Prop 11 (e). f, g are Riemann integrable, show that the $\min(f, g)$ is Riemann integrable.

Proof : *Given* $\varepsilon > 0$, $\exists$ *step functions*, $\underline{f}$, $\overline{f}$, $\underline{g}$, $\overline{g}$ *so that* $\underline{f} \le f \le \overline{f}$ *and* $\underline{g} \le g \le \overline{g}$.

$$\int_I f - \varepsilon \le \int_I \underline{f} \le \int_I \overline{f} \le \int_I f + \varepsilon \quad and \quad \int_I g - \varepsilon \le \int_I \underline{g} \le \int_I \overline{g} \le \int_I g + \varepsilon$$

Lemma If f, g *are step functions, then* $\min(f, g)$ *is also a step functions and hence integrable*

i.e we can talk about $\int_I \min(\overline{f}, \overline{g})$ *and* $\int_I \min(\underline{f}, \underline{g})$ *i.e.* $\min(\underline{f}, \underline{g}) \le \min(f, g) \le \min(\overline{f}, \overline{g})$

$$\int_I \min(\underline{f}, \underline{g}) \le \underline{\int_I} \min(f, g) \le \overline{\int_I} \min(f, g) \le \int_I \min(\overline{f}, \overline{g})$$

$$0 \le \overline{\int_I} \min(f, g) - \underline{\int_I} \min(f, g) \le \int_I \min(\overline{f}, \overline{g}) - \int_I \min(\underline{f}, \underline{g})$$

Define $h = (\overline{f} - \underline{f}) + (\overline{g} - \underline{g})$, h *is a step function*

$$0 \le \overline{\int_I} \min(f, g) - \underline{\int_I} \min(f, g) \le \int_I \min(\overline{f}, \overline{g}) - \int_I \min(\underline{f}, \underline{g})$$

$$0 \le \int_I h = \int_I (\overline{f} - \underline{f}) + \int_I (\overline{g} - \underline{g}) \le 2\varepsilon + 2\varepsilon = 4\varepsilon$$

Case 1. *let* $\min(\overline{f}, \overline{g}) = \overline{f}$ *and* $\min(\underline{f}, \underline{g}) = \underline{f}$, *and we have* $\overline{f} - \underline{f} \ge 0$, $\overline{g} - \underline{g} \ge 0$

$$0 \le \min(\overline{f}, \overline{g}) - \min(\underline{f}, \underline{g}) = (\overline{f} - \underline{f}) \le (\overline{f} - \underline{f}) + (\overline{g} - \underline{g}) = h$$

Case 2

let $\min(\overline{f}, \overline{g}) = \overline{g}$ *and* $\min(\underline{f}, \underline{g}) = \underline{g}$, *then* $0 \le \min(\overline{f}, \overline{g}) - \min(\underline{f}, \underline{g}) = \overline{g} - \underline{g} \le (\overline{g} - \underline{g}) + (\overline{f} - \underline{f}) = h$

Case 3

Let $\min(\overline{f}, \overline{g}) = \overline{f}$, *and* $\min(\underline{f}, \underline{g}) = \underline{g}$, *then* $\underline{g} \le \underline{f} \le \overline{f} \Rightarrow \overline{f} - \underline{g} \ge 0$. *Also* $\underline{f} \le \min(\overline{f}, \overline{g}) = \overline{f} \le \overline{g}$

$\Rightarrow \overline{g} - \underline{f} \ge 0$

$$0 \le \min(\overline{f}, \overline{g}) - \min(\underline{f}, \underline{g}) = \overline{f} - \underline{g} \le (\overline{f} - \underline{g}) + (\overline{g} - \underline{f}) = h$$

Case 4 *Let* $\min(\overline{f}, \overline{g}) = \overline{g}$, $\overline{f} \ge \overline{g} \ge \underline{g} \Rightarrow \overline{f} - \underline{g} \ge 0$. *Also Let* $\min(\underline{f}, \underline{g}) = \underline{f}$, $\underline{f} \le \overline{g} \Rightarrow \overline{g} - \underline{f} \ge 0$

$$0 \le \min(\overline{f}, \overline{g}) - \min(\underline{f}, \underline{g}) = \overline{g} - \underline{f} \le (\overline{g} - \underline{f}) + (\overline{f} - \underline{g}) = h$$

Conclusion

$$0 \le \min(\overline{f}, \overline{g}) - \min(\underline{f}, \underline{g}) \le h$$

$$0 \le \int_I (\min(\overline{f}, \overline{g}) - \min(\underline{f}, \underline{g})) \le \int_I h \le 4\varepsilon$$

$\forall \varepsilon > 0$, *we have* $0 \le \overline{\int_I \min(f, g)} - \underline{\int_I \min(f, g)} \le \int_I (\min(\overline{f}, \overline{g}) - \min(\underline{f}, \underline{g})) \le 4\varepsilon \to 0$

By Squeeze Theorem

$\overline{\int_I \min(f, g)} - \underline{\int_I \min(f, g)} = 0 \Rightarrow \min(f, g)$ *is Riemann integrable.*