Math 565: Introduction to Harmonic Analysis - Spring 2008

Homework # 2, Daewon Chung

1. Show that,

$$H\chi_{[a,b]}(x) := \lim_{\epsilon\to 0} H^\epsilon(\chi_{[a,b]})(x) = \frac{1}{\pi}\log\frac{|x-a|}{|x-b|}\,.$$

and that

$$H^*\chi_{[a,b]}(x) := \sup_{\epsilon>0}|H^\epsilon(\chi_{[a,b]})(x)| = \frac{1}{\pi}\left|\log\frac{|x-a|}{|x-b|}\right|.$$

[Solution] Let us pick $\epsilon < \min(|x-a|, |x-b|)$ and we consider three cases $x<a$, $x>b$, and $< a < x < b$. By definition,

$$\begin{aligned} H^\epsilon\chi_{[a,b]}(x) &= K_\epsilon * \chi_{[a,b]}(x) = \frac{1}{\pi}\int_{\mathbb{R}}\chi_{[a,b]}(x-y)\frac{1}{y}\chi_{\{|y|>\epsilon\}}(y)dy \\ &= \frac{1}{\pi}\int_{|y|>\epsilon}\frac{\chi_{[a,b]}(x-y)}{y}\,dy = \frac{1}{\pi}\int_{\substack{|y|>\epsilon \\ x-b<y<x-a}}\frac{1}{y}\,dy \end{aligned}$$

If $x<a$, then $x-b<y<x-a<-\epsilon<0$, and

$$H^\epsilon\chi_{[a,b]}(x) = \frac{1}{\pi}\int_{x-b}^{x-a}\frac{1}{y}\,dy = \frac{1}{\pi}\log\frac{|x-a|}{|x-b|}\,.$$

If $x>b$, then $\epsilon < x-b<y<x-a$, and we have the same result as above. If $a<x<b$, then $x-b<y<-\epsilon$ or $\epsilon<y<x-a$, thus

$$H^\epsilon\chi_{[a,b]}(x) = \frac{1}{\pi}\left(\int_{x-b}^{-\epsilon}\frac{1}{y}\,dy + \int_{\epsilon}^{x-a}\frac{1}{y}\,dy\right) = \frac{1}{\pi}\left(-\log\frac{|x-b|}{\epsilon} + \log\frac{|x-a|}{\epsilon}\right) = \frac{1}{\pi}\log\frac{|x-a|}{|x-b|},.$$

Thus, for every case, we have

$$H\chi_{[a,b]}(x) = \lim_{\epsilon\to 0} H^\epsilon(\chi_{[a,b]})(x) = \frac{1}{\pi}\log\frac{|x-a|}{|x-b|}\,.$$

Consider now $H^*\chi_{[a,b]}(x)$. If we consider $\epsilon > \max(|x-a|, |x-b|)$, then the integral region becomes an empty set. We consider the case when ϵ is between $|x-a|$ and $|x-b|$. If $x<a$, then $x-b<y<-\epsilon<x-a<0$, and

$$H^\epsilon\chi_{[a,b]}(x) = \frac{1}{\pi}\int_{x-b}^{-\epsilon}\frac{1}{y}\,dy = \frac{1}{\pi}\log\frac{\epsilon}{|x-b|} > \frac{1}{\pi}\log\frac{|x-a|}{|x-b|}\,.$$

Since $\epsilon < |x-b|$ and $\log\frac{\epsilon}{|x-b|} < 0$,

$$\left|H^\epsilon\chi_{[a,b]}(x)\right| = \left|\frac{1}{\pi}\log\frac{\epsilon}{|x-b|}\right| < \left|\frac{1}{\pi}\log\frac{|x-a|}{|x-b|}\right|.$$

If $x>b$, then $x-b<\epsilon<y<x-a$, and we have

$$\left|H^\epsilon\chi_{[a,b]}(x)\right| = \left|\frac{1}{\pi}\int_{\epsilon}^{x-a}\frac{1}{y}\,dy\right| = \left|\frac{1}{\pi}\log\frac{|x-a|}{\epsilon}\right| < \left|\frac{1}{\pi}\log\frac{|x-a|}{|x-b|}\right|.$$

If $a < x < b$, then there are two possibilities. One is $|x-b| < \epsilon < |x-a|$, i.e.

$$\left|H^{\epsilon}\chi_{[a,b]}(x)\right| = \left|\frac{1}{\pi}\int_{\epsilon}^{x-a}\frac{1}{y}\,dy\right| = \left|\frac{1}{\pi}\log\frac{|x-a|}{\epsilon}\right| < \left|\frac{1}{\pi}\log\frac{|x-a|}{|x-b|}\right|.$$

Another is $|x-a| < \epsilon < |x-b|$, i.e.

$$\left|H^{\epsilon}\chi_{[a,b]}(x)\right| = \left|\frac{1}{\pi}\int_{-\epsilon}^{x-b}\frac{1}{y}\,dy\right| = \left|\frac{1}{\pi}\log\frac{|x-b|}{\epsilon}\right| = \left|\frac{1}{\pi}\log\frac{\epsilon}{|x-b|}\right| < \left|\frac{1}{\pi}\log\frac{|x-a|}{|x-b|}\right|.$$

Hence, by first part, we can conclude that

$$H^{*}\chi_{[a,b]}(x) = \sup_{\epsilon\to 0}|H^{\epsilon}(\chi_{[a,b]})(x)| = \frac{1}{\pi}\left|\log\frac{|x-a|}{|x-b|}\right|,$$

2. Show that

$$\left|\{x\in\mathbb{R} : |H\chi_{[a,b]}(x)| > \lambda\}\right| = \frac{4|b-a|}{e^{\pi\lambda}-e^{-\pi\lambda}}.$$

More generally show that for any measurable subset E of $\mathbb{R}$ of finite measure $|E|$,

$$\left|\{x\in\mathbb{R} : |H\chi_{E}(x)| > \lambda\}\right| = \frac{4|E|}{e^{\pi\lambda}-e^{-\pi\lambda}} \le \frac{2|E|}{\pi\lambda}.$$

[Solution] By first problem, we know $H\chi_{[a,b]}(x) = \frac{1}{\pi}\log\frac{|x-a|}{|x-b|}$. First, we need to observe the function $\frac{|x-a|}{|x-b|}$. This function has a horizontal asymptote $y=1$, vertical asymptote $x=b$ and has a function value 1 at $x=(a+b)/2$. Therefore we can figure out the graph of $\log\frac{|x-a|}{|x-b|}$.

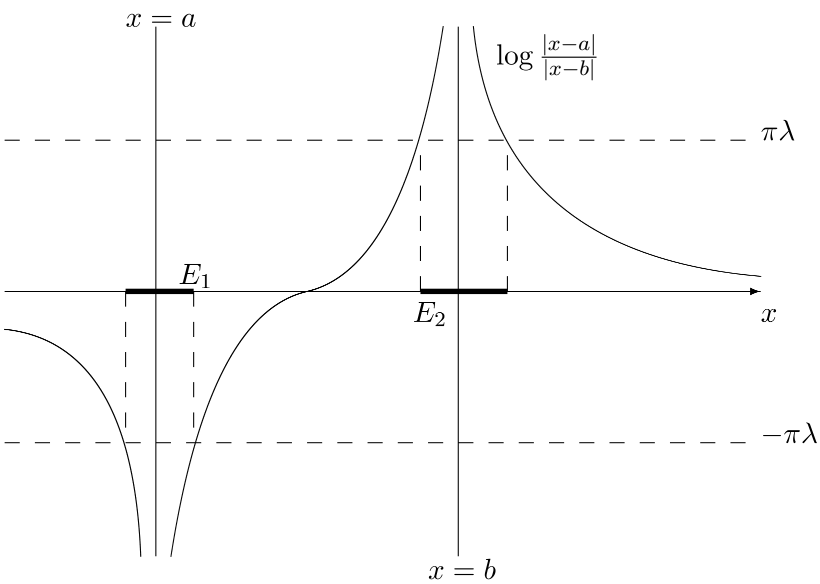

Figure 1. Graph of $\log\frac{|x-a|}{|x-b|}$.

Let $E = \{x\in\mathbb{R} : |H\chi_{[a,b]}(x)| > \lambda\} = E_1\cup E_2$, we can find $|E| = |E_1| + |E_2|$ as follows. Let $E_1 = [\alpha_1, \beta_1]$. Then

$$\alpha_1 = \frac{a-be^{-\pi\lambda}}{1-e^{-\pi\lambda}}$$

can be found with $\log \frac{x-a}{x-b} = -\pi\lambda$ because of $\alpha < a < b$, and

$$\beta_1 = \frac{be^{-\pi\lambda} + a}{1 + e^{-\pi\lambda}}$$

can be found with $\log \frac{a-x}{x-b} = e^{-\pi\lambda}$, $a < \beta_1 < b$. Also, if we set $E_2 = [\alpha_2, \beta_2]$ then we can find similarly

$$\alpha_2 = \frac{be^{\pi\lambda} + a}{1 + e^{\pi\lambda}}, \quad \beta_2 = \frac{be^{\pi\lambda} - a}{e^{\pi\lambda} - 1}.$$

Then

$$\begin{aligned}
|E| = \beta_1 - \alpha_1 + \beta_2 - \alpha_2 &= \frac{be^{-\pi\lambda} + a}{1 + e^{-\pi\lambda}} - \frac{a - be^{-\pi\lambda}}{1 - e^{-\pi\lambda}} + \frac{be^{\pi\lambda} - a}{e^{\pi\lambda} - 1} - \frac{be^{\pi\lambda} + a}{1 + e^{\pi\lambda}} \\
&= \frac{b + ae^{\pi\lambda}}{1 + e^{\pi\lambda}} - \frac{ae^{\pi\lambda} - b}{e^{\pi\lambda} - 1} + \frac{be^{\pi\lambda} - a}{e^{\pi\lambda} - 1} - \frac{be^{\pi\lambda} + a}{1 + e^{\pi\lambda}} = \frac{(b-a) + (a-b)e^{\pi\lambda}}{1 + e^{\pi\lambda}} + \frac{(b-a)e^{\pi\lambda} + (b-a)}{e^{\pi\lambda} - 1} \\
&= (b-a)\left(\frac{1 - e^{\pi\lambda}}{1 + e^{\pi\lambda}} + \frac{e^{\pi\lambda} + 1}{e^{\pi\lambda} - 1}\right) = (b-a)\frac{4e^{\pi\lambda}}{e^{2\pi\lambda} - 1} = \frac{4(b-a)}{e^{\pi\lambda} - e^{-\pi\lambda}}.
\end{aligned}$$

Now, to see the general case let us assume E is the union of finitely many disjoint intervals, each of finite length. We may express E in the form $E = \bigcup_{j=1}^{n} (a_j, b_j)$, where the a_j and b_j, $(j = 1, 2, ..., n)$, satisfy $a_1 < b_1 < a_2 < b_2 < \cdots < a_n < b_n$. It follows from the linearity of the Hilbert transform that

$$H\chi_E(x) = \frac{1}{\pi}\left(\log\left|\frac{x - a_1}{x - b_1}\right| + \log\left|\frac{x - a_2}{x - b_2}\right| + \cdots + \log\left|\frac{x - a_n}{x - b_n}\right|\right) = \frac{1}{\pi}\log\left|\prod_{j=1}^{n}\frac{x - a_j}{x - b_j}\right|.$$

Fix $\lambda > 0$ and set $F = \{x \in \mathbb{R} : |H\chi_E(x)| > \lambda\}$ Then F can be decomposed into the disjoint union

$$F = \{|g| > e^{\pi\lambda}\} \cup \{|g| < e^{-\pi\lambda}\} = F_1 \cup F_2,$$

where g is the rational function defined by

$$g(x) = \prod_{j=1}^{n}\frac{x - a_j}{x - b_j}.$$

Here, we claim that if $\mu \neq 1$, then the equation $g(x) = \mu$ has n distinct root $r_1, r_2, ..., r_n$ which satisfy

$$\sum_{j=1}^{n} b_j = \sum_{j=1}^{n} r_j + (1 - \mu)^{-1}\sum_{j=1}^{n}(b_j - a_j).$$

Furthermore, if $\mu > 1$, then

$$(\mu - 1)|\{g > \mu\}| = (\mu + 1)|\{g < -\mu\}| = \sum_{j=1}^{n}(b_j - a_j).$$

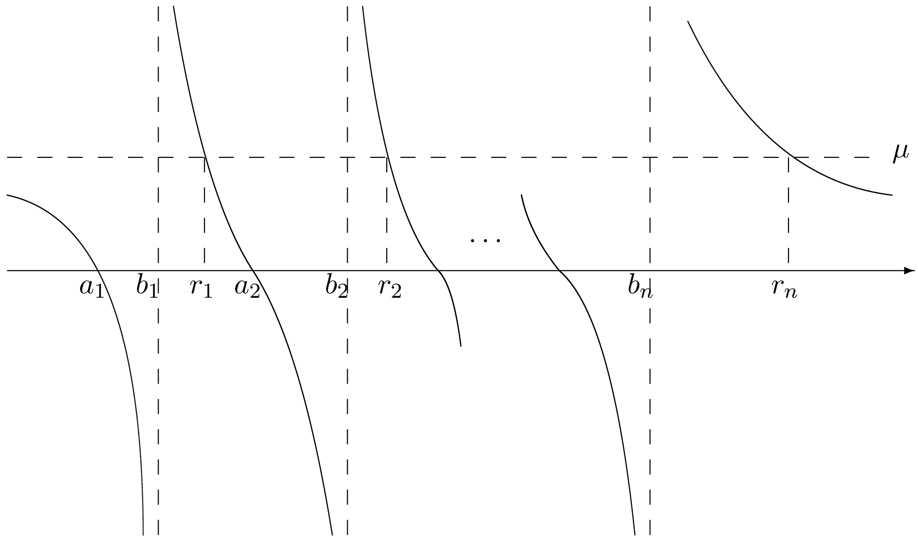

Figure 2. Graph of $g(x) = \prod_{j=1}^{n} \frac{x-a_j}{x-b_j}$.

Since g has a simple pole at each b_j, $(j = 1, 2, ..., n)$, and $g(x)$ approach to 1 as $|x|$ goes to infinity, there are exactly n distinct solutions, say r_1, $r_2, ..., r_n$, to the equation $g(x) = \mu$, $(\mu \neq 1)$ (Figure 2.). Since the numbers $r_1, r_2, ..., r_n$ are the roots of $g(x) = \mu$, those are also n roots of the n-th degree polynomial equation $p(x) = 0$, where

$$p(x) = \sum_{j=0}^{n} p_j x^j = \prod_{j=1}^{n}(x - a_j) - \mu \prod_{j=1}^{n}(x - b_j)\,.$$

The sum $\sum r_j$ of the roots is equal to $-p_{n-1}/p_n$ so, equating coefficients of x^n and of x^{n-1} in $p(x)$, we can have

$$p_n = 1 - \mu \quad \text{and} \quad p_{n-1} = -\sum_{j-1}^{n} a_j + \mu \sum_{j=1}^{n} b_j\,,$$

thus we obtain

$$\sum_{j=1}^{n} r_j = \frac{-1}{1-\mu}\Big(-\sum_{j=1}^{n} a_j + \mu \sum_{j=1}^{n} b_j \Big)\,,$$

which is equivalent to

$$\begin{aligned}
\sum_{j-1}^{n} b_j &= -\frac{1-\mu}{\mu}\Big(\sum_{j=1}^{n} r_j - \frac{1}{1-\mu} \sum_{j=1}^{n} a_j \Big) = \big(-\frac{1}{\mu} + 1 \big) \sum_{j=1}^{n} r_j + \frac{1}{\mu} \sum_{j=1}^{n} a_j \\
&= \sum_{j=1}^{n} r_j - \frac{1}{\mu} \sum_{j=1}^{n} r_j + \frac{1}{\mu} \sum_{j=1}^{n} a_j = \sum_{j=1}^{n} r_j - \frac{1}{\mu}\Big(\frac{1}{1-\mu} \sum_{j=1}^{n} a_j - \frac{\mu}{1-\mu} \sum_{j=1}^{n} b_j \Big) + \frac{1}{\mu} \sum_{j=1}^{n} a_j \\
&= \sum_{j=1}^{n} r_j - \Big(\frac{1}{\mu(1-\mu)} - \frac{1}{\mu} \Big) \sum_{j=1}^{n} a_j + \frac{1}{1-\mu} \sum_{j=1}^{n} b_j = \sum_{j=1}^{n} r_j + \frac{1}{1-\mu} \sum_{j=1}^{n} (b_j - a_j)\,.
\end{aligned}$$

Thus we prove the our first claim. If $\mu > 1$, then $\{\, g > \mu \} = \bigcup_{j=1}^{n} (b_j, r_j)$ (Figure 2.), so the identity; $(\mu - 1)|\{g > \mu\}\,| = \sum_{j=1}^{n} (b_j - a_j)$ follows from directly from the previous result. The other one can be established in similar fashion. Now we go back to our case, then we obtain from the claim,

$$|F_1| = |\{g > e^{\pi\lambda}\}| + |\{g < -e^{\pi\lambda}\}| = \frac{|E|}{e^{\pi\lambda} - 1} + \frac{|E|}{e^{\pi\lambda} + 1} = \frac{2|E|}{e^{\pi\lambda} - e^{-\pi\lambda}}\,.$$

By considering the rational function $1/g$ instead of g, and applying the analogous version of claim we obtain a similar estimate $|F_2| = 2|E|/(e^{\pi\lambda} - e^{-\pi\lambda})$. Since $|F| = |F_1| + |F_2|$, we have

$$\big|\{x \in \mathbb{R} : |H\chi_E(x)| > \lambda\}\big| = \frac{4|E|}{e^{\pi\lambda} - e^{-\pi\lambda}},$$

where E is the union of finitely many disjoint intervals. For given general measurable subset E of $\mathbb{R}$ with finite measure, we can find E_n such that each E_n is a finite union of intervals and $E_n \setminus E \to \varnothing$ as $n \to 0$. Then, as $n \to \infty$,

$$\|\chi_{E_n} - \chi_E\|_{L^2}^2 = \int_{\mathbb{R}} |\chi_{E_n} - \chi_E|^2 dx = \int_{E_n \setminus E} dx = |E_n \setminus E| \to 0.$$

Also, by L^2 boundedness of Hilbert transform, $H\chi_{E_n}$ converge to $H\chi_E$ in L^2-sense, also it converges in weak L^2. Thus, given $\epsilon > 0$ find an N such that for $n > N$, we have

$$\|H\chi_{E_n} - H\chi_E\|_{L^{2,\infty}} = \sup_{\alpha>0} \alpha \,|\{x \in \mathbb{R} : |H\chi_{E_n} - H\chi_E| > \alpha\}|^{\frac{1}{2}} < \epsilon^{\frac{1}{2}+1}.$$

Taking $\alpha = \epsilon$, we have $H\chi_{E_n}$ converge to $H\chi_E$ in measure. Then some subsequence of $H\chi_{E_n}$ converges to $H\chi_E$ almost everywhere. Hence, by Lebesgue's Dominated Convergence Theorem with dominate function $\chi_{\{|H\chi_{E_1}(x)|>\lambda\}}(x)$.

$$\begin{aligned}
\lim_{k\to\infty} |\{x \in \mathbb{R} : |H\chi_{E_{n_k}}(x)| > \lambda\}| &= \lim_{k\to\infty} \int_{\{x\in\mathbb{R}:|H\chi_{E_{n_k}}(x)|>\lambda\}} dx \\
&= \lim_{k\to\infty} \int_{\mathbb{R}} \chi_{\{|H\chi_{E_{n_k}}(x)|>\lambda\}} dx \\
&= \int_{\mathbb{R}} \lim_{k\to\infty} \chi_{\{|H\chi_{E_{n_k}}(x)|>\lambda\}} dx \\
&= \int_{\mathbb{R}} \chi_{\{|H\chi_E(x)|>\lambda\}} dx = |\{x \in \mathbb{R} : |H\chi_E(x)| > \lambda\}|.
\end{aligned}$$

This and the fact: $x \le \frac{1}{2}(e^x - e^{-x})$ give us desired result.

3. Let $P_t(x) = \frac{1}{\pi}\frac{t}{x^2+t^2}$ be the Poisson kernel, and $Q_t(x) = \frac{1}{\pi}\frac{x}{x^2+t^2}$ be the Conjugate Poisson kernel defined for all $t > 0$. Check that $\{P_t\}_{t>0}$ is an approximation of the identity as $t \to 0$, but $\{Q_t\}_{t>0}$ is not.
Verify that

$$\widehat{P_t}(\xi) = e^{-2\pi t|\xi|}, \quad \widehat{Q_t}(\xi) = -i\operatorname{sgn}\xi e^{-2\pi t|\xi|}.$$

[Solution] Let's try to see $\{P_t\}_{t>0}$ is an approximation of the identity as $t \to 0$. After substitute $y = x/t$ for fixed t, we can see

$$\int_{\mathbb{R}} P_t(x)dx = \frac{1}{\pi}\int_{\mathbb{R}} \frac{t}{x^2+t^2}\,dt = \frac{1}{\pi}\int_{\mathbb{R}} \frac{1}{y^2+1}\,dy = \frac{1}{\pi}\tan^{-1}(y)\Big|_{-\infty}^{\infty} = \frac{1}{\pi}\left(\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right) = 1.$$

Because $P_t(x)$ is positive for all $x \in \mathbb{R}$, $\|P_t\|_{L^1} = 1 < \infty$. And we need to show, for all $\delta > 0$,

$$\lim_{t\to 0} \int_{|x|>\delta} |P_t(x)|dx = 0.$$

For fixed δ,

$$\int_{|x|>\delta} |P_t(x)|dx = \int_{|x|>\delta} P_t(x)dx = \frac{2}{\pi}\int_{\delta}^{\infty} \frac{t}{x^2+t^2}\,dx$$
$$= \frac{2}{\pi}\int_{\tan^{-1}(\frac{\delta}{t})}^{\pi/2} \frac{\sec^2\theta}{\tan^2\theta+1}d\theta = \frac{2}{\pi}\left(\frac{\pi}{2} - \tan^{-1}\frac{\delta}{t}\right).$$

Since, $\delta/t \to \infty$ as $t \to 0$,

$$\lim_{t\to 0}\int_{|x|>\delta} |P_t(x)|dx = \lim_{t\to 0}\frac{2}{\pi}\left(\frac{\pi}{2} - \tan^{-1}\frac{\delta}{t}\right) = 0\,.$$

On the other hand,

$$\frac{1}{\pi}\int_{\mathbb{R}}\left|\frac{x}{x^2+t^2}\right|dx = \frac{1}{\pi}\left(\int_0^{\infty}\frac{x}{x^2+t^2}\,dx + \int_{-\infty}^{0}\frac{-x}{x^2+t^2}\,dx\right) = \frac{1}{\pi}\int_0^{\infty}\frac{2x}{x^2+t^2}\,dx$$
$$= \frac{1}{\pi}\log(x^2+t^2)\bigg|_0^{\infty} = \infty\,.$$

Thus $\{Q_t\}_{t>0}$ is not an approximation of the identity. Now, we'll see Fourier transform of each kernel. If we show that

$$\int_{\mathbb{R}} e^{-2\pi|\xi|t}e^{2\pi i\xi x}d\xi = P_t(x) = \frac{t}{\pi(x^2+t^2)}\,,$$

then , by the Fourier inversion theorem in the case of moderate decrease function, we can have

$$\int_{\mathbb{R}} P_t(x)e^{-2\pi i x\xi}dx = e^{-2\pi|\xi|t}\,.$$

This mean $e^{-2\pi|\xi|t}$ is a Fourier transform of $P_t(x)$. Now, we try to see the previous equality by calculate the integral separately.

$$\int_0^{\infty} e^{-2\pi\xi t}e^{2\pi i\xi x}d\xi = \int_0^{\infty} e^{2\pi i(x+it)\xi}d\xi = \frac{e^{2\pi i(x+it)\xi}}{2\pi i(x+it)}\bigg|_0^{\infty} = -\frac{1}{2\pi i(x+it)}\,,$$

and

$$\int_{-\infty}^{0} e^{2\pi\xi t}e^{2\pi i\xi x}d\xi = \int_{-\infty}^{0} e^{2\pi i(x-it)\xi}d\xi = \frac{e^{2\pi i(x-it)\xi}}{2\pi i(x-it)}\bigg|_{-\infty}^{0} = \frac{1}{2\pi i(x-it)}\,.$$

By adding these two integration, we get desired result,

$$\int_{\mathbb{R}} e^{-2\pi|\xi|t}e^{2\pi i\xi x}d\xi = -\frac{1}{2\pi i(x+it)} + \frac{1}{2\pi i(x-it)} = \frac{t}{\pi(x^2+t^2)}\,.$$

To avoid the sign confuse, we'll calculate the $\widehat{Q_t}(\xi)$ for the two cases. If $\xi \geq 0$, then

$$\widehat{Q_t}(\xi) = \frac{1}{\pi}\int_{\mathbb{R}}\frac{x}{x^2+t^2}\,e^{-2\pi i x\xi}dx = \frac{1}{\pi}\left(\int_{-\infty}^{\infty}\frac{x\cos(2\pi\xi x)}{x^2+t^2}dx - i\int_{-\infty}^{\infty}\frac{x\sin(2\pi\xi x)}{x^2+t^2}dx\right)$$
$$= \frac{1}{\pi}\left(\mathrm{Re}\int_{-\infty}^{\infty}\frac{x}{x^2+t^2}\,e^{2\pi i\xi x}dx - i\,\mathrm{Im}\int_{-\infty}^{\infty}\frac{x}{x^2+t^2}\,e^{2\pi i\xi x}dx\right).$$

Let

$$f(z) = \frac{ze^{2\pi i\xi z}}{z^2+t^2}.$$

Consider the integral of $f(z)$ over a closed semicircular contour $C_R = [-R,R]\cup\Gamma_R$ with radius R in the upper half plane. Then

$$\int_{C_R} \frac{ze^{2\pi i\xi z}}{z^2+t^2}dz = 2\pi i \mathrm{Res}(f(z), it) = 2\pi i \frac{z}{z+it}e^{2\pi i \xi z}\Big|_{z=it} = i\pi e^{-2\pi\xi t}\,.$$

On the other hand,

$$\int_{C_R} \frac{ze^{2\pi i\xi z}}{z^2+t^2}dz = \int_{-R}^{R}\frac{x}{x^2+t^2}\,e^{2\pi i\xi x}dx + \int_{\Gamma_R}\frac{z}{z^2+t^2}\,e^{2\pi i\xi z}dz \to \int_{-\infty}^{\infty}\frac{x}{x^2+t^2}\,e^{2\pi i\xi x}dx$$

as $R\to\infty$, because of

$$\left|\int_{\Gamma_R}\frac{z}{z^2+t^2}\,e^{2\pi i\xi z}dz\right| \le \int_0^{\pi}\left|\frac{Re^{i\theta}}{R^2e^{2i\theta}+t^2}\right| e^{-R2\pi\xi\sin\theta}Rd\theta \le 2\int_0^{\pi/2} e^{-R4\xi\theta}d\theta = \frac{1}{2R\xi}\left(1-e^{-2R\xi\pi}\right),$$

which tends to 0 as R approaches ∞. Thus,

$$\int_{-\infty}^{\infty}\frac{x}{x^2+t^2}\,e^{2\pi i\xi x}dx = i\pi e^{-2\pi\xi t}\,,$$

that implies, for $\xi\ge 0$,

$$\widehat{Q_t}(\xi) = -i\pi e^{-2\pi\xi t}\,.$$

If $\xi<0$, denote $\xi=-\eta \quad (\eta>0)$, then

$$\begin{aligned}\widehat{Q_t}(\xi) &= \frac{1}{\pi}\int_{\mathbb{R}}\frac{x}{x^2+t^2}\,e^{2\pi i x\eta}dx = \frac{1}{\pi}\left(\int_{-\infty}^{\infty}\frac{x\cos(2\pi\eta x)}{x^2+t^2}dx + i\int_{-\infty}^{\infty}\frac{x\sin(2\pi\eta x)}{x^2+t^2}dx\right)\\ &= \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{x}{x^2+t^2}\,e^{2\pi i\eta x}dx = \frac{1}{\pi}(i\pi e^{-2\pi\eta t}) = i\pi e^{2\pi t\xi}\,.\end{aligned}$$

by previous contour integral. Hence, we can see Fourier transform of Conjugate Poisson kernel is

$$\widehat{Q_t}(\xi) = -i\mathrm{sgn}\xi e^{-2\pi t|\xi|}\,.$$

4. Exercise 4.1.11 (a) in Grafakos (characterization of the Hilbert transform via invariances) Prove that if T is a bounded operator on $L^2(\mathbb{R})$ that commutes with translations and dilations and anticommutes with the reflection $f(x)\to\widetilde{f}(x)=f(-x)$, then T is a constant multiple of the Hilbert transform.

[Solution] Since T commutes with translations, then T is a convolution type operator.(Grafakos, Theorem 2.5.2.) Thus, there exist unique tempered distribution v such that

$$T(f) = f*v\,.$$

After taken Fourier transform, we can get

$$(\widehat{Tf})(\xi) = u(\xi)\widehat{f}(\xi)\,,$$

where $u(\xi) = \widehat{v}(\xi)$. Since T commutes with dilations, for $a > 0$, we have that

$$T(\delta_a f(x)) = \delta_a(Tf(x))\,.$$

However, T anticommutes with reflection, we have

$$T(\delta_a f(x)) = \mathrm{sgn} a\, \delta_a(Tf(x))\,,$$

for all values of a. Let $g(\xi) = \widehat{f}(\xi)$, and observe dilation of $u(\xi)g(\xi)$ with Time-Frequency Dictionary.

$$\begin{aligned}\frac{1}{a}u(\xi/a)g(\xi/a) &= \delta_a(u(\xi)g(\xi)) = \delta_a\big(\widehat{T\check{g}}\big)(\xi) = \frac{1}{a}\big(\widehat{\delta_{a^{-1}}T\check{g}}\big)(\xi) = \frac{1}{a}\big(\mathrm{sgn}a\,\widehat{T\delta_{a^{-1}}\check{g}}\big)(\xi) \\ &= \frac{\mathrm{sgn}a}{a}\big(\widehat{T\delta_{a^{-1}}\check{g}}\big)(\xi) = \frac{\mathrm{sgn}a}{a}u(\xi)\big(\widehat{\delta_{a^{-1}}\check{g}}\big)(\xi) = \frac{\mathrm{sgn}a}{a}u(\xi)a\delta_a g(\xi) = \mathrm{sgn}a\, u(\xi)\frac{1}{a}g(\xi/a)\,.\end{aligned}$$

This gives us, for $a \neq 0$,

$$u(\xi/a) = \mathrm{sgn}a\, u(\xi)\,.$$

Thus, $u(\xi)$ must be a constant multiple of $\mathrm{sgn}\xi$, and if we denote this constant with C then

$$Tf(x) = \big(-iD\mathrm{sgn}\xi\widehat{f}(\xi)\big)^{\vee}(x) = D\big(-i\mathrm{sgn}\xi\widehat{f}(\xi)\big)^{\vee}(x) = DHf(x)\,,$$

where $D = Ci$ is a constant.

[Remark.] In the class, we have a little problem to prove the weak type (1,1) of Hilbert transform. The question was for L^2 function, two definitions of Hilbert transform are coincide. Actually we've solved in the class. However, it is still interesting question for more general function which is in L^p. First, we may check this for characteristic function with

$$Hf(x) = \big(-i\mathrm{sgn}\xi\, \widehat{f}(\xi)\big)^{\vee}(x)\,.$$

We already found one in the problem 1. Now, we'll use the followings to see the other. For any number $c,\, d > 0$,

$$I = \int_c^d \int_0^\infty e^{-xy} dxdy = \int_c^d \frac{e^{-xy}}{-y}\bigg|_{x=0}^{x=\infty} dy = \int_c^d -\frac{1}{y} dy = \log\frac{|c|}{|d|}\,.$$

By Fubini's theorem,

$$I = \int_0^\infty \int_c^d e^{-xy} dydx = \int_0^\infty \frac{e^{-dx} - e^{-cx}}{x} dx = \log\frac{|c|}{|d|}\,.$$

We will use the last equality for following calculation.

$$\begin{aligned}
H\chi_{[a,b]}(x) &= \big(-i\mathrm{sgn}\xi\, \widehat{\chi_{[a,b]}}(\xi)\big)^{\vee}(x) \\
&= \bigg(-i\mathrm{sgn}\,\xi\, \frac{e^{-2\pi i\xi a} - e^{-2\pi i\xi b}}{2\pi i\xi}\bigg)^{\vee}(x) \\
&= \frac{1}{2\pi}\int_{\mathbb{R}} \frac{-\mathrm{sgn}\,\xi}{\xi}\Big(e^{2\pi i\xi(x-a)} - e^{2\pi i\xi(x-b)}\Big) d\xi \\
&= \frac{1}{2\pi}\bigg(\int_{\xi<0} \frac{e^{2\pi i\xi(x-a)} - e^{2\pi i\xi(x-b)}}{\xi} d\xi + \int_{\xi>0} \frac{e^{2\pi i\xi(x-b)} - e^{2\pi i\xi(x-a)}}{\xi} d\xi\bigg) \\
&= \frac{1}{2\pi}\int_{\xi>0} \frac{e^{-2\pi i\xi(x-b)} - e^{-2\pi i\xi(x-a)} + e^{-2\pi i\xi(b-x)} - e^{-2\pi i\xi(a-x)}}{\xi} d\xi \\
&= \frac{1}{2\pi}\bigg(\int_0^\infty \frac{e^{-2\pi i\xi(x-b)} - e^{-2\pi i\xi(x-a)}}{\xi} d\xi + \int_0^\infty \frac{e^{-2\pi i\xi(b-x)} - e^{-2\pi i\xi(a-x)}}{\xi} d\xi\bigg) \\
&= \frac{1}{2\pi}\bigg(\log\frac{|2\pi i(x-a)|}{|2\pi i(x-b)|} + \log\frac{|2\pi i(a-x)|}{|2\pi i(b-x)|}\bigg) = \frac{1}{\pi}\log\frac{|x-a|}{|x-b|}\,.
\end{aligned}$$

We may show, for any $f \in L^2$, that $H^\epsilon f(x)$ converge to $Hf(x) = (-i\mathrm{sgn}\xi \widehat{f}(\xi))^\vee$ in L^2-sense, as $\epsilon \to 0$. This give us strong type of $(2,2)$. Since the step functions are dense in $L^2(\mathbb{R})$, for $f \in L^2$, there is c_j and $A_j = [a_j, b_j]$ for $j = 1, 2, ...$, so that

$$\Big\| f - \sum_{j=1}^n c_j \chi_{[a_j,\, b_j]}(\cdot)\Big\|_{L^2} \to 0 \;\text{ as }\; n \to \infty\,.$$

Thus, for given $\epsilon' > 0$, we can choose N such that

$$\Big\| f - \sum_{j=1}^N c_j \chi_{[a_j,\, b_j]}(\cdot)\Big\|_{L^2} \le \frac{\epsilon'}{2}\,.$$

Also, we have seen, from the previous observation(problem 1), $\lim_{\epsilon\to 0} H^\epsilon \chi_{[a,b]}$ and $H\chi_{[a,b]}$ are coincide. By linearity of H^ϵ and H, we can have

$$\lim_{\epsilon\to 0} H^\epsilon \sum_{j=1}^n c_j \chi_{[a_j,\, b_j]}(x) = H \sum_{j=1}^n c_j \chi_{[a_j,\, b_j]}(x)\,.$$

Then we can have

$$
\begin{aligned}
\|\lim_{\epsilon\to 0} H^\epsilon f - Hf\|_{L^2} &= \Big\| \lim_{\epsilon\to 0} H^\epsilon f - \lim_{\epsilon\to 0} H^\epsilon\big(\sum_{j=1}^N c_j \chi_{[a_j,b_j]}(\cdot)\big) + H\big(\sum_{j=1}^N c_j\chi_{[a_j,b_j]}(\cdot)\big) - Hf\Big\|_{L^2} \\
&\le \Big\| \lim_{\epsilon\to 0} H^\epsilon f - \lim_{\epsilon\to 0} H^\epsilon\big(\sum_{j=1}^N c_j \chi_{[a_j,b_j]}(\cdot)\big)\Big\|_{L^2} + \Big\| Hf - H\big(\sum_{j=1}^N c_j\chi_{[a_j,b_j]}(\cdot)\big)\Big\|_{L^2} \\
&\le \Big\| \lim_{\epsilon\to 0} H^\epsilon\big(f - \sum_{j=1}^N c_j \chi_{[a_j,b_j]}(\cdot)\big)\Big\|_{L^2} + \Big\| H\big(f - \sum_{j=1}^N c_j\chi_{[a_j,b_j]}(\cdot)\big)\Big\|_{L^2} \\
&\le 2\Big\| f - \sum_{j=1}^N c_j \chi_{[a_j,b_j]}(\cdot)\big)\Big\|_{L^2} \le \epsilon' .
\end{aligned}
$$

Then, we can finished the proof of weak type (1,1) of H as we did in the class. Since the step functions are also dense in L^1, for $f \in L^1$, there is c_j and $A_j = [a_j, b_j]$ for $j = 1, 2, ...$ and we can choose some positive integer N so that

$$
\Big\| f - \sum_{j=1}^N c_j \chi_{[a_j,b_j]}(\cdot)\Big\|_{L^1} \le \epsilon' ,
$$

for any given ϵ'. Then, we claim that, for all $\delta > 0$,

$$
\Big| \{x \in \mathbb{R} : |\lim_{\epsilon\to 0} H^\epsilon f(x) - Hf(x)| > \delta\}\Big| = 0 .
$$

Choose N so that $\|f - \sum_{j=1}^N c_j\chi_{[a_j,b_j]}\|_{L^2} \le \delta\epsilon'/4$, then we have

$$
\begin{aligned}
&\Big| \{x \in \mathbb{R} : |\lim_{\epsilon\to 0} H^\epsilon f(x) - Hf(x)| > \delta\}\Big| \\
=&\Big| \{x \in \mathbb{R} : |\lim_{\epsilon\to 0} H^\epsilon f(x) - \lim_{\epsilon\to 0} H^\epsilon \sum_{j=1}^N c_j\chi_{[a_j,b_j]}(x) + H\sum_{j=1}^N c_j\chi_{[a_j,b_j]}(x) - Hf(x)| > \delta\}\Big| \\
\le&\Big| \{x \in \mathbb{R} : |\lim_{\epsilon\to 0} H^\epsilon \big(f(x) - \sum_{j=1}^N c_j\chi_{[a_j,b_j]}\big)(x)| > \delta/2\}\Big| + \Big| \{x \in \mathbb{R} : |H\big(f(x) - \sum_{j=1}^N c_j\chi_{[a_j,b_j]}\big)(x)| > \delta/2\}\Big| \\
\le&\frac{4C}{\delta}\Big\| f - \sum_{j=1}^N c_j\chi_{[a_j,b_j]}(\cdot)\Big\|_{L^1} \le C\epsilon' .
\end{aligned}
$$

Last inequality due to the weak type (1,1) of Hilbert transform. Therefore we can conclude, for any $f \in L^p(\mathbb{R})$, $p = 1, 2$

$$
\lim_{\epsilon\to 0} \frac{1}{\pi}\int_{|x-y|>\epsilon} \frac{f(y)}{x-y}\, dy = (-i\mathrm{sgn}\xi\, \widehat{f}(\xi))^\vee(x)
$$

almost everywhere. Then, by duality argument and interpolation, we can conclude that

$$
\lim_{\epsilon\to 0} \frac{1}{\pi}\int_{|x-y|>\epsilon} \frac{f(y)}{x-y}\, dy = (-i\mathrm{sgn}\xi\, \widehat{f}(\xi))^\vee(x) \quad \text{a.e. } x \in \mathbb{R} ,
$$

for given $f \in L^p$, $1 \le p < \infty$,