Harmonic Analysis: MATH–565
Homework #3

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1 Finite Dimensional Discrete Hilbert Transform

1.1 Definition on the Spatial Domain

For N a finite integer number, consider the finite dimensional Discrete Hilbert Transform (DHT) defined in terms of the following matrix $\mathbf{H}_N$:

$$\mathbf{H}_N := (h_{ij}) \ , \ \text{for } i, j \in I \tag{1}$$

where $h_{ij} = (i-j)^{-1}$ when $i \neq j$ and 0 otherwise, and $I := \{-N, -N+1, \ldots, 0, \ldots, N\}$. The matrix $\mathbf{H}_N$ can be written in terms of the column vectors $\mathbf{h}_i$, $i \in I$ and interms of $\mathbf{H}_{N-1}$ as follows:

$$\mathbf{H}_N = \begin{bmatrix} 0 & -1 & -\frac{1}{2} & \cdots & -\frac{1}{N} & \cdots & -\frac{1}{2N-1} & -\frac{1}{2N} \\ 1 & 0 & -1 & \cdots & -\frac{1}{N-1} & \cdots & -\frac{1}{2N-2} & -\frac{1}{2N-1} \\ \frac{1}{2} & 1 & 0 & \cdots & -\frac{1}{N-2} & \cdots & -\frac{1}{2N-3} & -\frac{1}{2N-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\ \frac{1}{N} & \frac{1}{N-1} & \frac{1}{N-2} & \cdots & 0 & \cdots & -\frac{1}{N-1} & -\frac{1}{N} \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ \frac{1}{2N-1} & \frac{1}{2N-2} & \frac{1}{2N-3} & \cdots & \frac{1}{N-1} & \cdots & 0 & -1 \\ \frac{1}{2N} & \frac{1}{2N-1} & \frac{1}{2N-2} & \cdots & \frac{1}{N} & \cdots & 1 & 0 \end{bmatrix} = - \begin{bmatrix} (\mathbf{h}_{-N})^T \\ (\mathbf{h}_{-N+1})^T \\ (\mathbf{h}_{-N+2})^T \\ \vdots \\ (\mathbf{h}_0)^T \\ \vdots \\ (\mathbf{h}_{N-1})^T \\ (\mathbf{h}_N)^T \end{bmatrix}$$

$$= [\mathbf{h}_{-N} \ \mathbf{h}_{-N+1} \ \ldots \ \mathbf{h}_0 \ \ldots \ \mathbf{h}_{N-1} \ \mathbf{h}_N] = \begin{bmatrix} 0 & -\mathbf{v}^T & -\frac{1}{2N} \\ \mathbf{v} & \mathbf{H}_{N-1} & -\mathbf{v}_{((2N-1))} \\ \frac{1}{2N} & \mathbf{v}^T_{((2N-1))} & 0 \end{bmatrix} ,$$

where T stands for transposition, $\mathbf{v} = [1 \; \frac{1}{2} \; \dots \; \frac{1}{2N-1}]^T$ and $\mathbf{v}_{((2N-1))}$ stands for a vector whose components are the same as $\mathbf{v}$ but shifted $2N-1$ positions. Note the following properties of H: i) $\mathbf{H}_N = -\mathbf{H}_N^T$, i.e., the matrix is skew-symmetric; ii) $(\mathbf{H}_N)^2 = -\mathbf{H}_N\mathbf{H}_N^T = -\mathbf{H}_N^T\mathbf{H}_N = (\mathbf{H}_N^T)^2$; and iii) the value $h^* = \max_{i\in I} ||\mathbf{h}_i||_1$, is attained at $i=0$ and it is equal to $h^* = 2\sum_{i=1}^{N} i^{-1}$.

Let us recall now the definition of the p-norms for matrices, which are induced from the p-norms of vectors: $||\mathbf{A}||_p := \sup_{||\mathbf{v}||_p=1} ||\mathbf{A}\mathbf{v}||_p$. From Linear Algebra, and from Functional Analysis also, we know that the following relationship holds: $||\mathbf{A}\mathbf{v}||_p \leq ||\mathbf{A}||_p||\mathbf{v}||_p$, for $p \geq 1$, $\mathbf{A}$ a matrix in $\mathbb{C}^{N\times N}$, and $\mathbf{v}$ a vector in $\mathbb{C}^N$ and appropriate vector and matrix norms. Let us introduce now the Harmonic numbers, defined as:

$$\mathsf{H}_k := \sum_{i=1}^{k} \frac{1}{i} = \gamma + \ln k + \frac{1}{2k} - \frac{1}{12k^2} + \mathcal{O}(k^{-4}) \,, \tag{2}$$

where $\gamma := \lim_{k\to\infty}(\mathsf{H}_k - \ln k) \approx 0.5772\dots$ is the so-called Euler-Mascheroni constant.

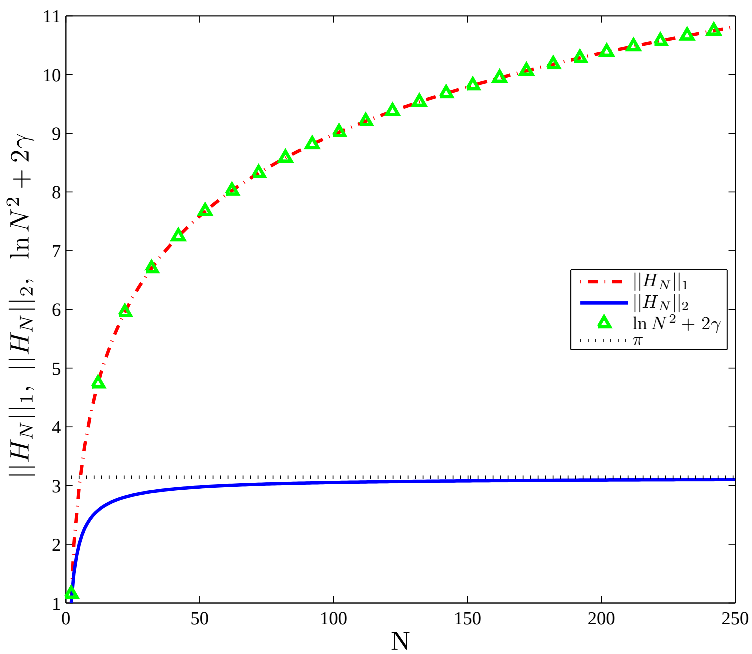


Fig. 1: Numerical evaluation of the ℓ_1-norm and the ℓ_2-norm of $\mathbf{H}_N$.

Motivation: A numerical evaluation of the ℓ_1-norm and the ℓ_2-norm of $\mathbf{H}_N$ gives the picture shown in Fig. 1. *Question*: Is it true that the ℓ_1-norm grows logarithmically as the dimension of the matrix, N, goes to infinity? In addition, is it true that the ℓ_2-norm is uniformly bounded for all the dimensions of the matrix?

Case 1: The ℓ_1-norm of $\mathbf{H}_N$. For $\mathbf{z} = [z_{-N}\ z_{-N+1} \ldots\ z_N]^T \in \mathbb{C}^{2N+1}$, we would like to prove that $||\mathbf{H}_N\mathbf{z}||_1 \leq ||\mathbf{H}_N||_1||\mathbf{z}||_1$, where $||\mathbf{H}_N||_1$ grows logarithmically as N goes to infinity. Using the definition we have:

$$||\mathbf{H}_N||_1 = \sup_{||\mathbf{z}||_1=1} ||\mathbf{H}\mathbf{z}||_1 = \sup_{||\mathbf{z}||_1=1} \sum_{i=-N}^{N} \Big| \sum_{\substack{j=-N\\ j\neq i}}^{N} \frac{z_j}{i-j} \Big| \leq \sup_{||\mathbf{z}||_1=1} \sum_{j=-N}^{N} |z_j| \sum_{\substack{i=-N\\ i\neq j}}^{N} \frac{1}{|i-j|}$$

$$\leq \sup_{||\mathbf{z}||_1=1} \sum_{j=-N}^{N} |z_j| \max_{-N\leq j\leq N} \sum_{\substack{i=-N\\ i\neq j}}^{N} \frac{1}{|i-j|} = \max_{-N\leq j\leq N} ||\mathbf{h}_j||_1 \sup_{||\mathbf{x}=1||_1} \sum_{j=-N}^{N} |z_j| = 2\mathsf{H}_N$$

$$\therefore ||\mathbf{H}_N||_1 \leq 2\gamma + 2\ln N + \frac{2}{N} + \mathcal{O}(N^{-2}) \ . \tag{3}$$

Taking the limit as N goes to infinity of the last expression we obtain the desired result.

Case 2: The ℓ_2-norm of $\mathbf{H}_N$. Our goal is to prove that $||\mathbf{H}_N\mathbf{z}||_2 \leq ||\mathbf{H}_N||_2||\mathbf{z}||_2$, where $||\mathbf{H}_N||_2$ is uniformly bounded by a constant for all N. Inspired[1] by the idea shown in the proof of the Hilbert's inequality, [1], we mimic here such a proof making a small but necessary change in order to achieve our goal.

Consider the positive integers $M > N$ and consider also the following matrices and vectors:

$$\mathbf{H}_M = \begin{bmatrix} \mathbf{M}_1 & \mathbf{M}_2 & \mathbf{M}_3 \\ \mathbf{M}_4 & \mathbf{H}_N & \mathbf{M}_5 \\ \mathbf{M}_6 & \mathbf{M}_7 & \mathbf{M}_8 \end{bmatrix} \text{ and } \mathbf{z}_M^N = \begin{bmatrix} \mathbf{0} \\ \mathbf{z} \\ \mathbf{0} \end{bmatrix} ,$$

where the $\mathbf{M}_i$'s are the appropriate matrices to construct $\mathbf{H}_M$ from $\mathbf{H}_N$, $\mathbf{0}$ is the zero column vector of dimension M, and $\mathbf{z}$ is a column vector of dimension N. Note that: $||\mathbf{z}||_2 = ||\mathbf{z}_M^N||_2$.

The square of the ℓ_2-norm of $\mathbf{H}_N$ can be bounded in the following way:

$$||\mathbf{H}_N||_2^2 = \sup_{||\mathbf{z}||_2=1} ||\mathbf{H}_N\mathbf{z}||_2^2 \leq \sup_{||\mathbf{z}_M^N||_2=1} ||\mathbf{H}_M\mathbf{z}_M^N||_2^2 \tag{4}$$

$$\leq \sup_{||\mathbf{z}_M^N||_2=1} \sum_{i=-M}^{M} \Big(\sum_{\substack{j=-M\\ j\neq i}}^{M} \frac{z_{M,j}^N}{i-j} \Big)^2 = \sup_{||\mathbf{z}_M^N||_2=1} \sum_{i=-M}^{M} \Big(\sum_{\substack{j=-N\\ j\neq i}}^{N} \frac{z_j}{i-j} \Big)^2 \tag{5}$$

[1] Actually, the idea of mimic Grafakos' proof was an inspiration of Prof. Pereyra. Thanks a lot!

$$||\mathbf{H}_N||_2^2 \leq \sup_{||\mathbf{z}_M^N||_2=1} \sum_{i=-M}^{M} \sum_{\substack{j=-N \\ j\neq i}}^{N} \sum_{\substack{k=-N \\ k\neq i}}^{N} \frac{z_j z_k}{(i-j)(i-k)} \tag{6}$$

$$\leq \sup_{||\mathbf{z}_M^N||_2=1} \sum_{\substack{j=-N \\ (j=k)}}^{N} z_j^2 \sum_{\substack{i=-M \\ i\neq j}}^{M} \frac{1}{(i-j)^2} + \sum_{j=-N}^{N} \sum_{\substack{k=-N \\ k\neq j}}^{N} z_j z_k \sum_{\substack{i=-M \\ i\neq j, i\neq k}}^{M} \frac{1}{(i-j)(i-k)} \,. \tag{7}$$

Now, we follow Grafakos' trick presented in [1]. One can show that for $k \neq j$:

$$\sum_{\substack{i=-M \\ i\neq j, i\neq k}}^{M} \frac{1}{(i-j)(i-k)} = \frac{2}{(j-k)^2} + \frac{1}{(j-k)}\Big(\sum_{\substack{i=-M \\ i\neq j}}^{M} \frac{1}{i-j} - \sum_{\substack{i=-M \\ i\neq k}}^{M} \frac{1}{i-k}\Big) \,. \tag{8}$$

Plugging the last result in (7), we have:

$$||\mathbf{H}_N||_2^2 \leq \sup_{||\mathbf{z}_M^N||_2=1} \left(\sum_{\substack{j=-N \\ (j=k)}}^{N} z_j^2 \sum_{\substack{i=-M \\ i\neq j}}^{M} \frac{1}{(i-j)^2} + \sum_{k=-N}^{N} \sum_{\substack{j=-N \\ j\neq k}}^{N} \frac{2 z_j z_k}{(j-k)^2} \right.$$

$$\left. + \sum_{j=-N}^{N} \sum_{\substack{k=-N \\ k\neq j}}^{N} \frac{z_j z_k}{(j-k)}\Big(\sum_{\substack{i=-M \\ i\neq j}}^{M} \frac{1}{i-j} - \sum_{\substack{i=-M \\ i\neq k}}^{M} \frac{1}{i-k}\Big) \right) \tag{9}$$

$$\leq \sup_{||\mathbf{z}_M^N||_2=1} \left(\sum_{\substack{j=-N \\ (j=k)}}^{N} z_j^2 \sum_{\substack{i=-M \\ i\neq j}}^{M} \frac{1}{(i-j)^2} + \sum_{j=-N}^{N} \sum_{\substack{k=-N \\ j\neq k}}^{N} \frac{z_j^2}{(j-k)^2} + \sum_{k=-N}^{N} \sum_{\substack{j=-N \\ j\neq k}}^{N} \frac{z_k^2}{(j-k)^2} \right.$$

$$\left. + \sum_{j=-N}^{N} \sum_{\substack{k=-N \\ k\neq j}}^{N} \frac{z_j z_k}{(j-k)}\Big(\sum_{\substack{i=-M \\ i\neq j}}^{M} \frac{1}{i-j} - \sum_{\substack{i=-M \\ i\neq k}}^{M} \frac{1}{i-k}\Big) \right) \tag{10}$$

where the last inequality holds from $2z_j z_k \leq z_j^2 + z_k^2$.

Recall the following convergent series: $\sum_{i=1}^{\infty} i^{-2} = \pi^2/6$. If we now take the limit as

M goes to infinity we have:

$$\begin{aligned}||\mathbf{H}_N||_2^2 \le &\sup_{||\mathbf{z}_M^N||_2=1} \lim_{M\to\infty} \Bigg(\sum_{j=-N}^{N} z_j^2 \sum_{\substack{i=-M\\ i\neq j}}^{M} \frac{1}{(i-j)^2} + \sum_{j=-N}^{N} \sum_{\substack{k=-N\\ j\neq k}}^{N} \frac{z_j^2}{(j-k)^2} \\ &+ \sum_{j=-N}^{N} \sum_{\substack{k=-N\\ k\neq j}}^{N} \frac{z_k^2}{(j-k)^2} + \sum_{j=-N}^{N} \sum_{\substack{k=-N\\ k\neq j}}^{N} \frac{z_j z_k}{(j-k)} \Big(\sum_{\substack{i=-M\\ i\neq j}}^{M} \frac{1}{i-j} - \sum_{\substack{i=-M\\ i\neq k}}^{M} \frac{1}{i-k} \Big) \Bigg) \\ &\le \sup_{||\mathbf{z}_M^N||_2=1} \Bigg(\frac{\pi^2}{3} \sum_{\substack{j=-N\\ (j=k)}}^{N} z_j^2 + \sum_{j=-N}^{N} z_j^2 \sum_{\substack{k=-N\\ k\neq j}}^{N} \frac{1}{(j-k)^2} + \sum_{k=-N}^{N} z_k^2 \sum_{\substack{j=-N\\ j\neq k}}^{N} \frac{1}{(j-k)^2} \Bigg) ,\end{aligned}$$

where the last inequality holds because for a fixed j and k and M large enough the summations $\sum_{i=-M,\ i\neq j}^{M} (i-j)^{-1}$ and $\sum_{i=-M,\ i\neq k}^{M} (i-k)^{-1}$ contain a large number of terms that overlap, so in the limit we can conclude that the both harmonic series cancel out. We can bound the summations in the last inequality one more time to obtain:

$$||\mathbf{H}_N||_2^2 \le \sup_{||\mathbf{z}_M^N||_2=1} \Bigg(\frac{\pi^2}{3} \sum_{j=-N}^{N} z_j^2 + \frac{\pi^2}{3} \sum_{j=-N}^{N} z_j^2 + \frac{\pi^2}{3} \sum_{k=-N}^{N} z_k^2 \Bigg) = \pi^2 ||\mathbf{z}||_2^2 \tag{11}$$

$$\therefore ||\mathbf{H}_N||_2^2 \le \pi^2 \ . \tag{12}$$

Consequently, we can conclude that $||\mathbf{H}_N||_2 \le \pi$ for all N. So, from the results on the ℓ_1- and ℓ_2-norms we can finally conclude that: The finite dimensional DHT defined in (1) is uniformly bounded in the dimension in the ℓ_2-norm but not in the ℓ_1-norm.

Bonus theorem: Hilbert's inequality. Given a real and square summable sequence a_k, the following inequality holds:

$$\sqrt{\sum_{j\in\mathbb{Z}} \Big(\sum_{\substack{k\in\mathbb{Z}\\ k\neq j}} \frac{a_k}{j-k} \Big)^2} \le \pi \sqrt{\sum_{j\in\mathbb{Z}} a_j^2} \ . \tag{13}$$

Moreover, the constant π is a sharp constant.

Proof. For an infinite square summable sequence, one can start considering both a sequence with finite support and an infinite matrix. Such a proof is essentially the same as the one shown here, but doesn't require the matrix vector definitions. So, we can say that the case of a square summable sequence with finite support is proved. Then, a limiting argument on a_k proves the theorem for an infinite square summable real sequence. For details and for the proof on the sharpness of π, please see reference [1]. $\square$

1.2 Definition on the Fourier Domain

In order to define the finite dimensional DHT we first need to define the finite dimensional Discrete Fourier Transform (DFT). Consider a sequence of complex numbers z_k, $k \in I = \{0, 1, \ldots, N-1\}$ then we define the sequence Z_n, $n \in I$, called the DFT of z_k, as:

$$Z_n := \sum_{k=0}^{N-1} z_k \exp\Big(-i\frac{2\pi}{N}kn\Big) \ , \ n \in I. \tag{14}$$

Similarly, we define the inverse formula (i.e. the discrete inverse Fourier transform) as:

$$z_k := \frac{1}{N}\sum_{n=0}^{N-1} Z_n \exp\Big(i\frac{2\pi}{N}kn\Big) \ , \ k \in I. \tag{15}$$

Recall that the Fourier multiplier corresponding to the Hilbert transform is: $(Hf)\hat{}(\xi) = -i\mathrm{sgn}(\xi)$. Now, the discrete version of the Fourier multiplier associated to the Hilbert transform can be defined as:

$$H_n^* := \begin{cases} 0 & , \quad n \in I_1 \\ -i & , \quad n \in I_2 \\ i & , \quad n \in I_3 \end{cases} \ , \tag{16}$$

where for N even (consequently, N odd) $I_1 = \{0, N/2\}$, $I_2 = \{1, 2, \ldots, N/2-1\}$, and $I_3 = \{N/2+1, \ldots, N-1\}$ (consequently, $I_1 = \{0\}$, $I_2 = \{1, 2, \ldots, (N-1)/2-1\}$, and $I_3 = \{(N+1)/2+1, \ldots, N-1\}$). Note that the definition takes care of the symmetry of the multiplier on the Fourier side. Consider the case when N is even and plug (16) in (15), hence we have:

$$\begin{aligned} h_k^* &= \frac{1}{N}\sum_{n=0}^{N-1} H_n^* \exp\Big(i\frac{2\pi}{N}kn\Big) \ , \ k \in \{0, 1, \ldots, N-1\} && (17) \\ &= \frac{1}{N}\left(\sum_{n=1}^{N/2-1} (-i)\exp\Big(i\frac{2\pi}{N}kn\Big) + \sum_{n=N/2+1}^{N-1} i\exp\Big(i\frac{2\pi}{N}kn\Big)\right) = \frac{2}{N}\sum_{n=1}^{N/2-1} \sin\Big(\frac{2\pi}{N}kn\Big) \ . && (18) \end{aligned}$$

Similarly, for N odd we obtain: $h_k^* = \dfrac{2}{N}\displaystyle\sum_{n=1}^{(N-1)/2} \sin\Big(\frac{2\pi}{N}kn\Big)$, $k \in I$. So, we can define the sequence corresponding to the finite dimensional DHT on the spatial domain as:

$$h_k^* = \frac{2}{N}\sum_{n=1}^{L} \sin\Big(\frac{2\pi}{N}kn\Big) \ , \ k \in \{0, 1, \ldots, N-1\} \ , \tag{19}$$

where $L = N/2 - 1$ for N even and $L = (N-1)/2$ for N odd.

Given a sequence of real numbers x_n, its Hilbert transformed version is given by the discrete convolution $H^*(x_n) = h_n^* * x_n$, which in matrix form can be represented as: $H^*(\mathbf{x}) = \mathbf{H}_N^* \mathbf{x}$ where $\mathbf{x} = [x_0 \; x_1 \; \dots \; x_{N-1}]^T$ and

$$\mathbf{H}_N^* = (h_{ij}) \equiv \left(\frac{2}{N} \sum_{n=1}^{L} \sin\left(\frac{2\pi}{N}(j-i)n \right) \right) \;, \text{ for } i,j \in \{1,2,\dots,N\} \;, \tag{20}$$

and $L = N/2 - 1$ for N even and $L = (N-1)/2$ for N odd.

Remark 1. There exists alternative closed forms for (20). When N is even:

$$h_k^* = \begin{cases} \frac{2}{N} \sin^2\left(\frac{\pi k}{2}\right) \cot\left(\frac{\pi k}{N}\right) & , \quad k \in \{1,2,\dots,N-1\} \\ 0 & , \quad k = 0 \end{cases} \tag{21}$$

and for N odd:

$$h_k^* = \begin{cases} \frac{1}{N} \cot\left(\frac{\pi k}{N}\right) - \frac{(-1)^k}{N} \sin^{-1}\left(\frac{\pi k}{N}\right) & , \quad k \in \{1,2,\dots,N-1\} \\ 0 & , \quad k = 0 \end{cases} \tag{22}$$

Remark 2. Note that by exploiting the L^2 isometry, we have that:

$$||\mathbf{H}_N^*||_2^2 = \sup_{||\mathbf{x}||_2=1} ||\mathbf{H}_N^* \mathbf{x}||_2^2 = \sup_{||\mathbf{x}||_2=1} ||\widehat{\mathbf{H}_N^* \mathbf{x}}||_2^2 \tag{23}$$

$$= \sup_{||\mathbf{x}||_2=1} \sum_{i=-N}^{N} |\widehat{\mathbf{H}_N^*}|^2 |\widehat{\mathbf{x}}|^2 = \sup_{||\mathbf{x}||_2=1} \sum_{i=-N}^{N} |\widehat{\mathbf{x}}|^2 = 1 \;. \tag{24}$$

What is the DFT of discrete DHT defined on the spatial domain?. If we consider the definition on the spatial domain for h_k then we have that: $h_k = (k-N)^{-1}$ for $k \in \{0,1,\dots,N-1\} \setminus \{N\}$ and $h_N = 0$. Then, if we compute the DFT of such a sequence, $H_N = 0$, while for $n \in \{0,1,\dots,N-1\} \setminus \{N\}$:

$$H_n = \sum_{\substack{k=0 \\ k \neq N}}^{2N} \frac{1}{k-N} \exp\left(-i \frac{2\pi k n}{2N+1} \right) = -2i \exp\left(i \frac{2\pi n N}{2N+1} \right) \sum_{j=1}^{N-1} \frac{1}{j} \sin\left(\frac{2\pi n j}{2N+1} \right) \;. \tag{25}$$

Note that the DFT version of h_k does not correspond to a discrete version of the multiplier $(Hf)^\wedge(\xi) = -i\mathrm{sgn}(\xi)$.

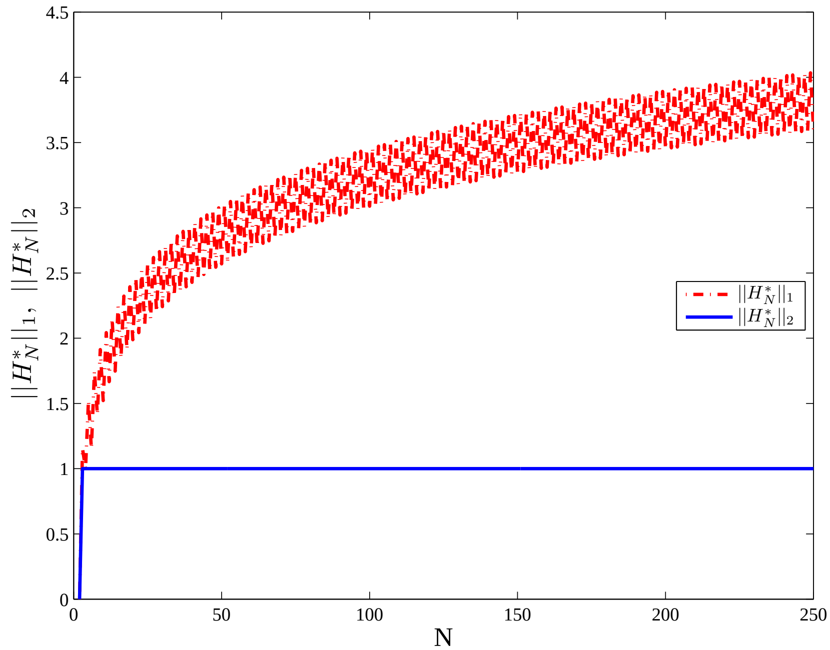


Fig. 2: Numerical evaluation of the ℓ_1-norm and the ℓ_2-norm of $\mathbf{H}_N^*$.

1.3 Applications

1.3.1 Analog/Digital Communications

Modulation Theory 101. Communications Theory deals with the problem of transmitting information from a sender to a receiver. Due to several technical reasons, an effective way of transmitting the information is to use a known waveform and embed such information in one of the parameters of the signal. Modulation is one way to introduce the information into the known signal. For example, if we introduce the information in the amplitude of a sinosoidal waveform, we talk about Amplitude Modulation (AM). In AM the information signal, say $s(t)$, is assumed to be band-limitted, i.e., the frequency components of $s(t)$ are compactly supported up to a certain known frequency f_{max}. The band-limitted signal is modulated using a sinusoidal waveform whose frequency is known to be much larger than $2f_{max}$, the bandwidth of $s(t)$. This modulating waveform is called the carrier signal and is denoted as $s_c(t)$. The frequency of $s_c(t)$ is termed as the carrier frequency and is denoted as f_c. The modulated signal is obtained by multiplying in the time-domain the information signal to the carrier signal, in symbols: $s_m = s(t)s_c(t) = s(t)\cos(2\pi f_c t)$. Note that the information signal turns out to be the (time-varying) amplitude of the carrier signal. One disadvantage of the AM modulation is that, when real valued signals are modulated, some bandwidth is wasted by having two identical side-bands on either side of the carrier frequency.

The so-called Single-Side Band (SSB) modulation solves the aforementioned problem. When a real signal is considered for transmission, one can exploit the symmetry of the Fourier transform of the real signal. In simple terms, given that we don't need a duplicated copy of the frequency information of $s(t)$ then we can eliminate such a redundant information! This elimination can be easily performed if one defines what is called the "analytic signal". The analytic signals associated to the information and the modulating signal are: $s_a(t) := s(t) + i\hat{s}(t)$ and $s_{c,a}(t) := s_c(t) + i\hat{s_c}(t)$, where $\hat{s}(t)$ and $\hat{s_c}(t)$ are the Hilbert transforms of $s(t)$ and $s_c(t)$, respectively. Therefore, if we modulate the analytic version of the information signal using the analytic version of the modulating signal we have:

$$
\begin{aligned}
s_{m,a}(t) &= s_a(t)s_{c,a}(t) = \big(s(t) + i\hat{s}(t)\big)\big(s_c(t) + i\hat{s_c}(t)\big) && (26)\\
&= \big(s(t) + i\hat{s}(t)\big)\big(\cos(2\pi f_c t) + i\sin(2\pi f_c t)\big) && (27)\\
&= s(t)\cos(2\pi f_c t) - \hat{s}(t)\sin(2\pi f_c t) + i\Big(s(t)\sin(2\pi f_c t) + \hat{s}(t)\cos(2\pi f_c t)\Big)\ . && (28)
\end{aligned}
$$

Finally, in applications the signal transmitted from sender to receiver in the case of SSB is the real part of the last equation: $s_m^{SSB} = s(t)\cos(2\pi f_c t) - \hat{s}(t)\sin(2\pi f_c t)$.

In Fig. 3 an example of SSB-AM is presented. For comparison purposes, a traditional or Double-Side Band (DSB) modulated signal is shown. The information signal considered is $s(t) = \cos(2\pi f_m t) + 0.5\cos(4\pi f_m t) + 0.25\cos(8\pi f_m t)$ for $f_m = 1$ [Hz] and the carrier signal has a frequency of f_c=40 [Hz].

Why this is important? Consider the case of AM radio broadcasting. If n is the maximum number of radio stations that can be allocated to the interval of frequencies associated to AM when DSB-AM is used, then by simply[2] switching to SSB modulation we can immediately allocated $2n$ radio stations.

1.3.2 Envelope Detection

Several applications result in a time-signal containing a rapidly oscillating component. The amplitude of the oscillation varies slowly with time, and the shape of the slow time variation is called the "envelope". The envelope often contains important information about the phenomenon of interest. One example of such applications is again an AM signal, see for example Fig. 3(a). By using the DHT, the rapid oscillations can be

[2]An additional technical detail has to be considered under this scenario: The AM radio receiver have to be aware that SSB is being used instead of DSB.

removed from the signal to produce a direct representation of the envelope. In terms of AM, the DHT can be used also to demodulate the transmitted signal.

As an example consider signal $x(t) = A\exp(-\lambda t)\cos(2\pi f_o t)$, which corresponds to the response of an exponentially dumped oscillating system, with decaying rate λ and oscillating frequency f_o. Assume that $A\exp(-\lambda t)$ is approximately constant with respect to $\cos(2\pi f_o t)$. If we form the analytic signal:

$$x_a(t) = x(t) + i\hat{x}(t) = A\exp(-\lambda t)\cos(2\pi f_o t) + iA\exp(-\lambda t)\sin(2\pi f_o t) \tag{29}$$

$$\therefore |x_a(t)| = \sqrt{A^2\exp(-2\lambda t)\big(\cos^2(2\pi f_o t) + \sin^2(2\pi f_o t)\big)} = A\exp(-\lambda t)\ , \tag{30}$$

which is the desired result. In Fig. 4 a numerical example is shown where $\lambda = 0.5$, $A = 10$, and f_o=4 [Hz].

1.3.3 Edge Detection in Image Processing

The DHT is used in image processing as a spatial filter because it can selectively emphasize features of an input object. In particular, the DHT produces an image that is edge enhanced with respect to the input image. However, given that the operation performed is one dimensional, then the DHT enhances edges along only a single direction. Hence, one has to apply the transform to the transpose image to obtain the edge enhancement in the other direction. Such a situation is shown in Fig. 5. Moreover, one should rotate the given image and apply the DHT in all the directions in order to obtain an image properly edge enhanced.

Why this is important? Edge detection is a commonly used approach for detecting discontinuities in an image. Moreover, edge detection is the first step in order to produce the segmentation of an image. Image segmentation is the process of partitioning an image into multiple regions of interest. For example, segmented images in medical imaging help the M.D.s to locate tumors and other pathologies, measure tissue volumes, etc.; in remote sensing applications a user can locate objects (called features or signatures) in the images such as roads, forests, etc.

2 Problem 2

Given $f \in L^p(\mathbb{R}^d)$, show that $\lim_{|h|\to\infty} ||f + \tau_h f||_p = 2^{1/p}||f||_p$.

Proof. We are going to use the fact that compactly supported functions (L^p_c) are dense in L^p, i.e. $\forall f \in L^p(\mathbb{R}^d)\ \exists \{f_n\}_{n=1}^{\infty}$, $f_n \in L^p_c(\mathbb{R}^d)$, such that $\|f - f_n\|_{L^p(\mathbb{R}^d)} \to 0$ as n goes to infinity.

Note that since τ_h is a linear operator and

$$\|\tau_h f - \tau_h f_n\|_{L^p(\mathbb{R}^d)} = \|\tau_h(f - f_n)\|_{L^p(\mathbb{R}^d)} = \|f - f_n\|_{L^p(\mathbb{R}^d)},$$

sequence $\{\tau_h f_n\}_{n=1}^{\infty}$ converges to $\tau_h f$ in $L^p(\mathbb{R}^d)$. Then we can write $\|f + \tau_h f\|_p$ as:

$$\begin{aligned}
\|f + \tau_h f\|_p &= \|f - f_n + f_n + \tau_h f_n + \tau_h f - \tau_h f_n\|_p \text{ (by triangle inequality)} && (31)\\
&\leq \|f - f_n\|_p + \|f_n + \tau_h f_n\|_p + \|\tau_h f - \tau_h f_n\|_p && (32)\\
&= 2\|f - f_n\|_p + \|f_n + \tau_h f_n\|_p . && (33)
\end{aligned}$$

Now note that since f_n's are compactly supported, for $|h|$ large enough f_n and $\tau_h f_n$ have disjoint support, so

$$\begin{aligned}
\|f_n + \tau_h f_n\|_p &= \left(\int_{\mathbb{R}^d} (f_n + \tau_h f_n)^p \, dx\right)^{1/p}\\
&= \left(\int_{supp\ f_n} f_n^p dx + \int_{supp\ \tau_h f_n} (\tau_h f_n)^p dx\right)^{1/p}\\
&= \left(2\int_{supp\ f_n} f_n^p dx\right)^{1/p} = 2^{1/p}\|f_n\|_p .
\end{aligned}$$

Hence, for all n and for $|h|$ large enough

$$\|f + \tau_h f\|_p \leq 2^{1/p}\|f_n\|_p + 2\|f - f_n\|_p .$$

In particular,

$$\lim_{|h|\to\infty} \|f + \tau_h f\|_p \leq 2^{1/p}\|f_n\|_p + 2\|f - f_n\|_p .$$

Letting $n \to \infty$ and using the fact that $f_n \to f$ in L^p we have:

$$\lim_{|h|\to\infty} \|f + \tau_h f\|_p \leq 2^{1/p}\|f\|_p .$$

Switching f and f_n we get the reverse inequality:

$$\begin{aligned}
2^{1/p}\|f\|_p &= \lim_{n\to\infty}\lim_{|h|\to\infty} \|f_n + \tau_h f_n\|_p\\
&\leq \lim_{n\to\infty}\lim_{|h|\to\infty} \Big(\|f - f_n\|_p + \|f + \tau_h f\|_p + \|f - \tau_h f\|_p\Big).
\end{aligned}$$

Since expression under the limit is positive we can interchange limits, obtaining

$$\lim_{|h|\to\infty} \|f + \tau_h f\|_p \geq 2^{1/p}\|f\|_p .$$

Which completes the proof of the exercise. □

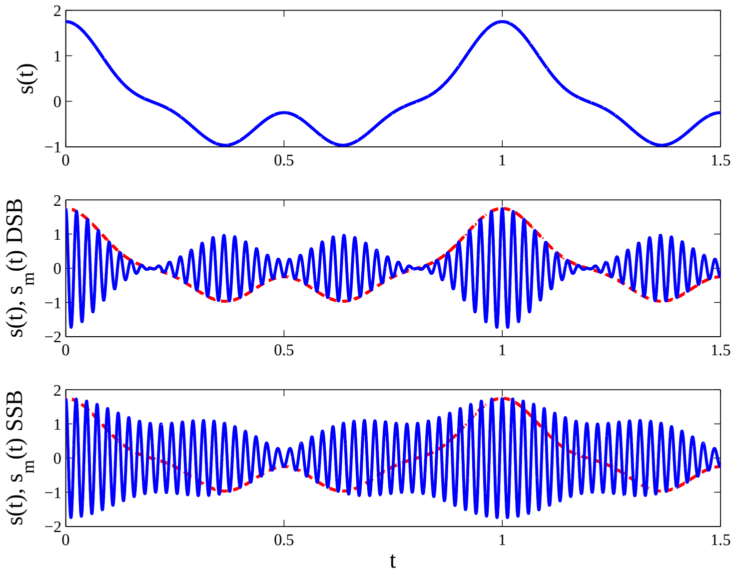
References

[1] Loukas Grafakos, "*An Elementary Proof of the Square Summability of the Discrete Hilbert Transform*," The American Mathematical Monthly, v. 101, N. 5, 1994, pp. 456–458.

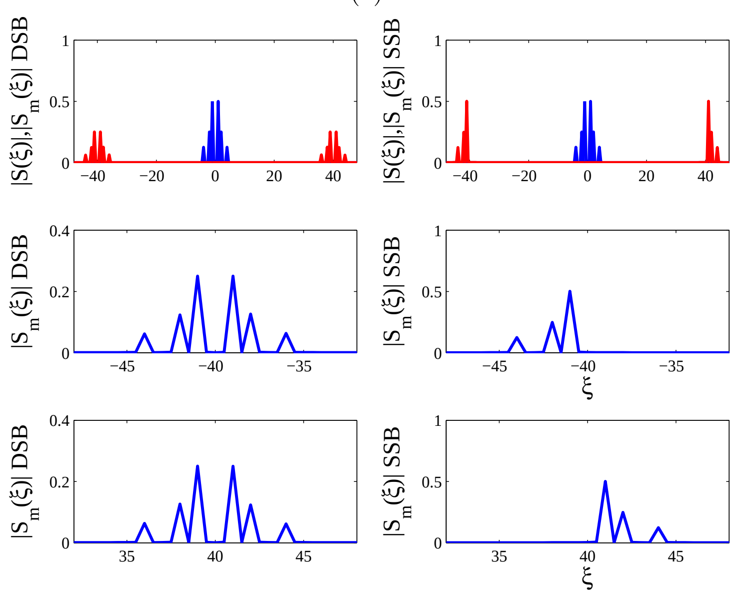
[2] Frank E. Burris, "*Matrix Formulation of the Discrete Hilbert Transform*," IEEE Transactions on Circuits and Systems, v. 22, N. 10, 1975, pp. 836–838.

[3] Javier Duoandicoetxea, "*Fourier Analysis*," AMS Graduate Studies in Mathematics Volume 29, 2001.

[4] G. M. Livadas and A. G. Constantinides, "*Image edge detection and segmentation based on the Hilbert transform*", In Proc. of ICASSP-88, v. 2, pp 1152–1155, 1988.



(a)



(b)

Fig. 3: (a) The information signal $s(t) = \cos(2\pi f_m t) + 0.5\cos(4\pi f_m t) + 0.25\cos(8\pi f_m t)$ for $f_m = 1$ [Hz] and f_c=40 [Hz]. Note that for DSB the information signal is the envelope of the modulated signal. (b) The power spectrum of the modulated signal when SSB and DSB modulation schemes are used. The four lower plots correspond to zoom in of the images on top.

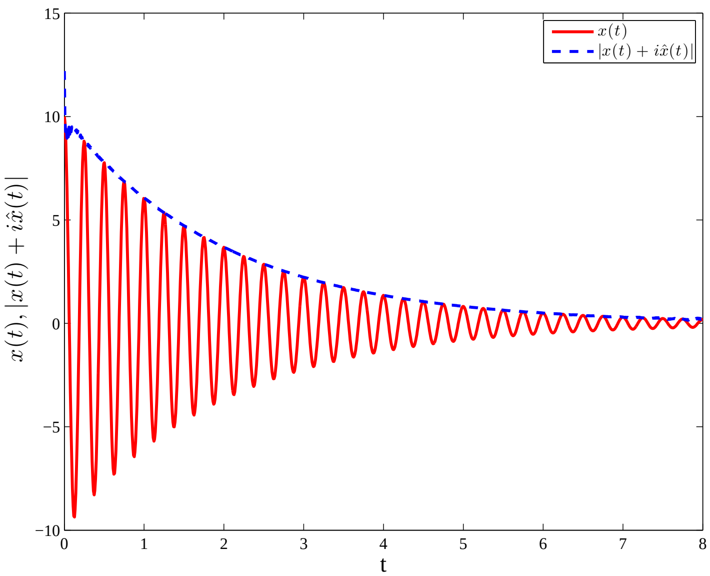


Fig. 4: Envelope detection using the DHT.

Fig. 5: Edge detection in image processing using the DHT. The left image is the original image, the center image is the edge enhanced image along the vertical direction and the right image is the edge enhanced image along the horizontal direction.