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1 Introduction

This report will address the boundedness of the discrete Hilbert transform in $\ell^2(\mathbb{Z})$. The argument makes use of Fourier-type arguments, in contrast to the article by Grafakos, which is available on the internet [1]. We will make use of the inverse Fourier transform to find a function, within $L^2(\mathbb{T})$ where $\mathbb{T} = [-\pi, \pi]$, associated to the discrete Hilbert transform.

2 Multiplier for Discrete Hilbert Transform

During the lectures, we were introduced to the Hilbert Transform on the Fourier-side. This definition was one of our first examples of a Fourier multiplier and, if you recall, it was given, at least for functions in the Schwartz class, as:

$$\widehat{H\phi}(\xi) = -i\text{sgn}(\xi)\widehat{\phi}(\xi).$$

For the discrete Hilbert transform, we will show that the analogous multiplier, $h(\xi)$ which is defined on $\mathbb{T}$, is given by:

$$h(\xi) = i(\pi\text{sgn}(\xi) - \xi).$$

3 Explanation

We begin by finding the Fourier coefficients of h; we note here that $h \in L^1(\mathbb{T})$ and, in fact, $\|h\|_1 = \pi^2$. As it will be required later, we also mention that $h \in L^2(\mathbb{T})$ and $h \in L^\infty(\mathbb{T})$ with associated norms of $\|h\|_2^2 = \frac{2\pi^3}{3}$ and $\|h\|_\infty = \pi$, respectively.

[1] http://www.math.missouri.edu/~loukas/preprints/monthly.pdf

By direct computation, we see that $\widehat{h}(0)=0$, since h is odd. For $n \neq 0 \in \mathbb{Z}$, we obtain that:

$$\begin{array}{rcl} \widehat{h}(n) & = & \frac{1}{2\pi}\int_{-\pi}^{\pi} h(\xi)e^{-in\xi}d\xi \\ & = & -\frac{1}{2\pi i}\{-\int_{-\pi}^{0}(\pi+\xi)e^{-in\xi}d\xi + \int_0^{\pi}(\pi-\xi)e^{-in\xi}d\xi\} \\ & = & -\frac{1}{2\pi i}\{-\int_0^{\pi}(\pi-\xi)e^{in\xi}d\xi + \int_0^{\pi}(\pi-\xi)e^{-in\xi}d\xi\} \\ & = & -\frac{1}{2\pi i}\{-\pi\int_0^{\pi}(e^{in\xi}-e^{-in\xi})d\xi + \int_0^{\pi}\xi(e^{in\xi}-e^{-in\xi})d\xi\} \\ & = & \int_0^{\pi}\sin(n\xi)d\xi - \frac{1}{\pi}\int_0^{\pi}\xi\sin(n\xi)d\xi \\ & = & \frac{1}{n}. \end{array}$$

The significance of this function is easily seen by examining the formula for the discrete Hilbert transform.

By Parseval, since $x \in \ell^2(\mathbb{Z})$, we can associate with it a function $f \in L^2(\mathbb{T})$ such that $\widehat{f}(j) = x_j$ for all $j \in \mathbb{Z}$. By definition, for this sequence and for $p \in \mathbb{Z}$:

$$\begin{array}{rcl} Hx(p) & = & \sum_{\substack{j=-\infty \\ j\neq p}}^{\infty} \frac{x_j}{p-j} \\ & = & \sum_{j=-\infty}^{\infty} \widehat{f}(j)\widehat{h}(p-j) \\ & = & (\widehat{f}\star\widehat{h})(p), \end{array}$$

where $\star$ denotes the discrete convolution of two sequences on $\mathbb{Z}$. We will be able to show that the discrete Hilbert transform is bounded by considering it as a Fourier coefficient. That is, if we can identify an $H \in L^2(\mathbb{T})$ such that $\widehat{H}(p) = Hx(p)$ for all $p \in \mathbb{Z}$, then we can use the identity of Parseval and obtain that $Hx \in \ell^2$.

Towards this end, we require the following fact concerning Fourier coefficients.

Theorem 3.1. *If $f \in L^2(\mathbb{T})$ and $g \in L^2(\mathbb{T})$, with $fg \in L^2(\mathbb{T})$, then $\widehat{fg}(n) = \sum_{j=-\infty}^{\infty} \widehat{f}(j)\widehat{g}(n-j) = (\widehat{f}\star\widehat{g})(n)$.*

Implicit in this theorem is the fact that, on the torus, $f \in L^2(\mathbb{Z}) \Rightarrow f \in L^1(\mathbb{Z})$; with this, each of the associated sequences of Fourier coefficients is $\ell^2(\mathbb{Z})$ summable. Since this result is well-known, we only provide a formal sketch of the proof. Assuming that $f(\xi) = \sum_{j=-\infty}^{\infty} \widehat{f}(j)e^{ij\xi}$ and $g(\xi) =$

$\sum_{k=-\infty}^{\infty} \widehat{g}(k) e^{ik\xi}$:

$$\begin{aligned}
\widehat{fg}(n) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\xi)g(\xi)e^{-in\xi}d\xi \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\sum_{j=-\infty}^{\infty}\widehat{f}(j)e^{ij\xi}\right)\left(\sum_{k=-\infty}^{\infty}\widehat{g}(k)e^{ik\xi}\right)e^{-in\xi}d\xi \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi}\sum_{j=-\infty}^{\infty}\sum_{k=-\infty}^{\infty}\widehat{f}(j)\widehat{g}(k)e^{ij\xi}e^{ik\xi}e^{-in\xi}d\xi \\
&= \frac{1}{2\pi}\sum_{j=-\infty}^{\infty}\sum_{k=-\infty}^{\infty}\widehat{f}(j)\widehat{g}(k)\int_{-\pi}^{\pi}e^{i(j+k-n)\xi}d\xi \\
&= \sum_{j=-\infty}^{\infty}\widehat{f}(j)\widehat{g}(n-j) \\
&= (\widehat{f}\star\widehat{g})(n),
\end{aligned}$$

by the orthogonality of the exponentials on $\mathbb{T}$.

Using this fact, we immediately obtain that $H(\xi) = f(\xi)h(\xi)$, where f and h have the same definitions as before. The theorem holds as $f \in L^2(\mathbb{T})$ (by Parseval) and $h \in L^\infty(\mathbb{T})$ (by direct computation) imply that $H = fh \in L^2(\mathbb{T})$. We finish by showing that the discrete Hilbert transform is bounded in $\ell^2(\mathbb{Z})$. We have, for $x \in \ell^2(\mathbb{Z})$:

$$\begin{aligned}
\|Hx\|_{\ell^2}^2 &= \sum_{p=-\infty}^{\infty}|Hx(p)|^2 \\
&= \|H\|_{L^2}^2 \\
&= \|fh\|_{L^2}^2 \\
&\leq \|h\|_{L^\infty}^2\|f\|_{L^2}^2 \\
&= \pi^2\|f\|_{L^2}^2 \\
&= \pi^2\|x\|_{\ell^2}^2,
\end{aligned}$$

which proves the boundedness. We point out that, in the second and sixth lines, we have used Parseval's identity; in the fourth, the result holds by a direct calculation.