WEIGHTED SCHUR'S LEMMA

TAYLOR DUPUY, JAIME HERNANDEZ, ARNAB SAHA

Lemma 0.1. *Let T be the linear operator defined by*

$$Tf(x) = \int K(x,y)f(y)dy. \tag{0.1}$$

If there exists measurable functions w_1 and w_2, constants $A, B\alpha$ and β such that

$$\int |K(x,y)w_1(y)^{\alpha q}|dy \leq C_1 w_2(x)^{\beta}, \tag{0.2}$$

$$\int |K(x,y)|w_2(x)^{\beta p/q}dx \leq C_2 w_1(y)^{\alpha p} \tag{0.3}$$

then $\|Tf\|_p \leq C_1^{1/q}C_2^{1/p}\|f\|_p$.

It is easiest to derive the theorem using a general α and β then come up with the conditions we need in order for the inequality to work, you'll see what I mean in the proof. Note that setting $\alpha = 1/q$ and $\beta = q/p$ where $\frac{1}{p} + \frac{1}{q} = 1$ gives a nice symmetry:

$$\int |K(x,y)|w_1(y)dy \leq C_1 w_2(x)^{q/p}, \tag{0.4}$$

$$\int |K(x,y)|w_2(x)dx \leq C_2 w_1(y)^{p/q}. \tag{0.5}$$

Proof. We do the same trick as in the case for Schur's Lemma in class. First,

$$\|Tf\|^p = \int |\int K(x,y)f(y)dy|^p dx \tag{0.6}$$

$$\leq \int \left[\int |K(x,y)f(y)|dy\right]^p dx. \tag{0.7}$$

Now the inner part is

$$\int K(x,y)f(y)dy = \int \underbrace{K(x,y)^{1/q}w_1(y)^{\alpha}}\,\underbrace{K(x,y)^{1/p}w_1(y)^{-\alpha}f(y)}\,dy \tag{0.8}$$

$$\leq \left[\int |K(x,y)w_1(y)^{\alpha q}|dy\right]^{1/q}\left[\int \frac{K(x,y)f(y)^p}{w_1(y)^{\alpha}}dy\right]^{1/p} \tag{0.9}$$

$$\leq \left[C_1 w_2(y)^{\beta}\right]^{1/q}\left[\int \frac{K(x,y)f(y)^p}{w_1(y)^{\alpha}}dy\right]^{1/p} \tag{0.10}$$

Where we applied Hölder's inequality to the two parts on the first line and then applied our hypothesis (The condition we choose to make this flipping work). Putting this back into our expression for $\|Tf\|_p^p$ gives

$$
\begin{aligned}
\|Tf\|^p &= \int \left[\left(C_1 w_2(x)^\beta\right)^{1/q} \left(\int \frac{K(x,y) f(y)^p}{w_1(y)^\alpha} dy \right)^{1/p} \right]^p dx && (0.11)\\
&= C_1^{p/q} \int w_2(x)^{\beta p/q} \int | \frac{K(x,y) f(y)^p}{w_1(y)^\alpha} | dy dx && (0.12)\\
&= C_1^{p/q} \int \frac{|f(y)|^p}{w_1(y)^\alpha} \left(\int |K(x,y)| w_2(x)^{\beta p/q} dx \right) dy && (0.13)\\
&\leq C_1^{p/q} C_2 \int \frac{|f(y)|^p}{w_1(y)^\alpha} w_2(y)^{\alpha p} dy = C_1^{p/q} C_2 \|f(y)\|_p^p. && (0.14)
\end{aligned}
$$

The third line is an application of Fubini's Theorem and the fourth is where we choose our second condition to make everything go nicely. □