The Borel-Cantelli Lemma and its Applications

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October 13, 2010

Abstract

We state and prove the Borel-Cantelli lemma and use the result to prove another proposition.

1 Definitions and Identities

Definition 1 Let \( \{E_k\}_{k=1}^{\infty} \) be a countable family of measurable subsets. The limit supremum of \( \{E_k\} \) is the set

\[
\limsup_{k \to \infty} (E_k) := \{ x \in \mathbb{R}^d : x \in E_k \text{ for infinitely many } k \}
\]
Proposition 1 The following identity holds:

\[
\limsup_{k \to \infty} (E_k) = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} E_k
\]

Proof. Assume that \(x \in \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} E_k\). So,

\[
x \in \left( \bigcup_{k=1}^{\infty} E_k \right) \cap \left( \bigcup_{k=2}^{\infty} E_k \right) \cap \left( \bigcup_{k=3}^{\infty} E_k \right) \cap \ldots
\]

Suppose that \(x \notin \limsup_{k \to \infty} (E_k)\). By definition, this means that there is a positive integer \(k_0\) such that for all \(k \geq k_0\), \(x \notin E_k\). Hence, \(x \notin \bigcup_{k=k_0}^{\infty} E_k\). Therefore, \(x \in \limsup_{k \to \infty} (E_k)\). This means that

\[
\limsup_{k \to \infty} (E_k) \supset \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} E_k
\]

Conversely, assume that \(x \in \limsup_{k \to \infty} (E_k)\). This means that \(x\) belongs to \(E_k\) for infinitely many \(k\). That is to say, \(x\) continuously reappears as an element in a set of the sequence \(E_k\). Then it is evident that \(x \in \bigcup_{k=1}^{\infty} E_k\). It is equally evident that \(x \in \bigcup_{k=2}^{\infty} E_k\), \(x \in \bigcup_{k=3}^{\infty} E_k\), and so on. Thus,

\[
x \in \left( \bigcup_{k=1}^{\infty} E_k \right) \cap \left( \bigcup_{k=2}^{\infty} E_k \right) \cap \left( \bigcup_{k=3}^{\infty} E_k \right) \cap \ldots
\]

\[
\in \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} E_k
\]

Therefore,

\[
\limsup_{k \to \infty} (E_k) \subset \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} E_k
\]

Thus,

\[
\limsup_{k \to \infty} (E_k) = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} E_k
\]
2 The Borel-Cantelli lemma and applications

Lemma 1 (Borel-Cantelli) Let \( \{E_k\}_{k=1}^{\infty} \) be a countable family of measurable subsets of \( \mathbb{R}^d \) such that

\[
\sum_{k=1}^{\infty} m(E_k) < \infty
\]

Then \( \limsup_{k \to \infty} (E_k) \) is measurable and has measure zero.

Proof. Given the identity,

\[
E = \limsup_{k \to \infty} (E_k) = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} E_k
\]

Since each \( E_k \) is a measurable subset of \( \mathbb{R}^d \), \( \bigcup_{k=n}^{\infty} E_k \) is measurable for each \( n \in \mathbb{N} \), and so \( \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} E_k \) is measurable as well, Stein [1]. Therefore, \( E \) is measurable.

Suppose that \( m(E) = \epsilon > 0 \). Then

\[
0 < \epsilon = m(E) = m\left( \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} E_k \right)
\]

Since for all \( n \in \mathbb{N} \), \( E \subset \bigcup_{k=n}^{\infty} E_k \), by the monotonicity property, Stein [1],

\[
m(E) \leq m\left( \bigcup_{k=n}^{\infty} E_k \right)
\]

By the countable sub-additivity property, Stein [1], for all \( n \in \mathbb{N} \),

\[
m(E) = m\left( \bigcup_{k=n}^{\infty} E_k \right) \leq \sum_{k=n}^{\infty} m(E_k)
\]

By assumption, \( \sum_{k=1}^{\infty} E_k < \infty \). It follows that the tail of the series can be made arbitrarily small. In other words, for any \( \delta > 0 \), there is an \( N' \in \mathbb{N} \) such that
\[ \sum_{k=N'}^\infty m(E_k) < \delta \]

However, if we choose \( \delta = \epsilon/2 \), we have

\[ 0 < \epsilon < m(E) \leq \sum_{k=N'}^\infty m(E_k) \leq \frac{\epsilon}{2} \]

Therefore, \( m(E) = 0 \).

\[ \square \]

**Proposition 2** Let \( \{f_n(x)\} \) be a sequence of measurable functions on \([0,1]\) with \( |f_n(x)| < \infty \) for a.e. \( x \in [0,1] \). Then there exists a sequence \( \{c_n\} \) of positive real numbers such that

\[ \frac{f_n(x)}{c_n} \to 0 \quad \text{a.e. } x \in [0,1] \]

**Proof.** Given a sequence of positive numbers \( \{c_n\} \), consider the set

\[ E_n = \{ x \in [0,1] : \frac{|f_n(x)|}{c_n} > \frac{1}{n} \} \]

Suppose that there is no sequence of positive numbers \( \{c_n\} \) such that \( m(E_n) \leq 2^{-n} \). Without loss of generally, we can assume that \( \{c_n\} \) is a sequence of positive numbers. Fix \( n \in \mathbb{N} \). Then it follows that for any \( N \in \mathbb{N} \),

\[ m(A_N) = m\left( \{ x \in [0,1] : \frac{|f_n(x)|}{N} > \frac{1}{n} \} \right) > 2^{-n} \]

\[ m(A_N) = m\left( \{ x \in [0,1] : |f_n(x)| > \frac{N}{n} \} \right) > 2^{-n} \]
So,

\[ A_1 = \{ x \in [0, 1] : |f_n(x)| > \frac{1}{n} \} \]
\[ A_2 = \{ x \in [0, 1] : |f_n(x)| > \frac{2}{n} \} \]
\[ A_3 = \{ x \in [0, 1] : |f_n(x)| > \frac{3}{n} \} \]

\vdots

\[ A_\infty = \{ x \in [0, 1] : |f_n(x)| = \infty \} \]

It is easy to see that this is a decreasing sequence of sets: \( A_1 \supset A_2 \supset A_3 \supset \ldots \). Since \( A_\infty \) is a subset of each \( A_N \), \( A_\infty = \bigcap_{N=1}^{\infty} A_N \). Hence, \( m(\bigcap_{N=1}^{\infty} A_N) = m(A_\infty) \), and so

\[ 2^{-n} < m \left( \bigcap_{N=1}^{\infty} A_N \right) = m(A_\infty) \]

However, by assumption \( m(A_\infty) = 0 \). Therefore, there is a sequence of positive numbers \( \{c_n\} \) such that

\[ m(E_n) = m(\{ x \in [0, 1] : \frac{|f_n(x)|}{c_n} > \frac{1}{n} \}) \leq 2^{-n} \]

Thus the series converges by comparison to a geometric series:

\[ \sum_{n=1}^{\infty} m(E_n) \leq \sum_{n=1}^{\infty} 2^{-n} \]
\[ \sum_{n=1}^{\infty} m(E_n) \leq 1 \]
\[ < \infty \]

According to the Borel-Cantelli lemma then, \( \limsup_{n \to \infty} E_n \) has measure zero. By definition,

\[ \limsup_{n \to \infty}(E_n) = \{ x : x \in E_n \text{ for infinitely many } n \} \]
So if $x \in \limsup_{n \to \infty} (E_n)$, then $x$ is a number in $[0,1]$ such that for infinitely many $n$,

$$\frac{|f_n(x)|}{c_n} > \frac{1}{n}$$

Negating this statement, if $x \notin \limsup_{n \to \infty} (E_n)$, then there is a $k_0 \in \mathbb{N}$ such that $|f_n(x)|/c_n \leq 1/n$ for all $n \geq k_0$. By comparison then, $\{|f_n(x)|/c_n\}$ would converge to 0 since $\{1/n\}$ converges to 0. Therefore, since $m(\limsup_{n \to \infty} (E_n)) = 0$, the conclusion is

$$\frac{f_n(x)}{c_n} \to 0 \quad \text{a.e. } x \in [0,1]$$

References