MATH 565: Introduction to Harmonic Analysis - Spring 2008
Homework/Project # 3

We have three teams that will work on different components analyzing the discrete and finite Hilbert transforms. Each team will also do a homework problem of the three listed in class some weeks ago, as assigned below (part (b), you are welcome to try all of them not just yours!)

Consider the "discrete" Hilbert transform

$$H = \{h_{mn}\}_{m,n\in\mathbb{Z}}, \quad h_m n = \begin{cases} \frac{1}{m-n} & m \neq n \\ 0 & m = n \end{cases}$$

whose action on a doubly infinite sequence $x = \{x_n\}_{n\in\mathbb{Z}}$ is given by

$$(Hx)(i) = \sum_{j\in\mathbb{Z}, j\neq i} \frac{x_j}{i-j}, \quad i \in \mathbb{Z}.$$

Consider also a finite dimensional analogue, defined for $z = (z_{-N}, \ldots, z_{-1}, z_0, z_1, \ldots, z_N) \in \mathbb{C}^{2N+1}$ by,

$$(Hz)(i) = \sum_{|j|\leq N, j\neq i} \frac{x_j}{i-j}, \quad |i| \leq N.$$

Finite Team 1 (Jorge, Oleksandra, Dae-won):

(a) Check that H_N is bounded in $\ell^2(\mathbb{Z}_N)$ with bounds independent of the dimension. Show that the $\ell^1(\mathbb{Z}_N)$ norms grow with the dimension, find the rate of growth. Find applications of the discrite/finite Hilbert transforms to showcase. Consider the finite Fourier transform and its inverse, and compare the inverse finite Fourier transform of the Fouirer multiplier given by multiplication by $i\text{sgn}(n)$

(b) Given $f \in L^p(\mathbb{R}^d)$ show that

$$\lim_{|h|\to\infty} \|f + \tau_h f\|_p = 2^{1/p}\|f\|_p.$$

Fourier Team 2 (Jaime, Taylor, Arnab)

(a) Provide Fourier arguments to show that the discrete Hilbert transform H is bounded in $\ell^2(\mathbb{Z})$, is it an isometry perhaps up to a constant)?

(b) Prove a weighted version of Schur's Lemma: Let $k : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C}$ be a measurable function. Suppose w_1, w_2 are measurable functions in $\mathbb{R}^d$, and M_1, M_2 positive constants such that

$$\begin{aligned} \int w_1(x)|k(x,y)|dx &\leq M_1 w_2(y), \quad \text{a.e. y} \\ \int w_2^{p/p'}(x)|k(x,y)|dy &\leq M_2 w_1^{p/p'}(x), \quad \text{a.e. x.} \end{aligned}$$

Then the integral operator $Tf(x) = \int f(y)K(x,y)dy$ is bounded in L^p, moreover,

$$\|Tf\|_p \leq M_1^{1/p} M_2^{1/p'} \|f\|_p.$$

Cotlar's Team 3 (Jean, Jim, Hector)

(a) Use Cotlar's Lemma to show that the discrete Hilbert transform H is bounded in $\ell^2(\mathbb{Z})$.

(b) Let k_ϵ be the truncated continuous Hilbert kernel, $k_\epsilon(x) = \frac{1}{x}\chi_{|x|>\epsilon}$. Verify that $k_\epsilon * k_\epsilon \in L^1(\mathbb{R})$ with L^1-bound independent of $\epsilon > 0$.

Presentations and reports due on the Review and final's Weeks May 5-9, May 12-17, 2008.