Exercise: Let I be a bounded interval, and let $f : I \to \mathbb{R}$ and $g : I \to \mathbb{R}$ be Riemann integrable functions on I. Let $a, b \in \mathbb{R}$, prove that $af + bg$ is Riemann integrable, and that $\int_I (af + bg) = a(\int_I f) + b(\int_I g)$.

We will prove this exercise as follows:
- First we will prove a lemma for linearity and integration of piecewise constant(step) functions.
- Then we will prove that $f + g$ is Riemann integrable, and that $\int_I (f + g) = \int_I f + \int_I g$.
- Next we will prove that cf is Riemann integrable, and that $\int_I cf = c(\int_I f)$ (first for $c \geq 0$, and then for $c < 0$).
- Together, this proves that $af + bg$ is Riemann integrable, and that $\int_I (af + bg) = a(\int_I f) + b(\int_I g)$.

First let us prove the following lemma.

Lemma 1. *Let I be a bounded interval, $a, b \in \mathbb{R}$, and $f : I \to \mathbb{R}$ and $g : I \to \mathbb{R}$ be piecewise constant functions on I, then $af + bg$ is also piecewise constant on I and $p.c.\int_I (af + bg) = a(p.c.\int_I f) + b(p.c.\int_I g)$.*

Proof. First note that we can assume that f, g are piecewise constant with respect to the same partition $\mathcal{P}$ since if they were not, we could take the common refinement.
Since f, g are piecewise constant w.r.t. $\mathcal{P}$ we have that $f(x) = c_J, g(x) = d_J \ \forall x \in J \in \mathcal{P}$. So $(af + bg)(x) = ac_J + bd_J \ \forall x \in J \in \mathcal{P}$. Thus $af + bg$ is piecewise constant on I.

$$\begin{aligned} a\left(p.c.\int_I f\right) + b\left(p.c.\int_I g\right) &= a\left(\sum_{J\in\mathcal{P}} c_J|J|\right) + b\left(\sum_{J\in\mathcal{P}} d_J|J|\right) \\ &= \sum_{J\in\mathcal{P}} (ac_J|J|) + \sum_{J\in\mathcal{P}} (bd_J|J|) \\ &= \sum_{J\in\mathcal{P}} (ac_J + bd_J)|J| \\ &= p.c.\int_I (af + bg) \end{aligned}$$

Giving us that $a(p.c.\int_I f) + b(p.c.\int_I g) = p.c.\int_I (af + bg)$, as desired. ◻

Now let us prove that $f + g$ is Riemann integrable and that $\int_I (f + g) = \int_I f + \int_I g$.

Proof. Since f, g are Riemann integrable, given $\epsilon > 0$ we can find piecewise constant functions $\underline{f}, \overline{f}, \underline{g}, \overline{g}$ such that:

$$\underline{f}(x) \leq f(x) \leq \overline{f}(x) \ \forall x \in I \tag{1}$$

and

$$\underline{g}(x) \le g(x) \le \overline{g}(x)\ \forall x \in I \tag{2}$$

and also that:

$$\int_I f - \epsilon \le \int_I \underline{f} \le \underline{\int_I} f = \int_I f = \overline{\int_I} f \le \int_I \overline{f} \le \int_I f + \epsilon \tag{3}$$

$$\int_I g - \epsilon \le \int_I \underline{g} \le \underline{\int_I} g = \int_I g = \overline{\int_I} g \le \int_I \overline{g} \le \int_I g + \epsilon \tag{4}$$

From (1) and (2) we get that

$$\underline{f}(x) + \underline{g}(x) \le f(x) + g(x) \le \overline{f}(x) + \overline{g}(x)\ \forall x \in I$$

which implies that we have:

$$\int_I (\underline{f} + \underline{g}) \le \underline{\int_I}(f+g) \le \overline{\int_I}(f+g) \le \int_I (\overline{f} + \overline{g}) \tag{5}$$

From (3),(4),(5), and **Lemma 1**, we get that:

$$0 \le \overline{\int_I}(f+g) - \underline{\int_I}(f+g) \le \int_I (\overline{f} + \overline{g}) - \int_I (\underline{f} + \underline{g}) = \left(\int_I \overline{f} + \int_I \overline{g}\right) - \left(\int_I \underline{f} + \int_I \underline{g}\right)$$

$$\le \left(\int_I f + \epsilon + \int_I g + \epsilon\right) - \left(\int_I f - \epsilon + \int_I g - \epsilon\right) = 4\epsilon$$

Hence, $0 \le \overline{\int_I}(f+g) - \underline{\int_I}(f+g) \le 4\epsilon$. Let $\epsilon \to 0$ then we get that $\overline{\int_I}(f+g) = \underline{\int_I}(f+g)$ and so $f+g$ is Riemann integrable.

To show that $\int_I (f+g) = \int_I f + \int_I g$ let us do the following:
Add (3) and (4), use (5), **Lemma 1**, and that $f+g$ is Riemann integrable to get the following inequalities:

$$\begin{aligned} \int_I f + \int_I g \ - 2\epsilon &\le \int_I \underline{f} + \int_I \underline{g} \\ &\le \underline{\int_I} f + \underline{\int_I} g = \int_I f + \int_I g = \overline{\int_I} f + \overline{\int_I} g \\ &\le \int_I \overline{f} + \int_I \overline{g} \le \int_I f + \int_I g \ + 2\epsilon \end{aligned}$$

Therefore we have that:

$$\int_I f + \int_I g \; - 2\epsilon \leq \int_I (f+g) \leq \int_I f + \int_I g \; + 2\epsilon$$

So now we can let $\epsilon \to 0$ (since we no longer have anything that depends on ϵ) and conclude that $\int_I (f+g) = \int_I f + \int_I g$.

Thus $f+g$ is Riemann integrable and $\int_I (f+g) = \int_I f + \int_I g$ □

Now let us prove that for $c \in \mathbb{R}, c \geq 0$ and f Riemann integrable, that cf is Riemann integrable, and that $\int_I cf = c\int_I f$.

Proof. Since f is Riemann integrable, we know that given $\epsilon > 0$ there are functions, say $\underline{f}, \overline{f}$ such that

$$\underline{f}(x) \leq f(x) \leq \overline{f}(x) \;\forall x \in I \tag{6}$$

and

$$\int_I f - \epsilon \leq \int_I \underline{f} \leq \underline{\int_I} f = \int_I f = \overline{\int_I} f \leq \int_I \overline{f} \leq \int_I f + \epsilon \tag{7}$$

Multiplying (6),(7) through by our constant c we get the following:

$$c\underline{f}(x) \leq cf(x) \leq c\overline{f}(x) \;\forall x \in I \tag{8}$$

$$c\int_I f - c\epsilon \leq c\int_I \underline{f} \leq c\underline{\int_I} f = c\int_I f = c\overline{\int_I} f \leq c\int_I \overline{f} \leq c\int_I f + c\epsilon \tag{9}$$

From (8), and **Lemma 1** we know that $c\underline{f}$ and $c\overline{f}$ are piecewise constant and so we have:

$$\int_I c\underline{f} \leq \underline{\int_I} cf \leq \overline{\int_I} cf \leq \int_I c\overline{f} \tag{10}$$

Now from (9),(10), and **Lemma 1**, we get:

$$0 \leq \overline{\int_I} cf - \underline{\int_I} cf \leq \int_I c\overline{f} - \int_I c\underline{f} = c\int_I \overline{f} - c\int_I \underline{f} \leq \left(c\int_I f + c\epsilon\right) - \left(c\int_I f - c\epsilon\right) = 2c\epsilon$$

Hence, $0 \leq \overline{\int_I} cf - \underline{\int_I} cf \leq 2c\epsilon$, let $\epsilon \to 0$ and we get that $\overline{\int_I} cf = \underline{\int_I} cf$ so we have that cf is Riemann integrable for $c \geq 0$.

Now let us show that $\int_I cf = c\int_I f$

From (9), (10), **Lemma 1**, and that cf is Riemann integrable we have the following inequalities:

$$c\int_I f - c\epsilon \le c\int_I \underline{f} = \int_I c\underline{f} \le \underline{\int_I} cf = \int_I cf = \overline{\int_I} cf \le \int_I c\overline{f} = c\int_I \overline{f} \le c\int_I f + c\epsilon$$

Therefore, we have:

$$c\int_I f - c\epsilon \le \int_I cf \le c\int_I f + c\epsilon$$

Now we can let $\epsilon \to 0$ and conclude that $\int_I cf = c\int_I f$.

Thus, cf is Riemann integrable and $\int_I cf = c\int_I f$. □

Now let us prove that cf is Riemann integrable and that $\int_I cf = c\int_I f$ for $c < 0$.

Proof. Since f is Riemann integrable, we know that given $\epsilon > 0$ there are functions, say $\underline{f}, \overline{f}$ such that

$$\underline{f}(x) \le f(x) \le \overline{f}(x)\ \forall x \in I \tag{11}$$

and

$$\int_I f - \epsilon \le \int_I \underline{f} \le \underline{\int_I} f = \int_I f = \overline{\int_I} f \le \int_I \overline{f} \le \int_I f + \epsilon \tag{12}$$

Multiplying (11),(12) through by our constant c we get the following:

$$c\overline{f}(x) \le cf(x) \le c\underline{f}(x)\ \forall x \in I \tag{13}$$

$$c\int_I f + c\epsilon \le c\int_I \overline{f} \le c\overline{\int_I} f = c\int_I f = c\underline{\int_I} f \le c\int_I \underline{f} \le c\int_I f - c\epsilon \tag{14}$$

Note think about flipping the upper and lower integrals.

From (13), and **Lemma 1** we know that $c\underline{f}$ and $c\overline{f}$ are piecewise constant and so we have:

$$\int_I c\overline{f} \le \underline{\int_I} cf \le \overline{\int_I} cf \le \int_I c\underline{f} \tag{15}$$

Now from (14),(15), and **Lemma 1**, we get:

$$0 \le \overline{\int_I} cf - \underline{\int_I} cf \le \int_I c\underline{f} - \int_I c\overline{f} = c\int_I \underline{f} - c\int_I \overline{f} \le \left(c\int_I f - c\epsilon\right) - \left(c\int_I f + c\epsilon\right) = -2c\epsilon$$

Hence we have, $0 \le \overline{\int_I} cf - \underline{\int_I} cf \le -2c\epsilon$, let $\epsilon \to 0$ and we get that $\overline{\int_I} cf = \underline{\int_I} cf$ so we have that cf is Riemann integrable for $c < 0$.

Now let us show that $\int_I cf = c\int_I f$ for $c < 0$.

From (14),(15), **Lemma 1**, and that cf is Riemann integrable we get the following inequalities:

$$c\int_I f + c\epsilon \leq c\int_I \overline{f} = \int_I c\overline{f} \leq \underline{\int_I} cf = \int_I cf = \overline{\int_I} cf \leq \int_I c\underline{f} = c\int_I \underline{f} \leq c\int_I f - c\epsilon$$

Therefore we have that:

$$c\int_I f + c\epsilon \leq \int_I cf \leq c\int_I f - c\epsilon$$

Now we have nothing that depends on ϵ, so we can let $\epsilon \to 0$ and conclude that $\int_I cf = c\int_I f$ for $c < 0$. $\square$

Together, this proves that $af + bg$ is Riemann integrable, and that $\int_I (af + bg) = a(\int_I f) + b(\int_I g)$.