Theorem 11.4.1 (g)

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We first prove the following lemma

Lemma 1 *Let I be a bounded interval, and let $f : I \to \mathbf{R}$ be a Riemann integrable function on I ($f \in \mathcal{R}(I)$) that is also a step or piecewise constant function (p.c.f.) with respect to a partition $\mathbf{P}_I$. Let J be a bounded interval containing I (i.e. $I \subseteq J$), then*

$$F(x) := \begin{cases} f(x) & \text{if } x \in I \\ 0 & \text{if } x \notin I \end{cases}$$

is a p.c.f. with respect to a partition $\mathbf{P}_J = \mathbf{P}_I \bigcup \{J \backslash I\}$. Moreover $F \in \mathcal{R}(J)$ and $\int_J F = \int_I f$.

Proof: If $f(x)$ is a p.c.f. with respect to a partition $\mathbf{P}_I$ then $f(x) = c_K$ for all $x \in K \in \mathbf{P}_I$, where c_K denotes a constant. Then by assumption $F(x) = f(x) = c_K$ for all $x \in K \in \mathbf{P}_I$ and $F(x) = 0$ for all $x \in K \in \{J \backslash I\}$, i.e. $F(x) = C_K$ for all $x \in K \in \mathbf{P}_I \bigcup \{J \backslash I\}$ where $C_K = c_K$ if $K \in \mathbf{P}_I$ and $C_K = 0$ if $K \in \{J \backslash I\}$. Since $\mathbf{P}_I \bigcap \{J \backslash I\} = \emptyset$, $F(x)$ is a p.c.f with respect to $\mathbf{P}_I \bigcup \{J \backslash I\} = \mathbf{P}_J$.

Consider

$$\int_I f = \sum_{K \in \mathbf{P}_I} c_K = \sum_{K \in \mathbf{P}_I} c_K + \sum_{K \in \{J \backslash I\}} 0 = \sum_{K \in \mathbf{P}_J} C_K = \int_J F$$

(The first equality follows from the definition of Riemann integral for a p.c.f., the second from adding zero, the third from the definition of F and because $\mathbf{P}_I \bigcap \{J \backslash I\} = \emptyset$ and $\mathbf{P}_I \bigcup \{J \backslash I\} = \mathbf{P}_J$, and the forth from the definition of the Riemann integral for a p.c.f.) Thus $F \in \mathcal{R}(J)$ and $\int_J F = \int_I f$. ■

We now prove the main theorem.

Theorem 1 (Theorem 11.4.1 (g)) *Let I be a bounded interval, and let $f : I \to \mathbf{R}$ be a Riemann integrable function on I ($f \in \mathcal{R}(I)$). Let J be a bounded interval containing I (i.e. $I \subseteq J$), and let $F : J \to \mathbf{R}$ be the function*

$$F(x) := \begin{cases} f(x) & \text{if } x \in I \\ 0 & \text{if } x \notin I \end{cases}$$

then F is Riemann integrable on J and $\int_J F = \int_I f$.

Proof: Given $\epsilon > 0$, there exists two p.c.f. $\underline{f}$ and $\overline{f}$ such that $\underline{f} \leq f \leq \overline{f}$ (note that $\underline{f}$ and $\overline{f}$ depend on $\epsilon > 0$,) and

$$\int_I \underline{f} \geq \int_I f - \epsilon \quad \text{and} \quad \int_I \overline{f} \leq \int_I f + \epsilon \tag{1}$$

Let

$$\underline{F}(x) := \begin{cases} \underline{f}(x) & \text{if } x \in I \\ 0 & \text{if } x \in J \backslash I \end{cases} \quad \text{and} \quad \overline{F}(x) := \begin{cases} \overline{f}(x) & \text{if } x \in I \\ 0 & \text{if } x \in J \backslash I \end{cases}$$

Note that $\underline{F}(x) \leq F(x) \leq \overline{F}(x)$, $\forall x \in J$ and

$$\int_I \underline{f} = \int_I \underline{f} + \int_{J \backslash I} 0 = \int_J \underline{F} \tag{2}$$

where the first inequality follows from adding zero and the second from Lemma 1. Similarly

$$\int_I \overline{f} = \int_I \overline{f} + \int_{J \backslash I} 0 = \int_J \overline{F} \tag{3}$$

Also note that

$$\int_J \underline{F} \le \sup_{\{\underline{F}\le F;\underline{F} \text{ p.c.f.}\}} \left[\int_J \underline{F}\right] = \underline{\int_J} F \quad \text{and} \quad \int_J \overline{F} \ge \inf_{\{\overline{F}\le F;\overline{F} \text{ p.c.f.}\}} \left[\int_J \overline{F}\right] = \overline{\int_J} F \tag{4}$$

and that (by Lemma 11.3.3)

$$\underline{\int_J} F \le \overline{\int_J} F \tag{5}$$

Putting together equations (1), (2), (3), (4), and (5) we obtain

$$\int_I f - \epsilon \le \int_I \underline{f} = \int_J \underline{F} \le \underline{\int_J} F \le \overline{\int_J} F \le \int_J \overline{F} = \int_I \overline{f} \le \int_I f + \epsilon \tag{6}$$

This implies that

$$\int_I f - \epsilon \le \underline{\int_J} F \le \overline{\int_J} F \le \int_I f + \epsilon$$

Since this inequality is true for all ϵ, this implies

$$\int_I f = \underline{\int_J} F = \overline{\int_J} F,$$

which means that $F \in \mathcal{R}(J)$ and that $\int_I f = \int_J F$. ■