compared with $\mu(Q_u)^{-1/2}$.

Property (viii): For $u = 0$, it is immediate from the definition that $\|h_0^Q\|_{L^1(X,\mu)}\|h_0^Q\|_{L^\infty(X,\mu)} = \mu(Q)^{1/2}\mu(Q)^{-1/2} = 1$. For $u \in \{1, \ldots, M_Q - 1\}$, by property (vii) we have

$$\|h_u^Q\|_{L^1(X,\mu)}\|h_u^Q\|_{L^\infty(X,\mu)} \simeq \mu(Q_u)^{1-1/2}\mu(Q_u)^{-1/2} = 1,$$

as required. □

We collect the convergence results of this section in the following theorem.

**Theorem 4.9.** *Let $(X, \rho)$ be a geometrically doubling quasi-metric space and suppose $\mu$ is a positive Borel measure on $X$ with the property that $\mu(B) < \infty$ for all balls $B \subseteq X$. For $1 < p < \infty$, for each $f \in L^p(X, \mu)$, and for each $m \in \mathbb{Z}$, we have*

$$f(x) = \sum_{k \geq m} \sum_{Q \in \mathscr{D}_k} \sum_{u=1}^{M_Q - 1} \langle f, h_u^Q \rangle h_u^Q(x) + \sum_{Q \in \mathscr{D}_m} \langle f, h_0^Q \rangle h_0^Q(x)$$

*where the first sum converges both in the $L^p(X, \mu)$-norm and pointwise $\mu$-a.e. For the second sum we have*

$$\sum_{Q \in \mathscr{D}_m} \langle f, h_0^Q \rangle h_0^Q(x) \to \begin{cases} 0, & \text{as } m \to -\infty \text{ if } \mu(X) = \infty; \\ \frac{1}{\mu(X)} \int_X f \, d\mu, & \text{as } m \to -\infty \text{ if } \mu(X) < \infty. \end{cases}$$

*As a consequence,*

$$f(x) = m_X(f) + \sum_{Q \in \mathscr{D}} \sum_{u=1}^{M_Q - 1} \langle f, h_u^Q \rangle h_u^Q(x).$$

Note that Theorem 4.9 completes the proof of Theorem 4.1.

*Proof of Theorem 4.9.* The first equality follows directly from (4.5) and (4.10). For the second sum, fix $x \in X$ and recall that there exists a unique $Q = Q^m \in \mathscr{D}_m$ such that $x \in Q$. Thus,

$$\sum_{Q\in\mathscr{D}_m} \langle f, h_0^Q\rangle h_0^Q(x) = \langle f, h_0^{Q^m}\rangle h_0^{Q^m}(x) = \frac{1_{Q^m}(x)}{\mu(Q^m)} \int_{Q^m} f\, d\mu.$$

If $\mu(X) = \infty$, then

$$\left|\frac{1_{Q^m}(x)}{\mu(Q^m)} \int_{Q^m} f\, d\mu\right| \leq \frac{\|f\|_{L^p(\mu)}}{\mu(Q^m)^{1/p}} \to 0 \text{ as } m \to -\infty$$

by Lemma 2.5. For the case $\mu(X) < \infty$, the asserted convergence is clear. □

## 5. Definitions and duality of dyadic $H^1_d(\widetilde{X})$ and dyadic $\mathrm{BMO}_d(\widetilde{X})$

We begin this section by recalling the definitions of continuous product $H^1(\widetilde{X})$ and $\mathrm{BMO}(\widetilde{X})$ developed in [HLW]. Here the underlying space $\widetilde{X} = X_1 \times \cdots \times X_n$ is a product space of homogeneous type. Then we introduce the definitions of the dyadic product Hardy and BMO spaces, $H^1_{d,d}(\widetilde{X})$ and $\mathrm{BMO}_{d,d}(\widetilde{X})$, associated to a system of dyadic cubes on $\widetilde{X}$. (For simplicity of notation, we work in the setting of two parameters, but everything goes through to the $n$-parameter setting, including the material we quote from [HLW, HLPW].) These dyadic spaces are defined by means of the Haar functions constructed in Section 4 above. We note that Tao used the same approach to define dyadic product $H^1$ and BMO on Euclidean underlying spaces, in the note [Tao]. Finally, we prove the duality relation

$$(H^1_{d,d}(\widetilde{X}))' = \mathrm{BMO}_{d,d}(\widetilde{X}).$$

Here our approach is to prove the duality of the product sequence spaces $s_1$ and $c_1$ which are models for $H^1_{d,d}(\widetilde{X})$ and $\mathrm{BMO}_{d,d}(\widetilde{X})$, and to pull this result across by means of the lifting and projection operators $T_L$ and $T_P$.

We recall from [HLW] the definitions of the Hardy space $H^1(\widetilde{X})$ and the bounded mean oscillation space $\mathrm{BMO}(\widetilde{X})$. These definitions rely on the orthonormal basis and the wavelet expansion in $L^2(X)$ which were recently constructed by Auscher and Hytönen [AH]. To state their result, we must first recall the set $\{x^k_\alpha\}$ of *reference dyadic points* as follows. Let $\delta$ be a fixed small positive parameter (for example, as noted in Section 2.2 of [AH], it suffices to take $\delta \leq 10^{-3}A_0^{-10}$). For $k = 0$, let $\mathscr{X}^0 := \{x^0_\alpha\}_\alpha$ be a maximal collection of 1-separated points in $X$. Inductively, for $k \in \mathbb{Z}_+$, let $\mathscr{X}^k := \{x^k_\alpha\} \supseteq \mathscr{X}^{k-1}$ and $\mathscr{X}^{-k} := \{x^{-k}_\alpha\} \subseteq \mathscr{X}^{-(k-1)}$ be $\delta^k$- and $\delta^{-k}$-separated collections in $\mathscr{X}^{k-1}$ and $\mathscr{X}^{-(k-1)}$, respectively.

Lemma 2.1 in [AH] shows that, for all $k \in \mathbb{Z}$ and $x \in X$, the reference dyadic points satisfy

$$\rho(x^k_\alpha, x^k_\beta) \geq \delta^k \ (\alpha \neq \beta), \qquad \rho(x, \mathscr{X}^k) = \min_\alpha \rho(x, x^k_\alpha) < 2A_0\delta^k. \tag{5.1}$$

Also, taking $c_0 := 1$, $C_0 := 2A_0$ and $\delta \leq 10^{-3}A_0^{-10}$, we see that $c_0$, $C_0$ and $\delta$ satisfy the conditions in Theorem 2.2 of [HK]. Therefore we may apply Hytönen and Kairema's construction (Theorem 2.2, [HK]), with the reference dyadic points $\{x^k_\alpha\}_{k\in\mathbb{Z},\alpha\in\mathscr{X}^k}$ playing the role of the points $\{z^k_\alpha\}_{k\in\mathbb{Z},\alpha\in\mathscr{A}_k}$, to conclude that there exists a set of half-open dyadic cubes

$$\{Q^k_\alpha\}_{k\in\mathbb{Z},\alpha\in\mathscr{X}^k}$$

associated with the reference dyadic points $\{x^k_\alpha\}_{k\in\mathbb{Z},\alpha\in\mathscr{X}^k}$. We call the reference dyadic point $x^k_\alpha$ the *center* of the dyadic cube $Q^k_\alpha$. We also identify with $\mathscr{X}^k$ the set of indices $\alpha$ corresponding to $x^k_\alpha \in \mathscr{X}^k$.

Note that $\mathscr{X}^k \subseteq \mathscr{X}^{k+1}$ for $k \in \mathbb{Z}$, so that every $x^k_\alpha$ is also a point of the form $x^{k+1}_\beta$. We denote $\mathscr{Y}^k := \mathscr{X}^{k+1}\backslash\mathscr{X}^k$, and relabel the points $\{x^k_\alpha\}_\alpha$ that belong to $\mathscr{Y}^k$ as $\{y^k_\alpha\}_\alpha$.

We now recall the orthonormal wavelet basis of $L^2(X)$ constructed by Auscher and Hytönen.

**Theorem 5.1** ([AH] Theorem 7.1)**.** *Let $(X, \rho, \mu)$ be a space of homogeneous type with quasi-triangle constant $A_0$, and let*

$$a := (1 + 2\log_2 A_0)^{-1}. \tag{5.2}$$

*There exists an orthonormal wavelet basis $\{\psi_\alpha^k\}$, $k \in \mathbb{Z}$, $y_\alpha^k \in \mathscr{Y}^k$, of $L^2(X)$, having exponential decay*

$$|\psi_\alpha^k(x)| \leq \frac{C}{\sqrt{\mu(B(y_\alpha^k, \delta^k))}} \exp\Big(-\nu\Big(\frac{\rho(y_\alpha^k, x)}{\delta^k}\Big)^a\Big), \tag{5.3}$$

*Hölder regularity*

$$|\psi_\alpha^k(x) - \psi_\alpha^k(y)| \leq \frac{C}{\sqrt{\mu(B(y_\alpha^k, \delta^k))}} \Big(\frac{\rho(x, y)}{\delta^k}\Big)^\eta \exp\Big(-\nu\Big(\frac{\rho(y_\alpha^k, x)}{\delta^k}\Big)^a\Big) \tag{5.4}$$

*for $\rho(x, y) \leq \delta^k$, and the cancellation property*

$$\int_X \psi_\alpha^k(x)\, d\mu(x) = 0, \qquad \text{for } k \in \mathbb{Z},\ \ y_\alpha^k \in \mathscr{Y}^k. \tag{5.5}$$

*Here $\delta$ is a fixed small parameter, say $\delta \leq 10^{-3}A_0^{-10}$, and $C < \infty$, $\nu > 0$ and $\eta \in (0, 1]$ are constants independent of $k$, $\alpha$, $x$ and $y_\alpha^k$.*

Moreover, the wavelet expansion is given by

$$f(x) = \sum_{k\in\mathbb{Z}} \sum_{\alpha\in\mathscr{Y}^k} \langle f, \psi_\alpha^k\rangle \psi_\alpha^k(x) = \sum_{k\in\mathbb{Z}} \Delta_k f(x),$$

in the sense of $L^2(X)$, where

$$\Delta_k f := \sum_{\alpha\in\mathscr{Y}^k} \langle f, \psi_\alpha^k\rangle \psi_\alpha^k(x)$$

is the orthogonal projection onto the subspace $W_k$ spanned by $\{\psi_\alpha^k\}_{\alpha\in\mathscr{Y}^k}$.

In what follows, we refer to the functions $\psi_\alpha^k$ as *wavelets*. Throughout Sections 5 and 6 of this paper, $a$ denotes the exponent from (5.2) and $\eta$ denotes the Hölder-regularity exponent from (5.4).

We now consider the product setting $(X_1, \rho_1, \mu_1) \times (X_2, \rho_2, \mu_2)$, where $(X_i, \rho_i, \mu_i)$, $i = 1, 2$, is a space of homogeneous type as defined in Section 1. For $i = 1, 2$, let $A_0^{(i)}$ be the constant in the quasi-triangle inequality (1.1), and let $C_{\mu_i}$ be the doubling constant as in inequality (1.2). On each $X_i$, by Theorem 5.1, there is a wavelet basis $\{\psi_{\alpha_i}^{k_i}\}$, with Hölder exponent $\eta_i$ as in inequality (5.4).

We refer the reader to [HLW], Definitions 3.9 and 3.10 and the surrounding discussion, for the definitions of the space $\overset{\circ}{G}$ of product test functions and its dual space $(\overset{\circ}{G})'$ of product distributions on the product space $X_1 \times X_2$. In [HLW], $\overset{\circ}{G}$ is denoted by $\overset{\circ}{G}(\beta_1, \beta_2; \gamma_1, \gamma_2)$ and $(\overset{\circ}{G})'$ is denoted by $\overset{\circ}{G}(\beta_1, \beta_2; \gamma_1, \gamma_2)'$, where the $\beta_i$ and $\gamma_i$ are parameters that quantify the size and smoothness of the test functions, and $\beta_i \in (0, \eta_i)$ where $\eta_i$ is the regularity exponent from Theorem 5.1. (In fact, in [HLW] the theory is developed for $\beta_i \in (0, \eta_i]$, but for simplicity here we only use $\beta_i \in (0, \eta_i)$ since that is all we need.) We note that the one-parameter scaled Auscher–Hytönen wavelets $\psi_\alpha^k(x)/\sqrt{\mu(B(y_\alpha^k, \delta^k))}$ are test functions, and that their tensor products $\psi_{\alpha_1}^{k_1}(x)\psi_{\alpha_2}^{k_2}(y)\big(\mu_1(B(y_{\alpha_1}^{k_1}, \delta_1^{k_1}))\mu_2(B(y_{\alpha_2}^{k_2}, \delta_2^{k_2}))\big)^{-1/2}$ are product test functions in $\overset{\circ}{G}$, for all $\beta_i \in (0, \eta_i]$ and all $\gamma_i > 0$, for $i = 1, 2$. These facts follow from the theory in [HLW], specifically Definition 3.1 and the discussion after it, Theorem 3.3, and Definitions 3.9 and 3.10 and the discussion between them.

As shown in Theorem 3.11 of [HLW], the reproducing formula

$$f(x_1,x_2)=\sum_{k_1}\sum_{\alpha_1}\sum_{k_2}\sum_{\alpha_2}\langle f,\psi_{\alpha_1}^{k_1}\psi_{\alpha_2}^{k_2}\rangle\psi_{\alpha_1}^{k_1}(x_1)\psi_{\alpha_2}^{k_2}(x_2) \tag{5.6}$$

holds in $\overset{\circ}{G}$ for all $\beta_i$, $\gamma_i\in(0,\eta_i)$ for $i=1$, 2, and hence by duality (5.6) also holds in $(\overset{\circ}{G})'$.

Next we define the product Hardy space $H^1$. This is the special case $p=1$ of the definition of $H^p$ in Definition 5.1 in [HLW].

**Definition 5.2** ([HLW], Definition 5.1)**.** The *product Hardy space* $H^1$ *on* $\widetilde{X}$ is defined as

$$H^1(\widetilde{X}):=\{f\in(\overset{\circ}{G})':S(f)\in L^1(\widetilde{X})\},$$

where $S(f)$ is the Littlewood–Paley $g$-function related to the orthonormal basis introduced in [AH] as follows:

$$S(f)(x_1,x_2):=\Big\{\sum_{k_1,\alpha_1}\sum_{k_2,\alpha_2}\Big|\langle\psi_{\alpha_1}^{k_1}\psi_{\alpha_2}^{k_2},f\rangle\widetilde{\chi}_{Q_{\alpha_1}^{k_1}}(x_1)\widetilde{\chi}_{Q_{\alpha_2}^{k_2}}(x_2)\Big|^2\Big\}^{1/2}, \tag{5.7}$$

with $\widetilde{\chi}_{Q_{\alpha_1}^{k_1}}(x_1):=\chi_{Q_{\alpha_1}^{k_1}}(x_1)\mu_1(Q_{\alpha_1}^{k_1})^{-1/2}$ and $\widetilde{\chi}_{Q_{\alpha_2}^{k_2}}(x_2):=\chi_{Q_{\alpha_2}^{k_2}}(x_2)\mu_2(Q_{\alpha_2}^{k_2})^{-1/2}$.

As noted in [HLPW], it follows from Definition 5.2 that $H^1(\widetilde{X})\cap L^2(\widetilde{X})$ is dense in $H^1(\widetilde{X})$ with respect to the $H^1(\widetilde{X})$ norm $\|f\|_{H^1}:=\|Sf\|_{L^1(\widetilde{X})}$.

We first give the definition of atoms for the product Hardy space. This definition is the special case with $q=2$, $p=1$ of Definition 4.7 in [HLPW].

**Definition 5.3** ([HLPW], Definition 4.7)**.** A function $a(x_1,x_2)$ defined on $\widetilde{X}$ is called an *atom of* $H^1(\widetilde{X})$ if $a(x_1,x_2)$ satisfies the following properties (1)–(3):

(1) supp $a\subset\Omega$, where $\Omega$ is an open set of $\widetilde{X}$ with finite measure;
(2) $\|a\|_{L^2}\le\mu(\Omega)^{-1/2}$; and
(3) $a$ can be further decomposed into *rectangle atoms* $a_R$ associated to dyadic rectangles $R=Q_1\times Q_2$, satisfying the following properties (i)–(iii):
  (i) supp $a_R\subset\overline{C}R$, where $\overline{C}$ is an absolute constant independent of $a$ and $R$;
  (ii) $\int_{X_1}a_R(x_1,x_2)\,d\mu_1(x_1)=0$ for a.e. $x_2\in X_2$ and $\int_{X_2}a_R(x_1,x_2)\,d\mu_2(x_2)=0$ for a.e. $x_1\in X_1$;
  (iii) $a=\sum\limits_{R\in m(\Omega)}a_R$ and $\sum\limits_{R\in m(\Omega)}\|a_R\|_{L^2}^2\le\mu(\Omega)^{-1}$.

The special case $p=1$, $q=2$ of the following result from [HLPW] shows that the atomic Hardy space defined in terms of the atoms from Definition 5.3 coincides with the dense subset $H^1(\widetilde{X})\cap L^2(\widetilde{X})$ of $H^1(\widetilde{X})$. This atomic decomposition will be used in Section 6.

**Theorem 5.4** ([HLPW], Theorem 4.8)**.** *Suppose that* $\max\big(\frac{\omega_1}{\omega_1+\eta_1},\frac{\omega_2}{\omega_2+\eta_2}\big)<p\le1<q<\infty$. *Then* $f\in L^q(X_1\times X_2)\cap H^p(X_1\times X_2)$ *if and only if* $f$ *has an atomic decomposition; that is,*

$$f=\sum_{i=-\infty}^{\infty}\lambda_ia_i, \tag{5.8}$$

*where* $a_i$ *are* $(p,q)$ *atoms,* $\sum_i|\lambda_i|^p<\infty$, *and the series converges in both* $H^p(X_1\times X_2)$ *and* $L^q(X_1\times X_2)$. *Moreover,*

$$\|f\|_{H^p(X_1\times X_2)}\sim\inf\Big\{\Big[\sum_i|\lambda_i|^p\Big]^{1/p}:f=\sum_i\lambda_ia_i\Big\},$$

*where the infimum is taken over all decompositions as above and the implicit constants are independent of the* $L^q(X_1\times X_2)$ *and* $H^p(X_1\times X_2)$ *norms of* $f$.

We turn to the definition of the product BMO space.

**Definition 5.5** ([HLW], Definition 5.2)**.** We define the *product* BMO *space on* $\widetilde{X}$ as

$$\mathrm{BMO}(\widetilde{X}) := \{f \in (\overset{\circ}{G})' : \mathcal{C}_1(f) \in L^\infty\},$$

with $\mathcal{C}_1(f)$ defined as follows:

$$\mathcal{C}_1(f) := \sup_{\Omega} \Big\{ \frac{1}{\mu(\Omega)} \sum_{R=Q^{k_1}_{\alpha_1}\times Q^{k_2}_{\alpha_2}\subset\Omega} \big|\langle \psi^{k_1}_{\alpha_1}\psi^{k_2}_{\alpha_2}, f\rangle\big|^2 \Big\}^{1/2}, \tag{5.9}$$

where $\Omega$ runs over all open sets in $\widetilde{X}$ with finite measure, and $R$ runs over all dyadic rectangles contained in $\Omega$.

We are now ready to define the dyadic versions of our function spaces.

**Definition 5.6.** We define the *dyadic product Hardy space* $H^1$ *on* $\widetilde{X}$ as

$$H^1_{d,d}(\widetilde{X}) := \{f \in L^1(X_1\times X_2) : S_{d,d}(f) \in L^1(\widetilde{X})\},$$

where $S_{d,d}(f)$ is the Littlewood-Paley $g$-function related to the Haar orthonormal basis built in the previous sections, as follows:

$$S_{d,d}(f)(x_1,x_2) := \Big\{ \sum_{Q_1\in\mathcal{D}_1}\sum_{Q_2\in\mathcal{D}_2}\sum_{u_1=1}^{M_{Q_1}-1}\sum_{u_2=1}^{M_{Q_2}-1} \big|\langle f, h^{u_1}_{Q_1}h^{u_2}_{Q_2}\rangle \widetilde{\chi}_{Q_1}(x_1)\widetilde{\chi}_{Q_2}(x_2)\big|^2 \Big\}^{1/2}, \tag{5.10}$$

where $\widetilde{\chi}_{Q_1}(x_1) := \chi_{Q_1}(x_1)\mu_1(Q_1)^{-1/2}$ and $\widetilde{\chi}_{Q_2}(x_2) := \chi_{Q_2}(x_2)\mu_2(Q_2)^{-1/2}$.

**Definition 5.7.** We define the *dyadic product* BMO *space on* $\widetilde{X}$ as

$$\mathrm{BMO}_{d,d}(\widetilde{X}) := \{f \in L^1_{\mathrm{loc}} : \mathcal{C}_{d,d}(f) \in L^\infty\},$$

with $\mathcal{C}_{d,d}(f)$ defined as follows:

$$\mathcal{C}_{d,d}(f) := \sup_{\Omega} \Big\{ \frac{1}{\mu(\Omega)} \sum_{\substack{R=Q_1\times Q_2\in\mathcal{D}_1\times\mathcal{D}_2,\\ R\subset\Omega}} \sum_{u_1=1}^{M_{Q_1}-1}\sum_{u_2=1}^{M_{Q_2}-1} \big|\langle f, h^{u_1}_{Q_1}h^{u_2}_{Q_2}\rangle\big|^2 \Big\}^{1/2}, \tag{5.11}$$

where $\Omega$ runs over all open sets in $\widetilde{X}$ with finite measure, and $R$ runs over all dyadic rectangles contained in $\Omega$.

**Remark 5.8.** We note that our definitions of $H^1_{d,d}(\widetilde{X})$ and $\mathrm{BMO}_{d,d}(\widetilde{X})$ can be rephrased in terms of martingales, as follows. However, in the rest of this paper we work with the definitions above in terms of Haar functions.

For $H^1_{d,d}(\widetilde{X})$: The dyadic square function $S_{d,d}(f)$ used in Definition 5.6 can be replaced by

$$\widehat{S}_{d,d}(f)(x_1,x_2) := \Big\{ \sum_{k_1,\alpha_1}\sum_{k_2,\alpha_2} \Big|\mathbb{D}_{Q^{k_1}_{\alpha_1}}\mathbb{D}_{Q^{k_2}_{\alpha_2}}(f)(x_1,x_2)\Big|^2 \Big\}^{1/2}, \tag{5.12}$$

where $\mathbb{D}_{Q^{k_1}_{\alpha_1}}\mathbb{D}_{Q^{k_2}_{\alpha_2}}(f)(x_1,x_2)$ means we first apply the orthogonal projection $\mathbb{D}_{Q^{k_2}_{\alpha_2}}$ to $f(x_1,\cdot)$ for a.e. $x_1$, and then apply $\mathbb{D}_{Q^{k_1}_{\alpha_1}}$ to the resulting function of $x_1$.

The space

$$\widehat{H}^1_{d,d}(\widetilde{X}) := \{f \in L^1(X_1\times X_2) : \widehat{S}_{d,d}(f) \in L^1(\widetilde{X})\}$$

coincides with $H^1_{d,d}(\widetilde{X})$, and moreover $\|S_{d,d}f\|_{L^1(X_1\times X_2)} \sim \|\widehat{S}_{d,d}f\|_{L^1(X_1\times X_2)}$; see [Tre, Section 3.2.3] and [FJ]. The advantage of using the square function $S_{d,d}$ is that only the absolute value of the individual coefficients appears in the definition, and there is no further cancellation involved. Also notice that using the local martingale differences and expectations we can write $S_{d,d}$ as follows:

$$S_{d,d}(f)(x_1,x_2) = \Big\{ \sum_{k_1,\alpha_1} \sum_{k_2,\alpha_2} \mathbb{E}_{Q^{k_1}_{\alpha_1}} \mathbb{E}_{Q^{k_2}_{\alpha_2}} \Big| \mathbb{D}_{Q^{k_1}_{\alpha_1}} \mathbb{D}_{Q^{k_2}_{\alpha_2}}(f)(x_1,x_2)\Big|^2 \Big\}^{1/2}. \tag{5.13}$$

For $\mathrm{BMO}_{d,d}(\widetilde{X})$: The quantity $\mathcal{C}_{d,d}(f)$ used in Definition 5.7 can be written as follows:

$$\mathcal{C}^{d,d}_1(f) = \sup_{\Omega} \Big\{ \frac{1}{\mu(\Omega)} \sum_{\substack{R=Q^{k_1}_{\alpha_1}\times Q^{k_2}_{\alpha_2}\in\mathcal{D}_1\times\mathcal{D}_2,\\ R\subset\Omega}} \int_{X_1\times X_2} \big|\mathbb{D}_{Q^{k_1}_{\alpha_1}} \mathbb{D}_{Q^{k_2}_{\alpha_2}}(f)(x_1,x_2)\big|^2 d\mu_1(x_1)d\mu_2(x_2)\Big\}^{1/2}. \tag{5.14}$$

where $\Omega$ runs over all open sets in $\widetilde{X}$ with finite measure, and $R$ runs over all dyadic rectangles contained in $\Omega$. This time we get equality because we are integrating the square of the local martingale differences (orthogonal projection onto the subspace generated by the Haar functions associated to the rectangle $R = Q^{k_1}_{\alpha_1} \times Q^{k_2}_{\alpha_2}$) and we can use Plancherel to replace the square of the $L^2$-norm by the sum of the squares of the Haar coefficients.

For each dyadic cube $Q$, the space

$$W(Q) := \mathrm{span}\{f(x)\colon \ \mathrm{supp}\, f \subset Q, f|_R \text{ is constant for each } R \in \mathrm{ch}(Q), \text{ and } \textstyle\int f = 0\}$$

of all functions supported on $Q$ that are constant on each child of $Q$ and have mean zero is a finite-dimensional vector space, with dimension $\dim(W(Q)) = M_Q - 1$. It is a subspace of the space $V(Q)$ defined in the proof of Theorem 4.8 above. There are many possible bases of Haar functions for $W(Q)$, in addition to the basis we chose above. However, it follows from the remarks in the three preceding paragraphs that the definitions of $H^1_{d,d}(\widetilde{X})$ and $\mathrm{BMO}_{d,d}(\widetilde{X})$ are independent of the Haar functions chosen on each finite-dimensional subspace $W(Q^{k_i}_{\alpha_i})$, since in the definitions we are using the orthogonal projections $\mathbb{D}_{Q^{k_1}_{\alpha_1}} \mathbb{D}_{Q^{k_2}_{\alpha_2}}(f)$.

We now establish the duality $\big(H^1_{d,d}(\widetilde{X})\big)' = \mathrm{BMO}_{d,d}(\widetilde{X})$. Following the approach used in [HLL] for the continuous case, we do so by passing to the corresponding duality result for certain sequence spaces $s^1$ and $c^1$, which are models for $H^1_{d,d}(\widetilde{X})$ and $\mathrm{BMO}_{d,d}(\widetilde{X})$ respectively. These sequence spaces are part of the homogeneous Triebel–Lizorkin family of sequence spaces adapted to the product setting. In the notation used in [FJ] and [Tre], $s^1 = \dot{f}^{0,2}_1$ and $c^1 = \dot{f}^{0,2}_\infty$.

We begin with the definitions and properties of these sequence spaces.

We define $s^1$ to be the sequence space consisting of the sequences $s = \{s_{Q_1,u_1,Q_2,u_2}\}$ of complex numbers with

$$\|s\|_{s^1} := \Big\| \Big\{ \sum_{Q_1\in\mathcal{D}_1} \sum_{Q_2\in\mathcal{D}_2} \sum_{u_1=1}^{M_{Q_1}-1} \sum_{u_2=1}^{M_{Q_2}-1} \big|s_{Q_1,u_1,Q_2,u_2} \widetilde{\chi}_{Q_1}(x_1)\widetilde{\chi}_{Q_2}(x_2)\big|^2 \Big\}^{1/2} \Big\|_{L^1(X_1\times X_2)} < \infty. \tag{5.15}$$

We define $c^1$ to be the sequence space consisting of the sequences $t = \{t_{Q_1,u_1,Q_2,u_2}\}$ of complex numbers with

$$\|t\|_{c^1} := \sup_{\Omega} \Big( \frac{1}{\mu(\Omega)} \sum_{R=Q_1\times Q_2\in\mathcal{D}_1\times\mathcal{D}_2,\ R\subset\Omega} \ \sum_{u_1=1}^{M_{Q_1}-1} \sum_{u_2=1}^{M_{Q_2}-1} |t_{Q_1,u_1,Q_2,u_2}|^2 \Big)^{1/2} < \infty, \tag{5.16}$$

where the supremum is taken over all open sets $\Omega$ in $\widetilde{X}$ with finite measure.

The main result for these sequence spaces is their duality.

**Theorem 5.9.** *The following duality result holds:* $\left(s^1\right)' = c^1$.

*Proof.* First, we prove that for each $t \in c^1$, if

$$L(s) = \langle s,t\rangle := \sum_{Q_1\in\mathcal{D}_1}\sum_{Q_2\in\mathcal{D}_2}\sum_{u_1=1}^{M_{Q_1}-1}\sum_{u_2=1}^{M_{Q_2}-1} s_{Q_1,u_1,Q_2,u_2}\cdot \bar{t}_{Q_1,u_1,Q_2,u_2}, \text{ for all } s\in s^1, \tag{5.17}$$

then

$$|L(s)| \le C\|s\|_{s^1}\|t\|_{c^1}. \tag{5.18}$$

To see this, define for each $k\in\mathbb{Z}$ the following subsets $\Omega_k$ and $\widetilde{\Omega}_k$ of $X_1\times X_2$ and the subcollection $B_k$ of certain rectangles in $\mathcal{D}_1\times\mathcal{D}_2$:

$$\Omega_k := \left\{(x_1,x_2)\in X_1\times X_2\ :\left\{\sum_{Q_1\in\mathcal{D}_1}\sum_{Q_2\in\mathcal{D}_2}\sum_{u_1=1}^{M_{Q_1}-1}\sum_{u_2=1}^{M_{Q_2}-1}\left|s_{Q_1,u_1,Q_2,u_2}\widetilde{\chi}_{Q_1}(x_1)\widetilde{\chi}_{Q_2}(x_2)\right|^2\right\}^{1/2} > 2^k\right\},$$

$$B_k := \left\{R = Q_1\times Q_2\in\mathcal{D}_1\times\mathcal{D}_2:\ \mu(\Omega_k\cap R) > \frac{1}{2}\mu(R),\ \mu(\Omega_{k+1}\cap R)\le \frac{1}{2}\mu(R)\right\},$$

and

$$\widetilde{\Omega}_k := \left\{(x_1,x_2)\in X_1\times X_2\ : M_s(\chi_{\Omega_k}) > \frac{1}{2}\right\},$$

where $M_s$ is the strong maximal function on $X_1\times X_2$.

**Remark 5.10.** First notice that since $s\in s^1$ we have $|\Omega_k|<\infty$. Second, each rectangle $R$ belongs to exactly one set $B_k$. Third, note that $|\widetilde{\Omega}_k|\le C|\Omega_k|$ because the strong maximal function is bounded on $L^2(\widetilde{X})$. Fourth, if $R\in B_k$ then $R\subset\widetilde{\Omega}_k$, therefore $\cup_{R\in B_k}R\subset\widetilde{\Omega}_k$ and $B_k\subset\{R: R\subset\widetilde{\Omega}_k\}$. Fifth, if $R\in B_k$ then $\mu((\widetilde{\Omega}_k\backslash\Omega_{k+1})\cap R)\ge\mu(R)/2$.

By Hölder's inequality, the linear functional $L(s)$ from (5.17) satisfies

$$\begin{aligned}
|L(s)| &\le \sum_k\Bigg(\sum_{R=Q_1\times Q_2\in\mathcal{D}_1\times\mathcal{D}_2,\ R\in B_k}\sum_{u_1=1}^{M_{Q_1}-1}\sum_{u_2=1}^{M_{Q_2}-1}|s_{Q_1,u_1,Q_2,u_2}|^2\Bigg)^{\frac12}\\
&\quad\times\Bigg(\sum_{R=Q_1\times Q_2\in\mathcal{D}_1\times\mathcal{D}_2,\ R\in B_k}\sum_{u_1=1}^{M_{Q_1}-1}\sum_{u_2=1}^{M_{Q_2}-1}|t_{Q_1,u_1,Q_2,u_2}|^2\Bigg)^{\frac12}\\
&\le \sum_k\mu(\widetilde{\Omega}_k)^{\frac12}\Bigg(\sum_{R=Q_1\times Q_2\in\mathcal{D}_1\times\mathcal{D}_2,\ R\in B_k}\sum_{u_1=1}^{M_{Q_1}-1}\sum_{u_2=1}^{M_{Q_2}-1}|s_{Q_1,u_1,Q_2,u_2}|^2\Bigg)^{\frac12} \qquad (5.19)\\
&\quad\times\Bigg(\frac{1}{\mu(\widetilde{\Omega}_k)}\sum_{R=Q_1\times Q_2\in\mathcal{D}_1\times\mathcal{D}_2,\ R\in B_k}\sum_{u_1=1}^{M_{Q_1}-1}\sum_{u_2=1}^{M_{Q_2}-1}|t_{Q_1,u_1,Q_2,u_2}|^2\Bigg)^{\frac12}\\
&\le \sum_k\mu(\widetilde{\Omega}_k)^{\frac12}\Bigg(\sum_{R=Q_1\times Q_2\in\mathcal{D}_1\times\mathcal{D}_2,\ R\in B_k}\sum_{u_1=1}^{M_{Q_1}-1}\sum_{u_2=1}^{M_{Q_2}-1}|s_{Q_1,u_1,Q_2,u_2}|^2\Bigg)^{\frac12}\|t\|_{c^1}.
\end{aligned}$$

In the last inequality we have used the fact that $B_k\subset\{R: R\subset\widetilde{\Omega}_k\}$, as stated in Remark 5.10.

Next we point out that from the definition of $\Omega_{k+1}$ and Remark 5.10, we have

$$\begin{aligned}
&\int_{\widetilde{\Omega}_k\backslash\Omega_{k+1}}\sum_{R=Q_1\times Q_2\in\mathcal{D}_1\times\mathcal{D}_2,\ R\in B_k}\sum_{u_1=1}^{M_{Q_1}-1}\sum_{u_2=1}^{M_{Q_2}-1}|s_{Q_1,u_1,Q_2,u_2}|^2\frac{\chi_{Q_1}(x_1)}{\mu_1(Q_1)}\frac{\chi_{Q_2}(x_2)}{\mu_2(Q_2)}d\mu_1(x_1)d\mu_2(x_2)\\
&\le 2^{2(k+1)}\mu(\widetilde{\Omega}_k\backslash\Omega_{k+1})\le C2^{2k}\mu(\Omega_k).
\end{aligned}$$

Moreover, by the Monotone Convergence Theorem and Remark 5.10, we have

$$\begin{aligned}
&\int_{\widetilde{\Omega}_k\backslash\Omega_{k+1}} \sum_{R=Q_1\times Q_2\in\mathcal{D}_1\times\mathcal{D}_2,\ R\in B_k} \sum_{u_1=1}^{M_{Q_1}-1}\sum_{u_2=1}^{M_{Q_2}-1} |s_{Q_1,u_1,Q_2,u_2}|^2 \frac{\chi_{Q_1}(x_1)}{\mu_1(Q_1)}\frac{\chi_{Q_2}(x_2)}{\mu_2(Q_2)} d\mu_1(x_1)d\mu_2(x_2)\\
&\geq \sum_{R=Q_1\times Q_2\in\mathcal{D}_1\times\mathcal{D}_2,\ R\in B_k} \sum_{u_1=1}^{M_{Q_1}-1}\sum_{u_2=1}^{M_{Q_2}-1} |s_{Q_1,u_1,Q_2,u_2}|^2 \frac{1}{\mu_1(Q_1)}\frac{1}{\mu_2(Q_2)}\mu(\widetilde{\Omega}_k\backslash\Omega_{k+1}\cap R)\\
&\geq \frac{1}{2}\sum_{R=Q_1\times Q_2\in\mathcal{D}_1\times\mathcal{D}_2,\ R\in B_k} \sum_{u_1=1}^{M_{Q_1}-1}\sum_{u_2=1}^{M_{Q_2}-1} |s_{Q_1,u_1,Q_2,u_2}|^2.
\end{aligned}$$

As a consequence, we obtain that

$$\sum_{R=Q_1\times Q_2\in\mathcal{D}_1\times\mathcal{D}_2,\ R\in B_k} \sum_{u_1=1}^{M_{Q_1}-1}\sum_{u_2=1}^{M_{Q_2}-1} |s_{Q_1,u_1,Q_2,u_2}|^2 \leq C2^{2k}\mu(\Omega_k).$$

Substituting this back into the last inequality of (5.19) yields that

$$|L(s)| \leq C\sum_k \mu(\widetilde{\Omega}_k)^{\frac12} 2^k\mu(\Omega_k)^{\frac12}\|t\|_{c^1} \leq C\sum_k 2^k\mu(\Omega_k)\|t\|_{c^1} \leq C\|s\|_{s^1}\|t\|_{c^1}.$$

Conversely, we need to verify that for any $L\in\left(s^1\right)'$, there exists $t\in c^1$ with $\|t\|_{c^1}\leq\|L\|$ such that for all $s\in s^1$, $L(s)=\sum_{Q_1,u_1,Q_2,u_2} s_{Q_1,u_1,Q_2,u_2}\overline{t_{Q_1,u_1,Q_2,u_2}}$.

Now let $s^{Q_1,u_1,Q_2,u_2}$ be the sequence that equals 0 everywhere except at the $(Q_1,u_1,Q_2,u_2)$ entry where it equals 1. Then it is easy to see that $\|s^{Q_1,u_1,Q_2,u_2}\|_{s^1}=1$. For all $s\in s^1$, we have $s=\sum_{Q_1,u_1,Q_2,u_2} s_{Q_1,u_1,Q_2,u_2}s^{Q_1,u_1,Q_2,u_2}$, where the limit holds in the norm of $s^1$. For each $L\in\left(s^1\right)'$, let $\overline{t_{Q_1,u_1,Q_2,u_2}}=L(s^{Q_1,u_1,Q_2,u_2})$. Since $L$ is a bounded linear functional on $s^1$, we see that

$$L(s)=L\Big(\sum_{Q_1,u_1,Q_2,u_2} s_{Q_1,u_1,Q_2,u_2}s^{Q_1,u_1,Q_2,u_2}\Big)=\sum_{Q_1,u_1,Q_2,u_2} s_{Q_1,u_1,Q_2,u_2}\overline{t_{Q_1,u_1,Q_2,u_2}}.$$

Let $t=\{t_{Q_1,u_1,Q_2,u_2}\}$. Then we only need to check that $\|t\|_{c^1}\leq\|L\|$.

For each fixed open set $\Omega\subset X_1\times X_2$ with finite measure, let $\bar\mu$ be a new measure such that $\bar\mu(R)=\dfrac{\mu(R)}{\mu(\Omega)}$ when $R\subset\Omega$, and $\bar\mu(R)=0$ when $R\not\subseteq\Omega$. And let $l^2(\bar\mu)$ be a sequence space such that when $s\in l^2(\bar\mu)$,

$$\Big(\sum_{R=Q_1\times Q_2\in\mathcal{D}_1\times\mathcal{D}_2,\ R\subset\Omega} \sum_{u_1=1}^{M_{Q_1}-1}\sum_{u_2=1}^{M_{Q_2}-1} |s_{Q_1,u_1,Q_2,u_2}|^2\frac{\mu(R)}{\mu(\Omega)}\Big)^{1/2}<\infty.$$

It is easy to see that $(l^2(\bar\mu))'=l^2(\bar\mu)$. Next,

(5.20)

$$\begin{aligned}
&\Big(\frac{1}{\mu(\Omega)}\sum_{R=Q_1\times Q_2\in\mathcal{D}_1\times\mathcal{D}_2,\ R\subset\Omega} \sum_{u_1=1}^{M_{Q_1}-1}\sum_{u_2=1}^{M_{Q_2}-1} |t_{Q_1,u_1,Q_2,u_2}|^2\Big)^{1/2}\\
&=\Big\|\big\{\mu(R)^{-1/2}t_{Q_1,u_1,Q_2,u_2}\big\}\Big\|_{l^2(\bar\mu)}\\
&=\sup_{s:\ \|s\|_{l^2(\bar\mu)}\leq1}\Big|\sum_{R=Q_1\times Q_2\in\mathcal{D}_1\times\mathcal{D}_2,\ R\subset\Omega} \sum_{u_1=1}^{M_{Q_1}-1}\sum_{u_2=1}^{M_{Q_2}-1} \mu(R)^{-1/2}t_{Q_1,u_1,Q_2,u_2}\cdot s_{Q_1,u_1,Q_2,u_2}\cdot\frac{\mu(R)}{\mu(\Omega)}\Big|
\end{aligned}$$

$$= \sup_{s:\ \|s\|_{l^2(\bar{\mu})}\le 1} \left| L\left( \left\{ s_{Q_1,u_1,Q_2,u_2} \frac{\mu(R)^{1/2}}{\mu(\Omega)} \right\}_{R=Q_1\times Q_2\in\mathcal{D}_1\times\mathcal{D}_2,\ R\subset\Omega, u_1=1,\ldots,M_{Q_1}-1, u_2=1,\ldots,M_{Q_2}-1} \right) \right|$$

$$\le \sup_{s:\ \|s\|_{l^2(\bar{\mu})}\le 1} \|L\| \cdot \left\| \left\{ s_{Q_1,u_1,Q_2,u_2} \frac{\mu(R)^{1/2}}{\mu(\Omega)} \right\}_{R=Q_1\times Q_2\in\mathcal{D}_1\times\mathcal{D}_2,\ R\subset\Omega, u_1=1,\ldots,M_{Q_1}-1, u_2=1,\ldots,M_{Q_2}-1} \right\|_{s^1}.$$

By (5.15) and the Cauchy–Schwarz inequality, we have

$$\left\| \left\{ s_{Q_1,u_1,Q_2,u_2} \frac{\mu(R)^{1/2}}{\mu(\Omega)} \right\}_{R=Q_1\times Q_2\in\mathcal{D}_1\times\mathcal{D}_2,\ R\subset\Omega, u_1=1,\ldots,M_{Q_1}-1, u_2=1,\ldots,M_{Q_2}-1} \right\|_{s^1}$$
$$\le \left( \sum_{R=Q_1\times Q_2\in\mathcal{D}_1\times\mathcal{D}_2,\ R\subset\Omega} \sum_{u_1=1}^{M_{Q_1}-1} \sum_{u_2=1}^{M_{Q_2}-1} |s_{Q_1,u_1,Q_2,u_2}|^2 \frac{\mu(R)}{\mu(\Omega)} \right)^{1/2}.$$

Hence, we obtain that (5.20) is bounded by

$$\sup_{s:\ \|s\|_{l^2(\bar{\mu})}\le 1} \|L\| \cdot \|s\|_{l^2(\bar{\mu})} \le \|L\|,$$

which implies that

$$\|t\|_{c^1} \le \|L\|. \qquad \square$$

We now define $s^2$ (corresponding to $\dot{f}_2^{0,2} \sim \ell^2$) to be the sequence space consisting of the sequences $s = \{s_{Q_1,u_1,Q_2,u_2}\}$ of complex numbers with

(5.21)

$$\|s\|_{s^2} := \left\| \left\{ \sum_{Q_1\in\mathcal{D}_1} \sum_{Q_2\in\mathcal{D}_2} \sum_{u_1=1}^{M_{Q_1}-1} \sum_{u_2=1}^{M_{Q_2}-1} \left| s_{Q_1,u_1,Q_2,u_2} \widetilde{\chi}_{Q_1}(x_1) \widetilde{\chi}_{Q_2}(x_2) \right|^2 \right\}^{1/2} \right\|_{L^2(X_1\times X_2)} < \infty.$$

Then we have the following density argument.

**Proposition 5.11.** *The set $s^1 \cap s^2$ is dense in $s^1$ in terms of the $s^1$ norm. Moreover, $c^1 \cap s^2$ is dense in $c^1$ in terms of the weak type convergence as follows: for each $t \in c^1$, there exists $\{t_n\}_n \subset c^1 \cap s^2$ such that $\langle s, t_n\rangle \to \langle s, t\rangle$ when $n$ tends to $\infty$, for all $s \in s^1$, where $\langle s, t\rangle$ is the inner product defined as in (5.17).*

*Proof.* For each sequence $s = \{s_{Q_1,u_1,Q_2,u_2}\} \in s^1$, we define a truncated sequence $s_n$ as follows:

$$s_n := \left\{ s^n_{Q_1,u_1,Q_2,u_2} \right\}, \quad s^n_{Q_1,u_1,Q_2,u_2} := s_{Q_1,u_1,Q_2,u_2} \chi_{A_n}(Q_1,Q_2), \tag{5.22}$$

where $A_n = \{(Q_1,Q_2) : k_1,k_2 \in [-n,n], Q_1 \in \mathscr{D}_{k_1}, Q_1 \subset B(x_1^0,n), Q_2 \in \mathscr{D}_{k_2}, Q_2 \subset B(x_2^0,n)\}$ for each positive integer $n$, where $x_1^0$ and $x_2^0$ are arbitrary fixed points in $X_1$ and $X_2$, respectively.

Then $s_n$ is in $s^1$ with $\|s_n\|_{s^1} \le \|s\|_{s^1}$. We also have that $s \in s^2$ with

$$\|s_n\|_{s^2} = \left( \sum_{k_1,k_2\in[-n,n], Q_1\in\mathscr{D}_{k_1}, Q_1\subset B(x_1^0,n), Q_2\in\mathscr{D}_{k_2}, Q_2\subset B(x_2^0,n)} |s_{Q_1,u_1,Q_2,u_2}|^2 \right)^{\frac{1}{2}} < \infty.$$

Moreover, it is easy to check that $s_n$ tends to $s$ in the sense of $s^1$ norm. As a consequence, we have that $s^1 \cap s^2$ is dense in $s^1$ in terms of the $s^1$ norm.

Now we turn to $c^1$. For each $t = \{t_{Q_1,u_1,Q_2,u_2}\} \in c^1$, we define $t_n$ analogously to $s_n$. Then, similarly, we have $t_n \in c^1 \cap s^2$ with $\|t_n\|_{c^1} \le \|t\|_{c^1}$ for each positive integer $n$. Moreover, since $|\langle s, t_n\rangle|$ and $|\langle s, t\rangle|$ are both bounded by $C\|s\|_{s^1}\|t\|_{c^1}$ for all $s \in s^1$, we have

$$\langle s, t_n\rangle = \sum_{k_1,k_2\in[-n,n], Q_1\in\mathscr{D}_{k_1}, Q_1\subset B(x_1^0,n), Q_2\in\mathscr{D}_{k_2}, Q_2\subset B(x_2^0,n)} \sum_{u_1=1}^{M_{Q_1}-1} \sum_{u_2=1}^{M_{Q_2}-1} s_{Q_1,u_1,Q_2,u_2} t_{Q_1,u_1,Q_2,u_2}$$

$$\to \sum_{Q_1\in\mathcal{D}_1}\sum_{Q_2\in\mathcal{D}_2}\sum_{u_1=1}^{M_{Q_1}-1}\sum_{u_2=1}^{M_{Q_2}-1} s_Q\, t_Q = \langle s, t\rangle$$

as $n\to+\infty$, which shows that $t_n$ tends to $t$ in the weak type convergence. $\square$

Now we define the lifting and projection operators as follows.

**Definition 5.12.** For functions $f\in L^2(X_1\times X_2)$, define the lifting operator $T_L$ by

$$T_L(f)=\big\{\langle f, h_{Q_1}^{u_1}h_{Q_2}^{u_2}\rangle\big\}_{Q_1,u_1,Q_2,u_2}. \tag{5.23}$$

**Definition 5.13.** For sequences $s=\{s_{Q_1,u_1,Q_2,u_2}\}$, define the projection operator $T_P$ by

$$T_P(s)=\sum_{Q_1\in\mathcal{D}_1}\sum_{Q_2\in\mathcal{D}_2}\sum_{u_1=1}^{M_{Q_1}-1}\sum_{u_2=1}^{M_{Q_2}-1} s_{Q_1,u_1,Q_2,u_2}h_{Q_1}^{u_1}(x_1)h_{Q_2}^{u_2}(x_2). \tag{5.24}$$

Then it is clear that

$$f=T_P\circ T_L(f)$$

in the sense of $L^2(X_1\times X_2)$.

Next we give the following two auxiliary propositions, which show that the lifting operator $T_L$ maps $H^1_{d,d}(X_1\times X_2)\cap L^2(X_1\times X_2)$ to $s^1\cap s^2$, and maps $\mathrm{BMO}_{d,d}(X_1\times X_2)\cap L^2(X_1\times X_2)$ to $c^1\cap s^2$.

**Proposition 5.14.** *For all $f\in L^2(X_1\times X_2)\cap H^1_{d,d}(X_1\times X_2)$, we have*

$$\|T_L(f)\|_{s^1}\lesssim \|f\|_{H^1_{d,d}(X_1\times X_2)}. \tag{5.25}$$

**Proposition 5.15.** *For all $f\in \mathrm{BMO}_{d,d}(X_1\times X_2)$, we have*

$$\|T_L(f)\|_{c^1}\lesssim \mathcal{C}_1^d(f). \tag{5.26}$$

**Proposition 5.16.** *For all $s\in s^1\cap s^2$, we have*

$$\|T_P(s)\|_{H^1_{d,d}(X_1\times X_2)}\lesssim \|s\|_{s^1}. \tag{5.27}$$

**Proposition 5.17.** *For all $t\in c^1\cap s^2$, we have*

$$\mathcal{C}_1^d(T_P(t))\lesssim \|t\|_{c^1}. \tag{5.28}$$

These four propositions follow directly from the definitions of the Hardy spaces $H^1_{d,d}(X_1\times X_2)$ and the sequence space $s^1$, and from the definitions of the BMO space $\mathrm{BMO}_{d,d}(X_1\times X_2)$ and the sequence space $c^1$.

**Theorem 5.18.** *For each $\varphi\in \mathrm{BMO}_{d,d}(X_1\times X_2)$, the linear functional given by*

$$\ell(f)=\langle\varphi,f\rangle:=\int \varphi(x_1,x_2)f(x_1,x_2)\,d\mu_1(x_1)d\mu_2(x_2), \tag{5.29}$$

*initially defined on $H^1_{d,d}(X_1\times X_2)\cap L^2(X_1\times X_2)$, has a unique bounded extension to $H^1_{d,d}(X_1\times X_2)$ with*

$$\|\ell\|\le C\mathcal{C}_1^d(\varphi).$$

*Conversely, every bounded linear functional $\ell$ on $H^1_{d,d}(X_1\times X_2)\cap L^2(X_1\times X_2)$ can be realized in the form of (5.29), i.e., there exists $\varphi\in \mathrm{BMO}_{d,d}(X_1\times X_2)$ with such that (5.29) holds for all $f\in H^1_{d,d}(X_1\times X_2)\cap L^2(X_1\times X_2)$, and*

$$\mathcal{C}_1^d(\varphi)\le C\|\ell\|.$$

*Proof.* Suppose $\varphi \in \mathrm{BMO}_{d,d}(X_1 \times X_2)$ and we define the linear functional $\ell$ as in (5.29) for $f \in H^1_{d,d}(X_1 \times X_2) \cap L^2(X_1 \times X_2)$. Then by the Haar expansion we have

$$\begin{aligned} \ell(f) &= \int \varphi(x_1,x_2) \sum_{Q_1\in\mathcal{D}_1}\sum_{Q_2\in\mathcal{D}_2}\sum_{u_1=1}^{M_{Q_1}-1}\sum_{u_2=1}^{M_{Q_2}-1} \langle f, h_{Q_1}^{u_1}h_{Q_2}^{u_2}\rangle h_{Q_1}^{u_1}(x_1)h_{Q_2}^{u_2}(x_2)\, d\mu_1(x_1)d\mu_2(x_2) \\ &= \sum_{Q_1\in\mathcal{D}_1}\sum_{Q_2\in\mathcal{D}_2}\sum_{u_1=1}^{M_{Q_1}-1}\sum_{u_2=1}^{M_{Q_2}-1} \langle f, h_{Q_1}^{u_1}h_{Q_2}^{u_2}\rangle\langle \varphi, h_{Q_1}^{u_1}h_{Q_2}^{u_2}\rangle \\ &\le C\Big\|\big\{\langle f, h_{Q_1}^{u_1}h_{Q_2}^{u_2}\rangle\big\}\Big\|_{s^1}\Big\|\big\{\langle \varphi, h_{Q_1}^{u_1}h_{Q_2}^{u_2}\rangle\big\}\Big\|_{c^1}, \end{aligned}$$

where the inequality follows from (5.18).

As a consequence, we obtain that

$$|\ell(f)| \;\le\; C\big\|T_L(f)\big\|_{s^1}\big\|T_L(\varphi)\big\|_{c^1} \le C\|f\|_{H^1_{d,d}(X_1\times X_2)}\mathcal{C}_1^d(\varphi),$$

which implies that $\varphi$ is a bounded linear functional on $H^1_{d,d}(X_1 \times X_2) \cap L^2(X_1 \times X_2)$ and hence has a unique bounded extension to $H^1_{d,d}(X_1 \times X_2)$ with $\|\ell\| \le C\mathcal{C}_1^d(\varphi)$ since $H^1_{d,d}(X_1 \times X_2) \cap L^2(X_1 \times X_2)$ is dense in $H^1_{d,d}(X_1 \times X_2)$.

Conversely, suppose $\ell$ is a bounded linear functional on $H^1_{d,d}(X_1 \times X_2) \cap L^2(X_1 \times X_2)$. We set $\ell_1 = \ell \circ T_P$. Then it is obvious that $\ell_1$ is a bounded linear functional on $s^1 \cap s^2$.

Note that $f = T_P \circ T_L(f)$ for every $f \in L^2(X_1 \times X_2)$. We have

$$\ell(f) = \ell\big(T_P \circ T_L(f)\big) = \ell_1\big(T_L(f)\big), \quad \text{where } \ell_1 = \ell \circ T_P.$$

Now by the duality of $s^1$ with $c^1$, we obtain that there exists $t \in c^1$ with $\|t\|_{c^1} \le \|\ell\|$ such that

$$\ell_1(s) = \langle t, s\rangle$$

for all $s \in s^1$. Hence, we have

$$\begin{aligned} \ell_1\big(T_L(f)\big) &= \langle t, T_L(f)\rangle \\ &= \sum_{Q_1\in\mathcal{D}_1}\sum_{Q_2\in\mathcal{D}_2}\sum_{u_1=1}^{M_{Q_1}-1}\sum_{u_2=1}^{M_{Q_2}-1} t_{Q_1,u_1,Q_2,u_2}\langle f, h_{Q_1}^{u_1}h_{Q_2}^{u_2}\rangle \\ &= \Big\langle f, \sum_{Q_1\in\mathcal{D}_1}\sum_{Q_2\in\mathcal{D}_2}\sum_{u_1=1}^{M_{Q_1}-1}\sum_{u_2=1}^{M_{Q_2}-1} t_{Q_1,u_1,Q_2,u_2}, h_{Q_1}^{u_1}h_{Q_2}^{u_2}\Big\rangle \\ &= \big\langle f, T_P(t)\big\rangle, \end{aligned}$$

which implies that

$$\ell(f) = \big\langle f, T_P(t)\big\rangle$$

and moreover,

$$\mathcal{C}_1^d(T_P(t)) \le C\|t\|_{c^1} \le C\|\ell\|. \qquad \square$$

## 6. Dyadic structure theorems for $H^1(\widetilde{X})$ and $\mathrm{BMO}(\widetilde{X})$

In this final section, we prove our dyadic structure results. Namely, the product Hardy space $H^1(\widetilde{X})$ is a sum of finitely many dyadic product Hardy spaces, and the product $\mathrm{BMO}(\widetilde{X})$ space is an intersection of finitely many dyadic product BMO spaces. We also establish the atomic decomposition of the dyadic product Hardy spaces $H^1_{\mathcal{D}_1^{t_1},\mathcal{D}_2^{t_2}}(\widetilde{X})$, which plays a key role in the proofs of the dyadic structure results. As in the previous section, these results all hold for $n$ parameters, but for notational simplicity we write in terms of two parameters.

**Theorem 6.1.** *Let $\widetilde{X} = X_1 \times X_2$ be a product space of homogeneous type. Then*

$$H^1(X_1 \times X_2) = \sum_{t_1=1}^{T_1} \sum_{t_2=1}^{T_2} H^1_{\mathcal{D}_1^{t_1}, \mathcal{D}_2^{t_2}}(X_1 \times X_2).$$

*The corresponding result holds for $n$ parameters.*

As a consequence of Theorem 6.1, we obtain the following by duality.

**Theorem 6.2.** *Let $\widetilde{X} = X_1 \times X_2$ be a product space of homogeneous type. Then*

$$\mathrm{BMO}(\widetilde{X}) = \bigcap_{t_1=1}^{T_1} \bigcap_{t_2=1}^{T_2} \mathrm{BMO}_{\mathcal{D}_1^{t_1}, \mathcal{D}_2^{t_2}}(\widetilde{X}).$$

*The corresponding result holds for $n$ parameters.*

To do this, we first give the definition of atoms for the dyadic product Hardy space $H^1_{\mathcal{D}_1^{t_1}, \mathcal{D}_2^{t_2}}(\widetilde{X})$ and then provide the atomic decomposition, where $t_1 = 1, \ldots, T_1$; $t_2 = 1, \ldots, T_2$. For brevity, we will drop the reference to the parameters $(t_1, t_2)$ until we return to the proof of Theorem 6.1.

**Definition 6.3.** A function $a(x_1, x_2)$ defined on $\widetilde{X}$ is called a *dyadic atom of* $H^1_{\mathcal{D}_1, \mathcal{D}_2}(\widetilde{X})$ if $a(x_1, x_2)$ satisfies:

(1) $\operatorname{supp} a \subset \Omega$, where $\Omega$ is an open set of $\widetilde{X}$ with finite measure;
(2) $\|a\|_{L^2} \le \mu(\Omega)^{-1/2}$;
(3) $a$ can be further decomposed into rectangle atoms $a_R$ associated to dyadic rectangle $R = Q_1 \times Q_2 \in \mathcal{D}_1 \times \mathcal{D}_2$, satisfying the following three conditions:
   (i) $\operatorname{supp} a_R \subset R$ (*localization*);
   (ii) $\int_{X_1} a_R(x_1, x_2)\, d\mu_1(x_1) = 0$ for a.e. $x_2 \in X_2$ and $\int_{X_2} a_R(x_1, x_2)\, d\mu_2(x_2) = 0$ for a.e. $x_1 \in X_1$ (*cancellation*);
   (iii) $a = \sum_{R \in m(\Omega)} a_R$ and $\sum_{R \in m(\Omega)} \|a_R\|^2_{L^2(\widetilde{X})} \le \mu(\Omega)^{-1}$ (*size*).

**Remark 6.4.** We note that the only difference between the dyadic atoms defined here and the "continuous" atoms defined earlier in Definition 5.3 is that here the rectangle atoms $a_R$ are supported on the rectangles $R$, while for the continuous atoms, the rectangle atoms $a_R$ are supported on a dilate of $R$.

Next we provide the atomic decomposition for $H^1_{\mathcal{D}_1, \mathcal{D}_2}(\widetilde{X})$.

**Theorem 6.5.** *If $f \in H^1_{\mathcal{D}_1, \mathcal{D}_2}(\widetilde{X}) \cap L^2(\widetilde{X})$, then*

$$f = \sum_k \lambda_k a_k,$$

*where each $a_k$ is an atom of $H^1_{\mathcal{D}_1, \mathcal{D}_2}(\widetilde{X})$ as in Definition 6.3, and $\sum_k |\lambda_k| \le C \|f\|_{H^1_{\mathcal{D}_1, \mathcal{D}_2}(\widetilde{X})}$.*

*Conversely, suppose $f := \sum_k \lambda_k a_k$ where each $a_k$ is an atom of $H^1_{\mathcal{D}_1, \mathcal{D}_2}(\widetilde{X})$ as in Definition 6.3, and $\sum_k |\lambda_k| \le C < \infty$, then $f \in H^1_{\mathcal{D}_1, \mathcal{D}_2}(\widetilde{X})$ and $\|f\|_{H^1_{\mathcal{D}_1, \mathcal{D}_2}(\widetilde{X})} \le C \sum_k |\lambda_k|$.*

*Proof.* Suppose $f \in H^1_{\mathcal{D}_1, \mathcal{D}_2}(\widetilde{X})$. Then $S_{d,d}(f) \in L^1(\widetilde{X})$. For each $k \in \mathbb{Z}$, we now define

$$\begin{aligned}
\Omega_k &= \{(x_1, x_2) \in \widetilde{X} : S_{d,d}(f)(x_1, x_2) > 2^k\}, \\
B_k &= \Big\{R = Q_1 \times Q_2 \in \mathcal{D}_1 \times \mathcal{D}_2 : \mu(\Omega_k \cap R) > \frac{1}{2}\mu(R), \text{ and } \mu(\Omega_{k+1} \cap R) \le \frac{1}{2}\mu(R)\Big\} \\
\widetilde{\Omega}_k &= \{(x_1, x_2) \in \widetilde{X} : M_s(\chi_{\Omega_k})(x_1, x_2) > \widetilde{C}\},
\end{aligned}$$

where $M_s$ is the strong maximal function on $\widetilde{X}$ and $\widetilde{C}$ is a constant to be determined later.

Now by the Haar expansion convergent in $L^2(\widetilde{X})$, and since each rectangle $R \in \mathcal{D}_1 \times \mathcal{D}_2$ belongs to exactly one set $B_k$, we have

$$\begin{aligned} f &= \sum_{Q_1\times Q_2\in\mathcal{D}_1\times\mathcal{D}_2} \sum_{u_1=1}^{M_{Q_1}} \sum_{u_2=1}^{M_{Q_2}} \langle f,\ h_{Q_1}^{u_1} h_{Q_2}^{u_2}\rangle h_{Q_1}^{u_1} h_{Q_2}^{u_2} \\ &= \sum_k \sum_{Q_1\times Q_2\in B_k} \sum_{u_1=1}^{M_{Q_1}} \sum_{u_2=1}^{M_{Q_2}} \langle f,\ h_{Q_1}^{u_1} h_{Q_2}^{u_2}\rangle h_{Q_1}^{u_1} h_{Q_2}^{u_2} \\ &=: \sum_k \lambda_k a_k, \end{aligned}$$

where

$$a_k(x_1,x_2) = \frac{1}{\lambda_k} \sum_{R=Q_1\times Q_2\in B_k} \sum_{u_1=1}^{M_{Q_1}} \sum_{u_2=1}^{M_{Q_2}} \langle f,\ h_{Q_1}^{u_1} h_{Q_2}^{u_2}\rangle h_{Q_1}^{u_1}(x_1) h_{Q_2}^{u_2}(x_2)$$

and

$$\lambda_k = \Big( \sum_{R=Q_1\times Q_2\in B_k} \sum_{u_1=1}^{M_{Q_1}} \sum_{u_2=1}^{M_{Q_2}} \big|\langle f,\ h_{Q_1}^{u_1} h_{Q_2}^{u_2}\rangle\big|^2 \Big)^{1/2} \mu(\widetilde{\Omega}_k)^{1/2}$$

To see that the atomic decomposition $\sum_{k=-\infty}^{\infty} \lambda_k a_k$ converges to $f$ in the $L^2$ norm, we only need to show that $\|\sum_{|k|>\ell} \lambda_k a_k\|_2 \to 0$ as $\ell \to \infty$. This follows from the following duality argument. Let $g \in L^2$ with $\|g\|_2 = 1$. Then

$$\big\| \sum_{|k|>\ell} \lambda_k a_k \big\|_2 = \sup_{\|g\|_2=1} \big|\langle \sum_{|k|>\ell} \lambda_k a_k, g\rangle\big|.$$

Note that

$$\big\langle \sum_{|k|>\ell} \lambda_k a_k, g \big\rangle = \sum_{|k|>\ell} \sum_{R=Q_1\times Q_2\in B_k} \sum_{u_1=1}^{M_{Q_1}} \sum_{u_2=1}^{M_{Q_2}} \langle f,\ h_{Q_1}^{u_1} h_{Q_2}^{u_2}\rangle \langle g, h_{Q_1}^{u_1} h_{Q_2}^{u_2}\rangle.$$

Applying Hölder's inequality gives

$$\begin{aligned} \big|\big\langle \sum_{|k|>\ell} \lambda_k a_k, g\big\rangle\big| &\le \Big( \sum_{|k|>\ell} \sum_{R=Q_1\times Q_2\in B_k} \sum_{u_1=1}^{M_{Q_1}} \sum_{u_2=1}^{M_{Q_2}} \big|\langle f,\ h_{Q_1}^{u_1} h_{Q_2}^{u_2}\rangle\big|^2 \Big)^{1/2} \\ &\quad \times \Big( \sum_{|k|>\ell} \sum_{R=Q_1\times Q_2\in B_k} \sum_{u_1=1}^{M_{Q_1}} \sum_{u_2=1}^{M_{Q_2}} \big|\langle g,\ h_{Q_1}^{u_1} h_{Q_2}^{u_2}\rangle\big|^2 \Big)^{1/2} \end{aligned}$$

Note again that

$$\Big( \sum_{|k|>\ell} \sum_{R=Q_1\times Q_2\in B_k} \sum_{u_1=1}^{M_{Q_1}} \sum_{u_2=1}^{M_{Q_2}} \big|\langle g,\ h_{Q_1}^{u_1} h_{Q_2}^{u_2}\rangle\big|^2 \Big)^{1/2} \le C\|g\|_2 \le C,$$

and

$$\Big( \sum_{|k|>\ell} \sum_{R=Q_1\times Q_2\in B_k} \sum_{u_1=1}^{M_{Q_1}} \sum_{u_2=1}^{M_{Q_2}} \big|\langle f,\ h_{Q_1}^{u_1} h_{Q_2}^{u_2}\rangle\big|^2 \Big)^{1/2}$$

tends to zero as $\ell$ tends to infinity. This implies that $\|\sum_{|k|>\ell} \lambda_k a_k\|_2 \to 0$ as $\ell \to \infty$ and hence, the series $\sum_{k=-\infty}^{\infty} \lambda_k a_k$ converges to $f$ in the $L^2$ norm.

Next, it is easy to see that for each $k$, $\operatorname{supp} a_k \subset \widetilde{\Omega}_k$ since $R \in B_k$ implies that $R \subset \widetilde{\Omega}_k$ when we choose $\widetilde{C} < 1/2$. Hence we see that condition (1) holds.

Now for each $k$, from the definition of $\lambda_k$ and the Hölder's inequality, we have

$$\begin{aligned}
\|a_k\|_{L^2(\widetilde{X})} &= \sup_{g:\ \|g\|_{L^2(\widetilde{X})}=1} \big|\langle a_k, g\rangle\big| = \frac{1}{\lambda_k}\Big|\sum_{R=Q_1\times Q_2\in B_k}\ \sum_{u_1=1}^{M_{Q_1}}\sum_{u_2=1}^{M_{Q_2}} \langle f,\ h_{Q_1}^{u_1}h_{Q_2}^{u_2}\rangle\langle g, h_{Q_1}^{u_1}h_{Q_2}^{u_2}\rangle\Big| \\
&\le \frac{1}{\lambda_k}\Big(\sum_{R=Q_1\times Q_2\in B_k}\ \sum_{u_1=1}^{M_{Q_1}}\sum_{u_2=1}^{M_{Q_2}} \big|\langle f,\ h_{Q_1}^{u_1}h_{Q_2}^{u_2}\rangle\big|^2\Big)^{1/2} \\
&\quad\times\Big(\sum_{R=Q_1\times Q_2\in B_k}\ \sum_{u_1=1}^{M_{Q_1}}\sum_{u_2=1}^{M_{Q_2}} \big|\langle g,\ h_{Q_1}^{u_1}h_{Q_2}^{u_2}\rangle\big|^2\Big)^{1/2} \\
&\le \mu(\widetilde{\Omega}_k)^{-1/2},
\end{aligned}$$

which implies that condition (2) holds.

It remains to check that $a_k$ satisfies condition (3) of Definition 6.3. To see this, we can further decompose $a_k$ as

$$a_k = \sum_{\overline{R}\in m(\widetilde{\Omega}_k)} a_{k,\overline{R}},$$

where $m(\widetilde{\Omega}_k)$ denotes the collection of maximal dyadic rectangles $\overline{R} \in \mathcal{D}_1 \times \mathcal{D}_2$ contained in $\widetilde{\Omega}_k$, and

$$a_{k,\overline{R}}(x_1,x_2) = \frac{1}{\lambda_k}\sum_{R=Q_1\times Q_2\in B_k,\ R\subset\overline{R}}\ \sum_{u_1=1}^{M_{Q_1}}\sum_{u_2=1}^{M_{Q_2}} \langle f,\ h_{Q_1}^{u_1}h_{Q_2}^{u_2}\rangle h_{Q_1}^{u_1}(x_1)h_{Q_2}^{u_2}(x_2).$$

By definition we can verify that

$$\operatorname{supp} a_{k,\overline{R}} \subset \overline{R}.$$

By the facts that $\int h_{Q_1}^{u_1}(x_1)d\mu_1(x_1) = \int h_{Q_2}^{u_2}(x_2)d\mu_2(x_2) = 0$, we have, for a.e. $x_2 \in X_2$,

$$\int_{X_1} a_{k,\overline{R}}(x_1,x_2)d\mu_1(x_1) = 0$$

and for a.e. $x_1 \in X_1$,

$$\int_{X_2} a_{k,\overline{R}}(x_1,x_2)d\mu_2(x_2) = 0,$$

which yield that the conditions (i) and (ii) of (3) in Definition 6.3 hold. It remains to show that $a_k$ satisfies the condition (iii) of (3).

To see this, we first note that

$$\|a_{k,\overline{R}}\|_2 = \frac{1}{\lambda_k}\Big\|\sum_{R=Q_1\times Q_2\in B_k,\ R\subset\overline{R}}\ \sum_{u_1=1}^{M_{Q_1}}\sum_{u_2=1}^{M_{Q_2}} \langle f,\ h_{Q_1}^{u_1}h_{Q_2}^{u_2}\rangle h_{Q_1}^{u_1}(x_1)h_{Q_2}^{u_2}(x_2)\Big\|_2.$$

From the definition of $\lambda_k$, by applying the same argument as for the estimates of $\|a_k\|_2$, we can obtain that

$$\sum_{\overline{R}\in m(\widetilde{\Omega}_k)} \|a_{k,\overline{R}}\|_{L^2}^2 \le \mu(\widetilde{\Omega}_k)^{-1/2},$$

which implies that condition (iii) holds.

We now prove that $\sum_k |\lambda_k| \le C\|f\|_{H^1_{\mathcal{D}_1,\mathcal{D}_2}(\widetilde{X})}$.

First note that by definition of $\Omega_{k+1}$ and Remark 5.10,

$$\int_{\widetilde{\Omega}_k\setminus\Omega_{k+1}} S_{d,d}(f)(x_1,x_2)^2\, d\mu_1(x_1)d\mu_2(x_2) \le 2^{2(k+1)}\mu(\widetilde{\Omega}_k) \le C2^{2(k+1)}\mu(\Omega_k).$$

Moreover, by the Monotone Convergence Theorem and by Remark 5.10, we have

$$\begin{aligned}
&\int_{\widetilde{\Omega}_k\setminus\Omega_{k+1}} S_{d,d}(f)(x_1,x_2)^2\, d\mu_1(x_1)d\mu_2(x_2)\\
&\qquad \ge \int_{\widetilde{\Omega}_k\setminus\Omega_{k+1}} \sum_{R=Q_1\times Q_2\in B_k}\sum_{u_1=1}^{M_{Q_1}}\sum_{u_2=1}^{M_{Q_2}} \big|\langle f, h_{Q_1}^{u_1}h_{Q_2}^{u_2}\rangle\big|^2 \widetilde{\chi}_{Q_1}(x_1)\widetilde{\chi}_{Q_2}(x_2)\, d\mu_1(x_1)d\mu_2(x_2)\\
&\qquad \ge \sum_{R=Q_1\times Q_2\in B_k}\sum_{u_1=1}^{M_{Q_1}}\sum_{u_2=1}^{M_{Q_2}} \big|\langle f, h_{Q_1}^{u_1}h_{Q_2}^{u_2}\rangle\big|^2 \frac{\mu(\widetilde{\Omega}_k\setminus\Omega_{k+1}\cap R)}{\mu(R)}\\
&\qquad \ge 1/2 \sum_{R=Q_1\times Q_2\in B_k}\sum_{u_1=1}^{M_{Q_1}}\sum_{u_2=1}^{M_{Q_2}} \big|\langle f, h_{Q_1}^{u_1}h_{Q_2}^{u_2}\rangle\big|^2,
\end{aligned}$$

since $R\in B_k$ implies that $\mu((\widetilde{\Omega}_{k+1}\setminus\Omega_{k+1})\cap R)\le \mu(R)/2$.

From the definition of $\lambda_k$, we have

$$\begin{aligned}
\sum_k |\lambda_k| &= C\sum_k\Big(\sum_{R=Q_1\times Q_2\in B_k}\sum_{u_1=1}^{M_{Q_1}}\sum_{u_2=1}^{M_{Q_2}} \big|\langle f,\ h_{Q_1}^{u_1}h_{Q_2}^{u_2}\rangle\big|^2\Big)^{1/2}\mu(\widetilde{\Omega}_k)^{1/2}\\
&\le C\sum_k 2^{k+1}\mu(\widetilde{\Omega}_k)^{1/2}\mu(\widetilde{\Omega}_k)^{1/2}\\
&\le C\sum_k 2^k\mu(\Omega_k)\\
&\le C\|f\|_{H^1_{\mathcal{D}_1,\mathcal{D}_2}(\widetilde{X})}.
\end{aligned}$$

Conversely, suppose $f := \sum_k \lambda_k a_k$ where each $a_k$ is an atom of $H^1_{\mathcal{D}_1,\mathcal{D}_2}(\widetilde{X})$ as in Definition 6.3, and $\sum_k|\lambda_k|\le C<\infty$.

Then

$$S_{d,d}(f)(x_1,x_2) = S_{d,d}\Big(\sum_k \lambda_k a_k\Big)(x_1,x_2) \le \sum_k |\lambda_k| S_{d,d}\big(a_k\big)(x_1,x_2).$$

It suffices to estimate the $L^1(\widetilde{X})$ norm of $S_{d,d}\big(a_k\big)(x_1,x_2)$. For the sake of simplicity, we drop the subscript $k$ for the atom and now estimate $\|S_{d,d}(a)\|_{L^1(\widetilde{X})}$, where $a$ is a dyadic atom as in Definition 6.3 with support $\Omega$.

Now let $\bar{\Omega} = \cup_{R\in m(\Omega)}100R$. We have

$$\|S_{d,d}\big(a\big)\|_{L^1(\widetilde{X})} = \|S_{d,d}\big(a\big)\|_{L^1(\bar{\Omega})} + \|S_{d,d}\big(a\big)\|_{L^1(\bar{\Omega}^c)}.$$

It is obvious that $\|S_{d,d}\big(a\big)\|_{L^1(\bar{\Omega})} \le \mu(\bar{\Omega})^{1/2}\|S_{d,d}\big(a\big)\|_{L^2(\bar{\Omega})} \le C\mu(\bar{\Omega})^{1/2}\|a\|_{L^2(\widetilde{X})} \le C$, where $C$ is a positive constant independent of $a$.

From the definition of the atom $a$, we have

$$\|S_{d,d}\big(a\big)\|_{L^1(\bar{\Omega}^c)} \le \sum_{R\in m(\Omega)}\int_{(100R)^c} S_{d,d}(a_R)(x_1,x_2)\, d\mu_1(x_1)\, d\mu_1(dx_2).$$

For each rectangle $R = Q_1 \times Q_2$ that appears in this sum, we have from the definition of the square function $S_{d,d}(a_R)$ that

$$S_{d,d}(a_R)(x_1,x_2) := \Big\{ \sum_{Q_1'\in\mathcal{D}_1} \sum_{Q_2'\in\mathcal{D}_2} \sum_{u_1=1}^{M_{Q_1'}-1} \sum_{u_2=1}^{M_{Q_2'}-1} \big|\langle a_R, h_{Q_1'}^{u_1} h_{Q_2'}^{u_2}\rangle \widetilde{\chi}_{Q_1'}(x_1)\widetilde{\chi}_{Q_2'}(x_2)\big|^2 \Big\}^{1/2}.$$

Now if $Q_2$ is a child of $Q_2'$, then $h_{Q_2'}$ is constant on $Q_2$, and so

$$\begin{aligned}\langle a_R, h_{Q_1'}^{u_1} h_{Q_2'}^{u_2}\rangle &= \int_{Q_1}\int_{Q_2} a_R(x_1,x_2) h_{Q_1'}^{u_1}(x_1) h_{Q_2'}^{u_2}(x_2)\, d\mu_1(x_1)\, d\mu_2(x_2) \\ &= \int_{Q_1} C\Big[\int_{Q_2} a_R(x_1,x_2)\, d\mu_2(x_2)\Big] h_{Q_1'}^{u_1}(x_1)\, d\mu_1(x_1) \\ &= 0\end{aligned}$$

by the cancellation property of $a_R$. Similarly, if $Q_1$ is a child of $Q_1'$, then $\langle a_R, h_{Q_1'}^{u_1} h_{Q_2'}^{u_2}\rangle = 0$. Clearly if $Q_i \cap Q_i' = \emptyset$ for $i = 1$ or $i = 2$ then the inner product is also zero. Thus

$$S_{d,d}(a_R)(x_1,x_2) = \Big\{ \sum_{\substack{Q_1'\in\mathcal{D}_1,\\ Q_1'\subseteq Q_1}} \sum_{\substack{Q_2'\in\mathcal{D}_2,\\ Q_2'\subseteq Q_2}} \sum_{u_1=1}^{M_{Q_1'}-1} \sum_{u_2=1}^{M_{Q_2'}-1} \big|\langle a_R, h_{Q_1'}^{u_1} h_{Q_2'}^{u_2}\rangle \widetilde{\chi}_{Q_1'}(x_1)\widetilde{\chi}_{Q_2'}(x_2)\big|^2 \Big\}^{1/2}.$$

This implies that $S_{d,d}(a_R)(x_1,x_2)$ is supported in $R$. Thus,

$$\int_{(100R)^c} S_{d,d}(a_R)(x_1,x_2)\, d\mu_1(x_1)\, d\mu_2(x_2) = 0.$$

As a consequence, we have

$$\|S_{d,d}(a)\|_{L^1(\widetilde{X})} \le C,$$

where $C$ is a positive constant independent of $a$. Then

$$\|S_{d,d}(f)\|_{L^1(\widetilde{X})} \le \sum_k |\lambda_k| \|S_{d,d}(a_k)\|_{L^1(\widetilde{X})} \le C\sum_k |\lambda_k|. \qquad \square$$

It remains to prove Theorem 6.1.

*Proof of Theorem 6.1.* First we prove the inclusion $H^1(X_1\times X_2) \subset \sum_{t_1=1}^{T_1}\sum_{t_2=1}^{T_2} H^1_{\mathcal{D}_1^{t_1},\mathcal{D}_2^{t_2}}(X_1\times X_2)$. Suppose $f \in H^1(X_1\times X_2)\cap L^2(X_1\times X_2)$. From Theorem 5.4, we obtain that

$$f = \sum_j \lambda_j a_j,$$

where each $a_j$ is an $H^1(X_1\times X_2)$ atom as defined in Definition 5.3, and $\sum_j |\lambda_j| \le C\|f\|_{H^1(X_1\times X_2)}$.

Thus, from the Definition 5.3, we see that $a_j$ is supported in an open set $\Omega_j \in X_1\times X_2$ with finite measure. Moreover, $a_j = \sum_{R\in m(\Omega_j)} a_{j,R}$, where each $a_{j,R}$ is supported in $\overline{C}R$. Since $R$ is a dyadic rectangle, we have $R = Q_{\alpha_1}^{k_1}\times Q_{\alpha_2}^{k_2}$ for some $k_1,k_2\in\mathbb{Z}$ and $\alpha_1\in\mathscr{A}_{k_1}$ and $\alpha_2\in\mathscr{A}_{k_2}$. Thus, from the definition of the dyadic cubes in Section 2.1, we further have $R \subset B(x_{\alpha_1}^{k_1}, C_1\delta^{k_1})\times B(x_{\alpha_2}^{k_2}, C_2\delta^{k_2})$. As a consequence, we obtain that $\mathrm{supp} a_{j,R}$ is contained in $B(x_{\alpha_1}^{k_1}, \overline{C}C_1\delta^{k_1})\times B(x_{\alpha_2}^{k_2}, \overline{C}C_2\delta^{k_2}) = \overline{C}R$.

Next, from Lemma 4.12 in [HK], we see that $B(x_{\alpha_1}^{k_1}, \overline{C}C_1\delta^{k_1})$ must be contained in some $Q_1^{t_1}(\overline{k}_1,\beta_1)\in\mathcal{D}^{t_1}$ for some $t_1$ in $\{1,2,\dots,T_1\}$. Similarly, $B(x_{\alpha_2}^{k_2}, \overline{C}C_2\delta^{k_2})$ must be contained in $Q_2^{t_2}(\overline{k}_2,\beta_2)\in\mathcal{D}^{t_2}$ for some $t_2$ in $\{1,2,\dots,T_2\}$. Moreover, there exists a constant $\widetilde{C}$ such that $\mu(Q_1^{t_1}(\overline{k}_1,\beta_1)) \le \widetilde{C}\mu(B(x_{\alpha_1}^{k_1}, \overline{C}C_1\delta^{k_1}))$ and that $\mu(Q_2^{t_2}(\overline{k}_2,\beta_2)) \le \widetilde{C}\mu(B(x_{\alpha_2}^{k_2}, \overline{C}C_2\delta^{k_2}))$. As a consequence, $\mathrm{supp}\, a_{j,R}$ is contained in $Q_1^{t_1}(\overline{k}_1,\beta_1)\times Q_2^{t_2}(\overline{k}_2,\beta_2)$.

Thus, we can divide the summation $a_j = \sum_{R\in m(\Omega_j)} a_{j,R}$ as follows:

$$a_j = \sum_{t_1=1}^{T_1}\sum_{t_2=1}^{T_2} \sum_{R\in m(\Omega_j), \overline{C}R\subset Q_1^{t_1}(\overline{k}_1,\beta_1)\times Q_2^{t_2}(\overline{k}_2,\beta_2)} a_{j,R}.$$

We now set

$$a_{j,t_1,t_2} = \sum_{R\in m(\Omega_j), \overline{C}R\subset Q_1^{t_1}(\overline{k}_1,\beta_1)\times Q_2^{t_2}(\overline{k}_2,\beta_2)} a_{j,R}.$$

Then we can verify that $a_{j,t_1,t_2}$ is an $H^1_{\mathcal{D}^{t_1},\mathcal{D}^{t_2}}(X_1\times X_2)$ atom. Moreover, $f_{t_1,t_2} := \sum_j \lambda_j a_{j,t_1,t_2} \in H^1_{\mathcal{D}^{t_1},\mathcal{D}^{t_2}}(X_1\times X_2)$ with

$$\|f_{t_1,t_2}\|_{H^1_{\mathcal{D}^{t_1},\mathcal{D}^{t_2}}(X_1\times X_2)} \leq \sum_j |\lambda_j| \leq C\|f\|_{H^1(X_1\times X_2)}.$$

Hence, we obtain that

$$f = \sum_j \lambda_j a_j = \sum_{t_1=1}^{T_1}\sum_{t_2=1}^{T_2} f_{t_1,t_2},$$

where $f_{t_1,t_2} \in H^1_{\mathcal{D}^{t_1},\mathcal{D}^{t_2}}(X_1\times X_2)$, and $\sum_{t_1=1}^{T_1}\sum_{t_2=1}^{T_2} \|f_{t_1,t_2}\|_{H^1_{\mathcal{D}^{t_1},\mathcal{D}^{t_2}}(X_1\times X_2)} \leq CT_1T_2\|f\|_{H^1(X_1\times X_2)}$.

This implies that $H^1(X_1\times X_2)\cap L^2(X_1\times X_2) \subset \sum_{t_1=1}^{T_1}\sum_{t_2=1}^{T_2} H^1_{\mathcal{D}_1^{t_1},\mathcal{D}_2^{t_2}}(X_1\times X_2)$. Moreover, note that $H^1(X_1\times X_2)\cap L^2(X_1\times X_2)$ is dense in $H^1(X_1\times X_2)$, we obtain that $H^1(X_1\times X_2) \subset \sum_{t_1=1}^{T_1}\sum_{t_2=1}^{T_2} H^1_{\mathcal{D}_1^{t_1},\mathcal{D}_2^{t_2}}(X_1\times X_2)$.

Second, we prove that $H^1_{\mathcal{D}_1^{t_1},\mathcal{D}_2^{t_2}}(\widetilde{X}) \subset H^1(\widetilde{X})$, for every $t_1 = 1,\ldots,T_1$; $t_2 = 1,\ldots,T_2$. To see this, we now apply the atomic decomposition for the dyadic product Hardy space $H^1_{\mathcal{D}_1^{t_1},\mathcal{D}_2^{t_2}}(\widetilde{X})$, i.e., Theorem 6.5.

For $f \in H^1_{\mathcal{D}_1^{t_1},\mathcal{D}_2^{t_2}}(\widetilde{X})\cap L^2(\widetilde{X})$, we have $f = \sum_k \lambda_k a_k$, where each $a_k$ is a dyadic atom as in Definition 6.3 and $\sum_k |\lambda_k| \leq C\|f\|_{H^1_{\mathcal{D}_1^{t_1},\mathcal{D}_2^{t_2}}(\widetilde{X})}$.

Note that the dyadic atom in Definition 6.3 is a special case of the $H^1(\widetilde{X})$ atom. We obtain that $f \in H^1(\widetilde{X})$ and that $\|f\|_{H^1(\widetilde{X})} \leq C\|f\|_{H^1_{\mathcal{D}_1^{t_1},\mathcal{D}_2^{t_2}}(\widetilde{X})}$, which yields that $H^1_{\mathcal{D}_1^{t_1},\mathcal{D}_2^{t_2}}(\widetilde{X})\cap L^2(\widetilde{X}) \subset H^1(\widetilde{X})$.

As a consequence, we obtain that $H^1_{\mathcal{D}_1^{t_1},\mathcal{D}_2^{t_2}}(\widetilde{X}) \subset H^1(\widetilde{X})$. $\square$

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