MATH 565: Introduction to Harmonic Analysis - Spring 2008
Homework # 2

Choose at least 2 out of the following four problems.

1. Show that,

$$H\chi_{[a,b]}(x) := \lim_{\epsilon\to 0} H^{\epsilon}(\chi_{[a,b]})(x) = \frac{1}{\pi}\log\frac{|x-a|}{|x-b|}.$$

and that

$$H^{*}\chi_{[a,b]}(x) := \sup_{\epsilon\to 0} |H^{\epsilon}(\chi_{[a,b]})(x)| = \frac{1}{\pi}\left|\log\frac{|x-a|}{|x-b|}\right|.$$

2. Show that

$$|\{x\in\mathbb{R} : |H\chi_{[a,b]}(x)| > \lambda\}| = \frac{4\,|b-a|}{e^{\pi\lambda}-e^{-\pi\lambda}}.$$

More generally show that for any measurable subset E of $\mathbb{R}$ of finite measure $|E|$,

$$|\{x\in\mathbb{R} : |H\chi_{E}(x)| > \lambda\}| = \frac{4\,|E|}{e^{\pi\lambda}-e^{-\pi\lambda}} \le \frac{2\,|E|}{\pi\lambda}.$$

Hint: See exercise 4.1.4. in Grafakos book.

3. Let $P_t(x) = \frac{1}{\pi}\frac{t}{x^2+t^2}$ be the Poisson kernel, and $Q_t(x) = \frac{1}{\pi}\frac{x}{x^2+t^2}$ be the Conjugate Poisson kernel dfined for all $t>0$. Check that $\{P_t\}_{t>0}$ is an approximation of the identity as $t\to 0$, but $\{Q_t\}_{t>0}$ is not.
Verify that

$$\widehat{P_t}(\xi) = e^{-2\pi t|\xi|}, \quad \widehat{Q_t}(\xi) = -i\,\mathrm{sgn}\xi\, e^{-2\pi t|\xi|}.$$

4. Exercise 4.1.11 (a) in Grafakos (characterization of the Hilbert transform via invariances)

Due on Thursday March 27, 2008.