MATH 565: Introduction to Harmonic Analysis - Spring 2008
Homework # 1

1. Let (X,μ) be a measure space. Let $\{T_t\}_{t>0}$ be a family of linear operators from $L^p(X,\mu)$ into $\mathcal{M}(X,\mu)$, $1 \le p$. Assume the maximal operator associated to the family, $T^*f(x) := \sup_{t>0} |T_t f(x)|$, is of weak-type (p,q), $1 \le q \le \infty$. Show that the the set

$$B = \{f \in L^p(X,\mu) : \lim_{t\to 0} T_t f(x) \quad \text{exists a.e.}\}$$

is closed.

2. Let $f \in L^p(X,\mu)$ and define its distribution function $d_f : [0,\infty) \to [0,\infty)$,

$$d_f(\lambda) := \mu\{x \in X : |f(x)| > \lambda\}.$$

Show that

$$\|f\|^p_{L^p(X,\mu)} = p \int_0^\infty \lambda^{p-1} d_f(\lambda)\, d\lambda.$$

3. Given $f \in L^p(X,\mu)$, and $1 \le p_0 < p < p_1$ define

$$f_0(x) := \begin{cases} f(x) & \text{if } \ |f(x)| > \lambda \\ 0 & \text{otherwise} \end{cases}, \qquad f_1(x) := f(x) - f_0(x).$$

Show that $f_0 \in L^{p_0}(X,\mu)$ and $f_1 \in L^{p_1}(X,\mu)$.

4. Let $1 \le p_0 < p < p_1 < \infty$, $A_0, A_1 > 0$. Verify that if you choose $C > 0$ such that $(2A_0C)^{p_0} = (2A_1C)^{p_1}$ then the quantity

$$\frac{p^{1/p}}{C}\left[\frac{(2A_0C)^{p_0}}{p-p_0} + \frac{(2A_1C)^{p_1}}{p_1-p}\right]^{1/p} = 2\left[\frac{p}{p-p_0} + \frac{p}{p_1-p}\right]^{1/p} A_0^{1-\theta}A_1^{\theta}.$$

This calculation gives the constant we claimed in Marcinkiewicz Interpolation Theorem.
Consider the left-hand-side as a function of $C > 0$. Can you find its global minimum? Does it coincide with the previous choice?

Due on Thursday February 28, 2008.