Exercise 11.4.2
Sara Pollock

Let $a < b \; \epsilon \; \mathbf{R}$ and $f : [a, b] \rightarrow \mathbf{R}$ a continuous, non-negative function: $f(x) \geq 0 \; \forall x \; \epsilon \; [a, b]$. Suppose $\int_{[a,b]} f = 0$. Show: $f(x) = 0$ for each $x \; \epsilon \; [a, b]$.

Proof by contradiction:

Assume $\exists \; c \; \epsilon \; [a, b] : f(c) = \alpha > 0$.

f is continuous on $[a, b] \implies f$ is continuous at $c \; \epsilon \; [a, b]$.
Then by the epsilon-delta definition of continuity, we may choose $\epsilon_1 : 0 < \epsilon_1 < \alpha$. Then $\exists \; \delta_1 > 0 : x \; \epsilon \; [a, b] \; s.t. \; |x - c| < \delta_1 \Longrightarrow |f(x) - \alpha| < \epsilon_1$.

Now, to be sure we stay within the interval $[a, b]$, define:

$$\delta_0 \equiv \begin{cases} min\{\frac{\delta_1}{2}, \frac{|c-a|}{2}, \frac{|c-b|}{2}\}, & c \neq a, \; c \neq b \\ min\{\frac{\delta_1}{2}, \frac{|b-a|}{2}\}, & c = a \text{ or } c = b \end{cases}$$

We have $\delta_0 \leq \frac{\delta_1}{2} < \delta_1$, so let's now define $I_0, \; |I_0| > 0$:

$$I_0 \equiv \begin{cases} [c - \delta_0, \; c + \delta_0], & c \neq a, \; c \neq b \\ [a, \; a + \delta_0], & c = a \\ [b - \delta_0, \; b], & c = b \end{cases}$$

Denote $I_0 = [a', b']$. It can be verified that $I_0 \subset [a, b]$.

We are assured that $\forall x \; \epsilon \; I_0 : 0 < \alpha - \epsilon < f(x) < \alpha + \epsilon$. In particular, $0 < \alpha - \epsilon < f(x)$. By property (h):

$$0 = \int_{[a,b]} f = \int_{[a,a']} f + \int_{[a',b']} f + \int_{[b',b]} f$$

Since $0 < \alpha - \epsilon < f(x) \; \forall x \; \epsilon \; I_0$, we have $\int_{[a',b']} f > |I_0|(\alpha - \epsilon) > 0$.

Now by property (d), $f(x) \geq 0 \; \forall x \; \epsilon \; I \Longrightarrow \int_I f \geq 0$, for I any subinterval of $[a, b]$.

So then

$$0 = \int_{[a,b]} f \geq \underbrace{\int_{[a,a']} f}_{\geq 0} + \underbrace{|I_0|(\alpha - \epsilon)}_{>0} + \underbrace{\int_{[b',b]} f}_{\geq 0} > 0, \text{ a contradiction!}$$

As our assumption that $\exists \; c \; \epsilon \; [a, b] : f(c) = \alpha > 0$ led to a contradiction, we conclude that $f(x) = 0$ for each $x \; \epsilon \; [a, b]$.