APPLICATION OF THE CONTRACTION MAPPING THEOREM TO EXISTENCE AND UNIQUENESS OF SOLUTION TO AN ODE MATH 402/502 - SPRING 2017

INSTRUCTOR: C. PEREYRA

ABSTRACT. In this note will see that in the setting of ordinary differential equations we are able to use the Contraction Mapping Theorem to prove existence and uniqueness of a solution to an initial value problem under certain conditions. In this case, X will be a closed ball in the complete metric space of continuous functions on a closed and bounded interval with the uniform metric, and $F : X \to X$ will be an integral operator. The technicalities come in determining the bounded interval, and the contraction mapping property for F.

1. The contraction mapping theorem

On Tuesday April 11, I stated and proved the Contraction Mapping Theorems for you. See Section 6.6 in Analysis II.

Theorem 1.1 (Contraction Mapping Theorem)**.** *Let (X, d) be a complete metric space, let $F : X \to X$ be continuous function such that there is a real number α, with $0 \le \alpha < 1$ and satisfying*

$$d\big(F(x), F(y)\big) \le \alpha d(x, y), \quad \text{for all } x, y \in X. \tag{1.1}$$

Then there exists a unique point $x_0 \in X$ such that $F(x_0) = x_0$ (i.e. x_0 is a fixed point for F). Moreover given any $y_0 \in X$, the sequence $\{y_n\}_{n\ge 0}$, defined recursively by $y_n := F(y_{n-1})$ for all $n \ge 1$, is a Cauchy sequence in X, convergent to x_0 the unique fixed point of F.

Remark 1.2. Functions that obey (1.1) with $0 \le \alpha \le 1$ are called *contractions*, and with $0 \le \alpha < 1$ are called *strict contractions* by Tao.

Sketch of the proof. (Existence) First show that the sequence $\{y_n\}_{n\ge 0}$ is Cauchy, since X is complete, the sequence is convergent to a point $x_0 \in X$. Second show that $x_0 = \lim_{n\to\infty} y_n$, is a fixed point. This takes care of the existence.

To show the sequence is Cauchy, use repeatedly the fact that F is a contraction map to conclude that for consecutive points we can estimate the distance

$$d(y_{k+1}, y_k) \le \alpha^k d(y_1, y_0).$$

The distance between y_n and y_m, for all $n > m \ge N$, can then be estimated using the triangle inequality to interspace consecutive terms by the tail of convergent geometric series:

$$\begin{aligned} d(y_n, y_m) &\le d(y_n, y_{n-1}) + d(y_{n-1}, y_{n-2}) + ... + d(y_{m+1}, y_m) \\ &\le (\alpha^{n-1} + \alpha^{n-2} + ... + a\alpha^m) d(y_1, y_0) \\ &\le \frac{\alpha^N}{1-a} d(y_1, y_0). \end{aligned}$$

The right-hand-side can be made less than a given $\epsilon > 0$ by choosing N large enough since $0 < \alpha < 1$ implies that α^N goes to zero as N goes to infinity. To show that the limit point x_0 is a fixed point use the fact that F is continuous, hence since y_n converges to x_0, then $F(y_n)$ converges to $F(x_0)$, however by definition of the sequence $\{y_n\}$, $F(y_n) = y_{n+1}$, and the shifted sequence converges to x_0 (why?). All together the sequence $F(y_n)$ converges to two limits: $F(x_0)$ and x_0, but limits are unique, therefore $F(x_0) = x_0$, and x_0 is a fixed point for F as claimed.

(Uniqueness) Assume there are two fixed points reach a contradiction (use the fact that F is a contraction map to reach the contradiction that $\alpha \geq 1$). $\square$

Example 1.3. Given real numbers $a < b$, let $f : [a, b] \to [a, b]$ be a continuous function, differentiable on (a, b) with $|f'(x)| \leq \alpha$ for all $x \in (a, b)$ and for some $\alpha < 1$. Then f is a (strict) contraction (why? by the Mean Value Theorem), and $X = [a, b]$ with the Euclidean metric, is a complete metric space. By the Contraction Mapping Theorem there exists a unique fixed point $c \in [a, b]$ such that $f(c) = c$, and we have an algorithm to find c.

Note that the existence of a fixed point in this case is a consequence of the Intermediate Value Theorem applied to the continuous function $g(x) = f(x) - x$. More precisely, since f maps into $[a, b]$ it means $a \leq f(x) \leq b$ for all $x \in [a, b]$, hence $g(b) \leq 0 \leq g(a)$, therefore there is $c \in [a, b]$ such that $g(c) = 0$, this implies $f(c) = c$. This argument is purely existencial, it does not give us an algorithm to find c.

The contraction mapping theorem is a extremely useful result, it will imply the inverse function theorem, which in turn implies the implicit function theorem (these two theorems, which imply each other, will be the subject of my last two lectures). Another application of the contraction mapping theorem is to the existence and uniqueness of solutions to an initial value problem for ordinary differential equations. I tried to illustrate this on Tuesday, I stated exactly what we were going to proof, and outlined most of the issues, but I did not have time to finish. I promised I will type this for you in laTeX and distribute it, as this is not in our textbook, for your future reference. You can find this and more in [Ro].

2. Application to an Initial Value Problem

Let E be an open subset of $\mathbb{R}^2$. Given a continuous function $f : E \to \mathbb{R}$ and a point $(a, b) \in E$, we seek a solution to the following Initial Value Problem:

$$\frac{dy}{dx} = f(x, y), \tag{2.1}$$

$$y(a) = b. \tag{2.2}$$

This means we want to find an $h > 0$ and a differentiable function $\phi : [a - h, a + h] \to \mathbb{R}$ such that $\phi'(x) = f\big(x, \phi(x)\big)$ for all $x \in [a-h, a+h]$ and $\phi(a) = b$. Necessarilly $\big(x, \phi(x)\big) \in E$ for all $x \in [a - h, a + h]$. If $E = \mathbb{R}^2$ this is no longer an issue, since for all x in the domain of ϕ, it will hold that $\big(x, \phi(x)\big)$ is in $\mathbb{R}^2$ the domain of f.

Remark 2.1. Even if f is defined on all $\mathbb{R}^2$ and a very nice function, the domain of the solution can be very small ($h > 0$ maybe very small). Also, there could be more than one solution. We will stipulate certain hypothesis which will guarantee the existence of a unique solution.

Theorem 2.2. *Let E be an open subset of $\mathbb{R}^2$. Given a continuous function $f : E \to \mathbb{R}$ and a point $(a, b) \in E$, assume f is Lipschitz on the y-variable, that is, there is $M > 0$ such that*

$$|f(x, y) - f(x, z)| \leq M|y - z|, \quad \textit{for all } (x, y), (x, z) \in E. \tag{2.3}$$

Then there is $h > 0$ such that there is a unique function $\phi : (a - h, a + h) \to \mathbb{R}$ solving the initial value problem $\phi'(x) = f\big(x, \phi(x)\big)$ for all $x \in [a - h, a + h]$ and $\phi(a) = b$.

Proof. The first observation is that by the Fundamental Theorem of calculus, we can replace the initial value problem by an equivalent integral equation. If indeed $\phi'(x) = f\big(x, \phi(x)\big)$ for all $x \in [a - h, a + h]$ and $\phi(a) = b$, then

$$\phi(x) - b = \phi(x) - \phi(a) = \int_a^x \phi'(t)\,dt = \int_a^x f\big(t, \phi(t)\big)\,dt.$$

Hence the function ϕ that we are looking for solves the following integral equation,

$$\phi(x) = b + \int_a^x f\big(t, \phi(t)\big)\,dt. \tag{2.4}$$

Notice that if a function ϕ satisfies (2.4) then ϕ is differentiable by the FTC and $\phi'(x) = f\big(x, \phi(x)\big)$ for all $x \in [a - h, a + h]$ and $\phi(a) = b$. Also note that since f is assumed to be continuous on E then the function $g : [a - h, a + h] \to \mathbb{R}$ by $g(t) = f(t, \psi(t))$ is a continuous function, hence Riemann integrable.

Our goal then is to find $h > 0$ so that if $\psi : [a - h, a + h] \to R$ then $(x, \psi(x)) \in E$ for all $x \in [a - h, a + h]$. To achieve that we will enforce that ψ be continuous and that $|\psi(x) - b| \leq Nh$ for all $|x - a| \leq h$ and for some $N > 0$ to be specified in a couple paragraphs, in other words, ψ is in the closed ball B centered at the constant (hence continuous) function $\psi_0(x) = b$ and with radius Nh in the complete metric space $C([a - h, a + h])$ of continuous functions on $[a - h, a + h]$. The closed ball B is closed in this complete metric space, hence B is itself a complete metric space with uniform distance $d : B \times B \to [0, \infty)$ given by

$$d(\psi_1, \psi_2) = \sup_{x \in [a - h, a + h]} |\psi_1(x) - \psi_2(x)|.$$

Suppose we have found h with the properties described in the previous paragraph. Then let $F : B \to C[a - h, a + h]$ be defined by

$$F(\psi)(x) := b + \int_a^x f\big(t, \psi(t)\big)\,dt, \quad \text{for all } \psi \in B, x \in [a - h, a + h]. \tag{2.5}$$

Since f is continuous, then $F(\psi)$ is well defined and continuous on $[a - h, a + h]$ by Fundamental Theorem of Calculus.

Claim: (i) $F : B \to B$, that is if $\psi \in B$ then $F(\psi) \in B$.

(ii) F is a (strict) contraction provided h is chosen small enough ($h < 1/M$ suffices, where M is the Lipschitz constant of f).

Assuming the claim (hence the existence of h small enough, given $N > 0$, so that everything said holds) we can now use the Contraction Mapping Theorem for $X = B = \{\psi \in C([a - h, a + h)) : |\psi(x) - b| \leq Nh \text{ for all } x \in [a - h, a + h]\}$ a complete metric space with the uniform metric d, and $F : B \to B$ the integral operator defined in (2.5). Hence there is a unique function $\phi \in B$ such that $F(\phi) = \phi$, but this is precisely the integral equation (2.4), and as a consequence, $\phi'(x) = f\big(x, \phi(x)\big)$ for all $x \in [a - h, a + h]$ and $\phi(a) = b$.

To find $h > 0$. Choose $N > 0$ such that $|f(a,b)| < N$, by continuity of f at (a,b), there is $r > 0$ such that $B_{\mathbb{R}^2}\big((a,b),r\big) \subset E$ and $|f(x,y)| \leq N$ for all $(x,y) \in B_{\mathbb{R}^2}\big((a,b),r\big)$. Let

$$h := \min\{r/2, r/2N, 1/M\} > 0, \tag{2.6}$$

this choice enforces that $h \leq r/2$, $Nh \leq r/2$, and $Mh \leq 1$. The first two constraints on h imply that the rectangle $R = [a-h, a+h] \times [b-Nh, b+Nh]$ is a subset of E. Indeed, R is contained in the square $[a-r/2, a+r/2] \times [b-r/2, b+r/2]$ which is clearly contained in the ball $B_{\mathbb{R}^2}\big((a,b),r\big) \subset E$. The third constarint ensures that $\alpha = Mh < 1$, where α will be the contraction constant for the mapping F.

Proof of the Claim. (i) Assume $\psi \in B \subset C([a-h, a+h])$ we need to show that $F(\psi) \in B$. If $\psi \in B$ then $|\psi(x) - b| \leq Nh$ for all $|x-a| \leq h$ and this implies $(x, \psi(x)) \in R \subset B_{\mathbb{R}^2}\big((a,b),r\big) \subset E$ for all $|x-a| \leq h$. Hence we can evaluate $f(t,\psi(t))$ for all $t \in [a-h, a+h]$ and moreover $|f(t,\psi(t))| \leq N$, by definition of the ball B. With this in mind we estimate $|F(\psi)(x) - b|$ for each $x \in [a-h, a+h]$ using the triangle inequality for integrals,

$$|F(\psi)(x) - b| \leq \int_a^x |f\big(t, \psi(t)\big)|\, dt \leq Nh.$$

This is precisely saying that $F(\psi) \in B$.

(ii) To show that $F : B \to B$ is a contraction we estimate the difference between the images under F of two functions $\psi_1, \psi_2 \in B$, we use the triangle inequality for integrals and the Lipschitz condition (2.3) for f, namely, for all $|x-a| \leq h$,

$$\begin{aligned} |F(\psi_1)(x) - F(\psi_2)(x)| &= \left| \int_a^x \Big(f\big(t, \psi_1(t)\big) - f\big(t, \psi_2(t)\big) \Big)\, dt \right| \\ &\leq \int_a^x |f\big(t, \psi_1(t)\big) - f\big(t, \psi_2(t)\big)|\, dt \\ &\leq \int_a^x M|\psi_1(t) - \psi_2(t)|\, dt \\ &\leq hMd(\psi_1, \psi_2). \end{aligned}$$

Taking the supremum over all $x \in [a-h, a+h]$ we conclude that

$$d\big(F(\psi_1), F(\psi_2)\big) \leq hMd(\psi_1, \psi_2),$$

where $\alpha := hM < 1$ by our definition of $h > 0$.

□

References

[Ro] Maxwell Rosenlicht, *Introduction to Analysis.* Dover Publications (1986).