Errata/Clarifications
HARMONIC ANALYSIS. FROM FOURIER TO WAVELETS
by María Cristina Pereyra and Lesley Ward
updated August 23, 2021

- Chapter 1

1. Page 4, line -4 (last displayed formula): Let as agree that $\mathbb{N} := \{0, 1, 2, 3, \dots\}$, therefore the displayed line should be replaced by

$$\{\sin(nt) : n \in \mathbb{N} \setminus \{0\}\} \quad \cup \quad \{\cos(nt) : n \in \mathbb{N}\}$$

2. Page 7, two lines before Exercise 1.2 and in Exercise 1.2, twice replace "$\{a_n\}$" by "$\{a_n : n \in \mathbb{Z}\}$" and "$\{b_n\}$" and "$\{c_n\}$" by "$\{b_n : n \in \mathbb{N}\}$" and "$\{c_n : n \in \mathbb{N}\}$".
3. Page 8, line 12, where Taylor coefficients are displayed should say for $n \in \mathbb{N}$.
4. Page 9, last line in Definition 1.6, it should say "$n \in \mathbb{Z}$" not "$n \in \mathbb{N}$".
5. Page 10, bottom line, should add to (1.7): "$n, k \in \mathbb{Z}$".
6. Page 11, line 3: replace "$\{e^{in\theta}\}_{n\in\mathbb{N}}$" by "$\{e^{in\theta}\}_{n\in\mathbb{Z}}$" or "$\{e^{in\theta} : n \in \mathbb{Z}\}$".
7. Page 13, 2nd line after Definition 1.15: replace "2π-periodic functions: $\sin(n\theta)$, $\cos(n\theta)$, $e^{-in\theta}$" by "2π-periodic functions: $\sin(n\theta)$ and $\cos(n\theta)$ where $n \in \mathbb{N}$, or $e^{in\theta}$ where $n \in \mathbb{Z}$".
8. Page 20, in each row, specify in the sum where n is. In the first 2 cases $n \in \mathbb{N}$, in the third case $n \in \mathbb{Z}$, and in the fourth case $k \in \mathbb{N}$.

- Chapter 2

1. Page 33, Figure 2.3, line 5: replace $2 < p_1 < \infty$ by $2 < p_2 < \infty$.
2. Page 40, Definition 2.41. first line: repalce "A sequence of bounded functions" by "A sequence of functions".
3. Page 47, equation (2.7): the argument in the left-hand-side limit should be x not x_0.
4. Page 47, in Theorem 2.60, the domain of the functions must be compact, in this case that is equivalent to asking the interval I to be closed and bounded.
5. Page 51, line 4: replace "$g_0(x) \leq M$" by "$|g_0(x)| \leq M$".
6. Page 52, Theorem 2.75: should be "for all p with $1 \leq p < \infty$" instead of "for all p with $1 \leq p \leq \infty$"

- Chapter 3

1. Page 56, 2nd displayed formula: sum should over $0 < |n| \leq N$.

2. Section 3.2.2: replace n by $|n|$ in the following places:

   * page 63 line -7 in last displayed formula in denominator on the right-hand-side replace n^k by $|n|^{-k}$

   * page 63, last line: replace n^{-k} by $|n|^{-k}$.

   * page 64, first line: replace n^{-k} by $|n|^{-k}$.

   *page 64, line 8: replace $(\pi n)^{-1}$ by $(\pi|n|)^{-1}$ and replace n^{-1} by $|n|^{-1}$.

   * page 64, line 11: replace n^{-k} by $|n|^{-k}$.

   * page 66, Exercise 3.22 displayed formula: replace $\frac{C}{n}$ by $\frac{C}{|n|}$.

3. Page 64 third line in Example 3.15: replace “Its Fourier coefficients are given by” by “Its Fourier coefficients are given, when $n \neq 0$, by”.

4. Page 64, Theorem 3.16 and Corollary 3.17: missing hypothesis of continuity (counterexample f equal to zero except at one point, as pointed out by Paul Tupper, Associate Professor, Canada Research Chair, Department of Mathematics, Simon Fraser University). Statements should read as follows,

   * Theorem 3.16. Let $f : \mathbb{T} \to \mathbb{C}$ be 2π-periodic and continuous,...

   * Corollary 3.17. Assume $f \in C(\mathbb{T})$. If ...

   A remark can be made: If all we know is that $f \in L^1(\mathbb{T})$, then by the uniqueness principle in L^1 (Theorem 4.34) we conclude that $f = Sf$ a.e. where Sf is a continuous function, in other words, under the assumptions of Theorem 3.16 assuming that $f \in L^1(\mathbb{T})$ then f is equal almost everywhere to the continuous function Sf.

5. Page 66, last quote replace by: *The smoother f is, the faster the rate of convergence of the partial Fourier sums to f is.*

6. Page 67, line 9: “proof” instead of “proofs”.

7. Page 70,after Theorem 3.30: the inclusions for the Hölder spaces should be reversed, that is $C^\beta(\mathbb{T}) \subset C^\alpha(\mathbb{T}) \subset C(\mathbb{T})$ instead of $C(\mathbb{T}) \subset C^\alpha(\mathbb{T}) \subset C^\beta(\mathbb{T})$.

- Chapter 4

  1. Page 81, Exercise 4.5: specify $N \geq 2$.

  2. Page 88, second displayed formula: missing dy in the second integral.

  3. Page 88, David: ”Exercise 4.16, starts with an induction argument with $k \geq 1$ and $m = 0$. It then says to check by induction when $k \geq 0$ and $m = 0$. For induction, shouldn’t this be $k \geq 1$ (start with the base case) and go up? Now you already know that if f and g are continuous then so is $f * g$, but that is not really germane to the problem.”

     Perhaps rephrase, we want induction, base case $n = m = 0$ is already known, do the case $k \geq 1$ and $m = 0$ to gain insight in how to go about the inductive step.

  4. Page 89, Table 4.1: first item derivative/polynomial in Frequency column must assume $n \neq 0$.

5. Page 91, Exercise 4.22, 2nd line: remove absolute values in the integrand, that is replace $\int_{-\pi}^{\pi} |K(\theta)|\, d\theta = 2\pi$ by $\int_{-\pi}^{\pi} K(\theta)\, d\theta = 2\pi$.
6. Page 95, line 10: replace "For $y \leq \delta$" by "For $|y| \leq \delta$" (missing absolute values).
7. Page 96, line -1, Page 97, line 8: replace $|k| \leq N$ by $|k| < N$.
8. Page 97, line 8: summation shoul dbe over $|k| < N$ not over $|k| \leq N$.
9. Page 97, Figure 4.5: the functions pictured are NF_N not F_N. The maximum value of F_N is N, the graphs peak at N^2.
10. Page 99, line 8: [SS3, Chapter 2, Sections 5.3 and 5.4] (missing Chapter).
11. Page 100, Exercise 4.39: replace "integrable" by "Riemann integrable".

- Chapter 5

1. Page 108, line 7: replace $\mathbb{T}$ by $L^2(\mathbb{T})$.
2. Page 108, last line and last displayed formula: remove 2π from the exponent, that is write: $S_N f(\theta) = \sum_{|n|\leq N} \widehat{f}(n)e^{in\theta}$.
3. Page 109, line 6: David Cruz-Uribe suggests to include a short explanation on why the L^2-norm of f is referred as the energy of f...

   "It took me a while (with the help of the PDEs person in the next office) to remember why: if u is the solution of the wave equation on $\mathbb{R}^n$, then $\|\nabla u\|_2$ is constant in time because the integral of the gradient (in both space and time) is the total energy in the system.

   You should probably define this somewhere! It would probably make a good project to have students go off and read about the wave equation and the computation of total energy."
4. Page 111, Exercise 5.8: Notation was confusing for students (some took A^N for the Nth power not the Nth sequence). Are the last two instructions inverted?
5. Page 121, first line: replace "continuous functions g. [..], take $g \in C(\mathbb{T})$" by "twice-continuously differentiable functions g. [..], take $g \in C^2(\mathbb{T})$" .
6. Page 121, Exercise 5.28:

   * Replace "for continuous functions $g \in C(\mathbb{T})$" by "for twice continuously differentiable functions $g \in C^2(\mathbb{T})$".

   * in the limit replace $\|S_n g - g\|_{L^p(\mathbb{T})}$ by $\|S_N g - g\|_{L^p(\mathbb{T})}$.
7. Page 122, last line: replace $X^{\perp} = 0$ by $X^{\perp} = \{0\}$.

- Chapter 6

1. Page 138, Exercise 6.20: Add comment "Note that in matrix language $\langle v, w\rangle = v^t\overline{w}$ when v, w are considered as column vectors".
2. Page 143, line -10: replace "in the decomposition of F_N" by "in the decomposition of $\overline{F_N}$".

3. Page 144 Exercise 6.28: in line 4 replace "$v(0)$, $v(1)$ and of the numbers $v(2)$, $v(3)$" by "$v(0)$, $v(2)$ and of the numbers $v(1)$, $v(3)$". In line 6: replace "column vectors $[v(0), v(1)]^t$ and $[v(2), v(3)]^t$" by "column vectors $[v(0), v(2)]^t$ and $[v(1), v(3)]^t$".
4. Page 145 in Section 6.42, line 4: replace "$-N \leq n-k < 0$. Define $w(n-k) := w(N-(n-k)$" by "$-N < n-k < 0$. Define $w(n-k) := w(N+(n-k)$".
5. Page 148, in Exercise 6.37, line 3: replace "analytically" by "by hand".
6. Page 154 line -5, replace "2×2 multiplications" by "2×3 multiplications".
7. Page 155, Exercise 6.47, in the displayed matrices: in the 8x8 middle permutation matrix in the right-hand-side, the 6th and 7th rows maybe should be permuted (check the calculation).
8. Page 159, Project 6.9(e) last line: replace $\chi : G \to \mathbb{C}$ by $\chi : G \to S_1$ where $S_1 := \{z \in \mathbb{C} : |z| = 1\}$, the unit circle.
9. pages 159-160, Project 6.9 add at the begining: a good reference is [SS03] Section 7.2 and also see Chapter 2 for an application to the distribution of primes (Dirichlet's problem).

- Chapter 7

1. Page 164 line -10: replace "stablish" by "establish".
2. Page 167, line -6: "In Section 7.5 we say more" instead of "In Section 7.8 we say more".
3. Page 169, Table 7.1: in item (g) Frequency column need $\xi \neq 0$.
4. Page 176, Exercise 7.20: for (j) we need inversion formula. Replace by:

   *Exercise 7.20: Verify property (i) in Table 7.1: if $f, g \in \mathcal{S}(\mathbb{R})$, then $\widehat{f * g}(\xi) = \widehat{f}(\xi)\widehat{g}(\xi)$.

   Add a new exercise after Exercise 7.30.

   New exercise: Verify property (j) in Table 7.1: if $f, g \in \mathcal{S}(\mathbb{R})$, then $\widehat{fg}(\xi) = \widehat{f} * \widehat{g}(\xi)$.
5. Page 179, Exercise 7.25: Replace by:

   *Exercise 7.25: Replace stement by "Prove Theorem 7.24. Does Theorem 7.24 remain true if f is only assumed to be continuous and integrable, so that the convolution is well defined? If not, what additional assumptions on f you could add to make it valid?"

   Need assumptions of bounded and uniformly continuous, here is a counterexample constructed by David Cruz-Uribe and his student: Let f be defined as zero except on the intervals $[n, n + 2/n^3]$, and on these intervals, let the graph of f be an isosceles triangle of height n. Then

$$\int f(x)\, dx = \sum_{n=1}^{\infty} n^{-2} < \infty,$$

so f is continuous and in $L^1(\mathbb{R})$.

Let $K_t(x) = t^{-1}K(x/t)$, where K is a C^∞ function with $\mathrm{supp}(K) \in (-1,1)$. Then for all t sufficiently small ($t < 1/n^3$ should work) I get that

$$K_t * f(n) = n^4 t \int_0^1 K(u)du$$

and this does not converge to 0 uniformly for all n.

6. Page 183, Exercise 7.32: * replace "real-valued" by "complex-valued",

   * revise Hint, since these are complex valued one needs to consider $f \pm ig$ also... in fact rewrite as follows:

   * Hint: use Plancherel on $f+g$ to conclude the real parts are equal and on $(f+ig)$ to conclude the imaginary parts are equal.

7. Page 188, Project 7.8(e) in lines -1, -3 and -5, replace $1/(i\pi z)$ by $i/(\pi z)$ (check before doing it!).

- Chapter 8

1. Page 199, Table 8.1(d)(e): missing 2π in the exponents for both Fourier and inverse Fourier transforms on the left-hand-side column.

2. Page 201, Table 8.2

   * In the legend $a, b \in \mathbb{C}$, $h \in \mathbb{R}$

   * (e) in Frequency column: replace $\widehat{\tilde{T}} = -\tilde{\widehat{T}}$ by $\widehat{\tilde{T}} = \tilde{\widehat{T}}$.

3. Page 202, line 9 (in equation (8.7)): the argument in the middle function should be $\widehat{\phi}$ not $\widehat{\psi}$.

4. Page 209, line -14: replace "times $e^{2\pi i\xi n}$." by "times $e^{2\pi i\xi x}$.".

5. Page 209, line -12: it should say "integral over $[-1/2, 1/2]$" instead of "integral over $[-2\pi, 2\pi]$".

6. Page 215, Exercise 8.53: False, $\widehat{f}(\xi) \sim 1/\xi^2$ which is of moderate decrease. This is because the hat function is the convolution of $\chi_{[0,1]}$ with itself, and the Fourier transform of $\chi_{[0,1]}$ is essentially $\mathrm{sinc}(\xi) = \sin(\xi)/xi$. Finally convolution under Fourier transform becomes product. Need a different example.

7. Page 218-219, Project 8.8: Can add sources based on Cameron's Honor's Thesis. She found some very good references including Tao's trick to prove famous inequalities in the group context... (check her thesis).

- Chapter 9

1. Page 225, Definition 9.3, line 3: replace $\{g_{n,k}\}_{j,k\in\mathbb{Z}}$ by $\{g_{n,k}\}_{n,k\in\mathbb{Z}}$.

2. Page 229 in Theorem 9.11 (Balian-Low Theorem): in the second displayed equality missing a factor $\frac{1}{4\pi}$ in front of the integral of the square of the derivative.

3. Page 241, Figure 9.7: replace twice $Q_0 f$ by $Q_{-1} f$ (in the second graph and in the legend for the figure.
4. Page 242, line 10: replace $j = 0$ by $j = -1$.
5. Page 243, Last line Section 9.4.4 replace by: "We conclude that $Q_j f(x) = \sum_{I\in\mathcal{D}_j} \langle f, h_I\rangle h_I(x)$ since for each $x \in \mathbb{R}$ there is exactly one $I \in \mathcal{D}_j$ such that $x \in I$."
6. Page 246, last displayed inequality/equalities: both sums should be adding over $\mathcal{D}_j$ instead of over $\mathcal{D}_n$.
7. Page 247, line 3: "$|P_{-j}g(x)| \leq \frac{1}{2^j}\int_0^K |g(t)|dt$" (inequality not equality).

- Chapter 10

1. Page 262, Key Property: replace by
   * Key Properties: If $f(x)$ is in V_0 then all integer translates $f(x-k)$ are in V_0. If $f(x)$ is in V_j then $f(2x)$ is in V_{j+1}.
2. Page 263, line 6: kill twice argment (t), write "The piece of f in V_j is $S_j f$,".
3. Page 265, add to Exercise 10.6: "Explain why V_j and W_j are the spaces described in Example 10.5."
4. Page 274, in only displayed formula (line -6): specify under the sum that $k \in \mathbb{Z}$.
5. Page 279, last paragraph in the Proof of Lemma 10.25: Add to the first sentence "and interchanging the sum and the integral over $[0,1]$ by the Lebesgue Dominated Convergence Theorem with dominating function $g(\eta) = \sum_{n\in\mathbb{Z}} |\widehat{f}(\eta+n)|^2$ is in $L^2([0,1])$ with L^1-norm equal to $\|\widehat{f}\|^2_{L^2(\mathbb{R})}$.
6. Page 281, in Lemma 10.30: add "such that $e^{\pi i\xi}v(\xi)\overline{H(\frac{\xi}{2}+\frac{1}{2})} \in L^2([0,1])$ and" [displayed formula]
7. Page 281, in the proof of Lemma 10.30: this is an if and only if statement and the proof has only the forward direction. For the backward direction all needed is the observation that when $\sigma(\xi) = e^{2\pi ik\xi}$ then $f \in W_0$.
8. Page 283, line -8: $\lambda(\xi) = e^{-2\pi i\xi}\sigma(\xi)$ (add a minus sign to the exponent).
9. Page 284, In last paragraph: one could also choose $\sigma(\xi) = e^{-4\pi i\xi}$ with period 1/2 and that will give $G(\xi) = e^{-2\pi i\xi}\overline{H(\xi+1/2)}$, see Exercise 10.36.
10. Page 286, first displayed formula in line 2: missing a negative sign in the exponential, should be $e^{-2\pi ik\xi}$ instead of $e^{2\pi ik\xi}$.
11. Page 286, line -6: add absolute values. Replace $G(\pm 1/2) = 1$ by $|G(\pm 1/2)| = 1$.
12. Page 287, Example 10.38 first and second displayed formulas, almost interchange right-hand-sides, that is $H(\xi) = \chi_{[-1/4,1/4)(\xi)}$ and $G(\xi) = e^{2\pi i\xi}H(\xi+1/2) = e^{2\pi i\xi}\chi_{[-1/2,-1/4)\cup[1/4,1/2]}(\xi)$.
13. Page 291, last line, Page 292, line 5: replace $\widehat{\varphi}(0) = 1$ by $|\widehat{\varphi}(0)| = 1$.

- Chapter 11

  1. Page 311, line 6: "If for all x the sum..." (replace t by x).
  2. Page 316, in first set of displayed equalities, missing argument (x) twice in the third line.
  3. Page 317, line 3: replace "holds" by "hold".
  4. Page 317, lines 5 and 6 (displayed formulas): replace n in the sum and in the summands by m (for aesthetic reasons, m is used in the previous page).
  5. Page 320, Example 11.23, line 6: should say "for all $j \geq 0$ instead of "for all $j \geq 1$" because $V_0 \perp W_0$.

- Chapter 12

  1. Page 336, Section 12.3.1, line 4: replace "$\mathcal{D} = \bigcup_{j\in\mathbb{Z}} \mathcal{D}$, where $I \in \mathcal{D}$" by "$\mathcal{D} = \bigcup_{j\in\mathbb{Z}} \mathcal{D}_j$, where $I \in \mathcal{D}_j$"
  2. Page 339, line -7, item (ii) Dilation: in the right-hand-side of the equality should be $\mathrm{Sha}_{\delta_a(r,\beta)}(\delta_a f)$ instead of $\mathrm{Sha}_{\delta_a}(r,\beta)(\delta_a f)$.
  3. Page 355, line -10: replace twice $\|g\|_p$ by $\|g\|_q$.
  4. Page 359, Section 12.7.1. : be careful with the analytic extension of f and the Cauchy formula... We say that F is twice the analytic extension.... so that explains the missing 1/2 in the formula... (check!).
  5. Page 381, Example A.48. twice replace $\mathcal{P}_n$ by $\mathcal{P}_N$ i nthe first and last lines of the example.

- Appendix

  1. Page 374, end of line 4: $\langle z, y\rangle$ instead of $\langle y, x\rangle$.
  2. Page 381 in Example A.48: twice replace $\mathcal{P}_n$ by $\mathcal{P}_N$ (in the first and last lines of the example).

People that have contributed to this list

- David Cruz-Uribe (Trinity College, taught a reading course in Spring 2013), and his student Philip Cho.

- Paul Tupper (Simon Fraser University, teaching in Summer 2014 a class of about 12 students with 2 courses advanced calculus + some applied Fourier in PDE classes).

- Jens Lorenz (UNM, Spring 2015).

- UNM students Math 472/572.

  * Fall 2012: Wang, David Weirich, Cameron Lavigne (also wrote Honor's Thesis defended with honors Dec 2013).

* Fall 2013: Nuriye Atasaver (Wrote MS thesis on Petermichl's representation of Hilbert transform defended in Dec 2014).

* Fall 2016: Cairn Overturf.

* Fall 2017: Steven Kao.

* Fall 2019: Rob Dukes.

* Fall 2021: Brandon Conder, Jose Agudelo.

- Marti Prats (Professor at Universitat de Barcelona who will be teaching a course on "Anàlisi harmònica i teoria del senyal" in Spring 2022).
- Rene Girard (Graduate Student, Concordia University, Montreal, Canada).