Practice for Second Exam - MATH 402 - Spring 2019

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1. Exercise 3.2.1(b) (connection between uniform continuity and uniform convergence).

2. Exercises 3.3.6 and 3.3.7 (Uniform convergence preserves boundedness but pointwise convergence does not necessarily preserve boundedness).

3. Exercise 3.4,3 (the metric space of continuous functions with the uniform distance is complete).

4. Recall that a function $f : [a, b] \to \mathbb{R}$ is piecewise constant or is a step function if there is a finite partition $\mathcal{P}$ of subintervals of the interval $I = [a, b]$ such that $f(x) = \sum_{J\in\mathcal{P}(I)} c_J \chi_J(x)$. Prove that every continuous function on $[a, b]$ is a uniform limit of piecewise constant or step functions. (Hint: remember that continuous functions on compact sets are uniformly continuous).

5. Compute the radius of convergence of the power series and expand the given function in a power series:

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{n^4}{n!}\, x^n$, (b) $\displaystyle\sum_{n=1}^{\infty} \sqrt{n}\, 2^n\, x^n$, (c) $f(x) = \dfrac{1}{1+x^2}$.

6. Given $f : \mathbb{R}^n \to \mathbb{R}^m$, with components $f = (f_1, \ldots, f_m)$. Show that f is differentiable at $\mathbf{a} \in \mathbb{R}^n$ if and only if $f_i : \mathbb{R}^n \to \mathbb{R}$ is differentiable at $\mathbf{a}$ for all $i = 1, \ldots, m$. Moreover $Df(\mathbf{a})$, the matrix representation of the linear transformation $f'(\mathbf{a})$, is the matrix whose rows are $Df_i(\mathbf{a})$, the matrix representation of $f_i'(\mathbf{a})$ for each $i = 1, \ldots, m$.

7. Show that the function $f : \mathbb{R}^2 \to \mathbb{R}$ given by $f(x, y) = |xy|$ is differentiable at $(0, 0)$ but is not of class C^1 in any neighborhood of $(0, 0)$.

8. (Mean Value Theorem) Let D be an open subset of $\mathbb{R}^n$, let $f : D \to \mathbb{R}$ be differentiable on D. If D contains a line segment with end points $\mathbf{a}$ and $\mathbf{a} + \mathbf{h}$, then there is a point $\mathbf{c} = \mathbf{a} + \xi\mathbf{h}$ with $0 < \xi < 1$ on the line segment such that
$$f(\mathbf{a} + \mathbf{h}) - f(\mathbf{a}) = f'(\mathbf{c}) \cdot \mathbf{h}.$$

9. Let D be an open subset of $\mathbb{R}^n$, let $f : D \to \mathbb{R}^n$ be differentiable on D. Let $\mathbf{a} \in D$ and $f(\mathbf{a}) = \mathbf{b}$. Suppose that g maps a neighborhood V of $\mathbf{b}$ into $\mathbb{R}^n$, that $g(\mathbf{b}) = \mathbf{a}$ and that
$$g(f(\mathbf{x})) = \mathbf{x}$$
for all $\mathbf{x}$ in a neighborhood U of $\mathbf{a}$. If f is differentiable at $\mathbf{a}$ and g is differentiable at $\mathbf{b}$ then
$$g'(\mathbf{b})f'(\mathbf{a}) = I_n.$$
Where I_n denotes the identity linear transformation in $\mathbb{R}^n$, that is $I(\mathbf{x}) = \mathbf{x}$. This shows that $f'(\mathbf{a})$ is an invertible linear transformation moreover $g'(\mathbf{b}) = [f'(\mathbf{a})]^{-1}$.

10. Exercises 6.6.1, 6.6.2, 6.6.3, and 6.6.4 (differentiability vs contraction).

11. Show that every contraction defined on a metric space X is a continuous function.

12. Consider $y(t) = \tan t$ for $|t| < \pi/2$ assume known basic properties.

    (a) Show that $y' = 1 + y^2$ and $y(0) = 0$. Show that $\tan(t) = \int_0^t (1 + \tan(u)^2)\, du$ for $|t| < \pi/2$.

    (b) Consider the map $F(f)(t) = \int_0^t \big(1 + f^2(u)\big)\, du$, show that it takes continuous functions bounded by $M > 0$ on $[-\delta, \delta]$ into continuous functions bounded by $M > 0$ in $[-\delta, \delta]$ for $\delta > 0$ small enough and is a strict contraction in the metric space $C([-\delta, \delta])$ with the uniform metric for sufficiently small δ.

    (c) By the contraction mapping theorem F has a unique fixed point, $F(f) = f$ and by (b) that unique fixed point is $f(t) = \tan t$. The contraction mapping theorem provides an algorithm to find the fixed point. Initiate the algorithm with $f_0(t) = 0$ and compute several iterations, what do you get?