Practice for Midterm Exam - MATH 402/502 - Spring 2019

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1. Let $a < b$ be real numbers and let $f : [a, b] \to \mathbb{R}$ be Riemann integrable. Let $g : [-b, -a] \to \mathbb{R}$ be defined by $g(x) := f(-x)$. Show that g is Riemann integrable, moreover
$$\int_{[-b,-a]} g = \int_{[a,b]} f.$$

2. Let $A = (1, 3) \cup \{0\} \cup \{1/n : n \geq 1, n \in \mathbb{N}\}$ a subset of $\mathbb{R}$. Decide whether A is open, closed, bounded, connected, and/or compact. Explain why or why not the set has any of the given properties. Find the closure, the interior, and the boundary of A, explain your answers.

3. Let (X, d_X) and (Y, d_Y) be metric spaces. Show that the set $Z = X \times Y$ is a metric space with metric $d : Z \times Z \to \mathbb{R}$ defined by
$$d\big((x, y), (x', y')\big) = d_X(x, x') + d_Y(y, y').$$

4. Show that the finite union of compact sets in metric space (X, d) is compact.

5. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by
$$f(x, y) = \begin{cases} y \sin(1/x) & \text{if } xy \neq 0 \\ 0 & \text{otherwise} \end{cases}.$$
Show that f is continuous function at $(0, 0)$.

6. Let $P : \mathbb{R}^2 \to \mathbb{R}$ be a polynomial in two variables, namely a function of the form
$$P(t, s) = \sum_{i=1}^{n} \sum_{j=1}^{m} c_{ij} t^i s^j.$$
   (a) Show that P is continuous on $\mathbb{R}^2$.
   (b) Let (X, d) be a metric space, $f, g : X \to \mathbb{R}$ continuous functions in X. Show that the function $H : X \to \mathbb{R}$ defined by $H(x) = P(f(x), g(x))$ is continuous in X.

7. Let (X, d_X) and (Y, d_Y) be metric spaces, f, f_n be functions from $X \to Y$ such that f_n converges uniformly to f and each f_n is bounded. Show that f is bounded and then show that the functions f_n are uniformly bounded.

8. Let (X, d_X) be a compact metric space, f, f_n, g, g_n be functions from $X \to R$ such that f_n converges uniformly to f, g_n converges uniformly to g. Assume also that f_n and g_n are continuous. Show that the sequence $f_n g_n$ converges uniformly to fg.

9. Let the function $f, f_n : (-1, 1) \to \mathbb{R}$ be given by the geometric sum $f_n(x) = \sum_{k=1}^{n} x^k$ and $f(x) = \frac{x}{1-x}$. Show that f_n converges uniformly to f on $[-r, r]$ for each $0 < r < 1$ (Hint: use the M-Weierstrass test). Justify differentiation and integration term by term formulas to conclude that for all $x \in (-1, 1)$,
$$\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} n x^{n-1}, \qquad -\log(1-x) - x = \sum_{n=1}^{\infty} \frac{x^{n+1}}{n+1}.$$

10. Show that the functions $f_n : \mathbb{R} \to \mathbb{R}$ defined for each $n \geq 1$ by $f_n(x) = \sqrt{x^2 + 1/n^2}$ converge uniformly to the function $f(x) = |x|$. Observe that each of the functions f_n is differentiable, do their derivatives f_n' converge uniformly?

11. Prove the Weierstrass M-test: Given $f_n : X \to R$, assume that for each $n \geq 0$ we have $|f_n(x)| \leq a_n$ for all $x \in X$ and the numerical series $\sum_{n=0}^{\infty} a_n$ is convergent. Then $\sum_{n=0}^{\infty} f_n(x)$ converges uniformly to a bounded function f on X. (If you assume the functions f_n are continuous then f is a continuous functions, see Theorem 3.5.7).