Review and Practice Problems for Exam # 2 - MATH 401/501 - Spring 2020

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Real numbers

- Real numbers are closed under addition, multiplication, negation, subtraction and division by non-zero real numbers. You are free to use usual arithmetic properties (commutative and associative properties of addition and multiplication, distributive property, etc).
- Real numbers have an order, and obey a trichotomy if x, y are real numbers then exactly one of the following holds: $x = y$, $x < y$ or $x > y$.
- Should know and be able to use
  (i) the definition of absolute value of a real number,
  (ii) the triangle inequality (and "reverse" triangle inequality).
- Understand Archimedean properties and their implications: interspersing of integers by $\mathbb{R}$, density of rationals and irrationals.
- Understand the meaning of upper and lower bounds for a set in $\mathbb{R}$, and the meaning of the supremum (least upper bound or l.u.b.) and infimum (greatest lower bound or g.l.b.) of a set of real numbers.
- Be able to show that a given number is the supremum (infimum) of a set by showing that
  (i) it is an upper (lower) bound for the set,
  (ii) it is the smallest upper (largest lower) bound.
- Appreciate the Least Upper Bound (l.u.b.) and Greatest Lower Bound (g.l.b.) properties of real numbers: every non-empty and bounded set of real numbers has a unique l.u.b. called the supremum of the set and a unique g.l.b. called the infimum of the set.

Sequences of real numbers

- Know the definition of bounded sequences, bounded above sequences and bounded below sequences. More precisely a sequence $\{x_n\}_{n\geq 0}$ is bounded (respectively bounded above or bounded below) iff there is $M > 0$ such that for all $n \geq 0$ we have $|x_n| \leq M$ (respectively $x_n \leq M$ or $M \leq x_n$).
- Know the ϵ, N definition of Cauchy sequences and of convergent sequences in $\mathbb{R}$ to a limit $L \in \mathbb{R}$. More precisely, a sequence $\{x_n\}_{n\geq 0}$ of real numbers
  - is Cauchy iff given $\epsilon > 0$ there is $N > 0$ such that for all $n, m \geq N$ then $|x_n - x_m| \leq \epsilon$,
  - converges to L iff given $\epsilon > 0$ there is $N > 0$ such that for all $n \geq N$ then $|x_n - L| \leq \epsilon$.
- Be able to show that limits are unique (that is if a sequence converges it converges to a unique limit).
- Be able to prove or disprove that a given sequence converges or is Cauchy by using the "ϵ, N definition". E.g. $a_n = 1/n$, $b_n = 2^{-n}$.
- Be able to show that a convergent sequence is a Cauchy sequence.
- Be able to show and use that Cauchy sequences (and hence convergent sequences) are bounded sequences. However not all bounded sequences are convergent, e.g . $b_n = (-1)^n$ for all $n \geq 0$.
- Be able to show that the sum/product of two Cauchy sequences (or two convergent sequences) is a Cauchy sequence (or a convergent sequence and convergent to the sum/product of the limits of the given convergent sequences "limit laws").
- Understand that if a Cauchy (convergent) sequence is bounded away from zero then the sequence of reciprocals is Cauchy (convergent and to the reciprocal of the limit which is necessarily non-zero, another "limit law").

- Be able to prove or disprove that a given sequence converges by appealing to additive/multiplicative/reciprocal properties of limits (limit laws), and using known basic limits.
- Know and be able to use the Monotone Bounded Sequence Convergence Theorem:
  (i) an increasing and bounded above sequence is convergent and to the sequence's supremum,
  (ii) a decreasing and bounded below sequence is convergent and to the sequence's infimum.
- You should know and use some basic limits :
  - $\lim_{n\to\infty} x^n = 0$ if $|x| < 1$, is 1 if $x = 1$, and does not exist if $x = -1$ or $|x| > 1$;
  - $\lim_{n\to\infty} x^{1/n} = 1$ if $x > 0$;
  - $\lim_{n\to\infty} 1/n^{1/k} = 0$ for all integers $k \geq 1$.
  - $\lim_{n\to\infty} n^{1/n} = 1$;
  - Let $a_n > 0$, $q \in \mathbb{R}$, if $\lim_{n\to\infty} a_n = 1 > 0$ then $\lim_{n\to\infty} a_n^q = 1$.
- Appreciate and use the deep fact that Cauchy sequences are convergent sequences in $\mathbb{R}$ (completeness of the real numbers). [You don't need to remember how to prove this fact, this was hard to prove.]

Limit points, limsup, liminf

- Appreciate the definition of "limit points" of a sequence as the collection of "subsequencial limits" (the limits of convergent subsequences of the sequence.
- Know that c is a limit point for a sequence $\{x_n\}$ if for all $\epsilon > 0$ there are "infinitely many" terms of the sequence in the interval $[c-\epsilon, c+\epsilon]$. More precisely, for all $\epsilon, N > 0$ there is an $n_N \geq N$ such that $|x_{n_N} - c| \leq \epsilon$ (necessarily the set of labels $\{n_N\}_{N\geq 0}$ is an infinite set!).
- Be able to identify the "limit points" (or "subsequencial limits") of a concrete sequence e.g: $a_n = 3$ for all $n \geq 0$, $b_n = (-1)^n$ for all $n \geq 0$, $c_n = (-1)^n n$ for all $n \geq 0$.
- Know that bounded sequences in $\mathbb{R}$ have limit superior/inferior in $\mathbb{R}$, defined as
  $$\limsup\{x_n\} := \lim_{N\to\infty} \sup_{n\geq N} x_n \text{ and } \liminf\{x_n\} := \lim_{N\to\infty} \inf_{n\geq N} x_n.$$
- Be aware of the ϵ-N characterization of limsup (similarly liminf, not recorded here): for all $\epsilon > 0$
  (i) Finitely many terms of the sequence $\{x_n\}$ are larger than $\limsup\{x_n\} + \epsilon$. More precisely for all $\epsilon > 0$ there is $N > 0$ such that for all $n \geq N$ we have $x_n \leq \limsup\{x_n\} + \epsilon$.
  (ii) Infinitely many terms of the sequence $\{x_n\}$ are in between $\limsup\{x_n\} - \epsilon$ and $\limsup\{x_n\} + \epsilon$. More precisely, for all $\epsilon > 0$ and $N > 0$ there is $n_N \geq N$ such that $|x_{n_N} - \limsup\{x_n\}| \leq \epsilon$.

  And its consequences:
  - $\limsup\{x_n\}$ and $\liminf\{x_n\}$ are limit points (subsequencial limits) of the sequence.
  - A sequence of real numbers converges if and only if the limsup and the liminf coincide.
  - The limsup is the "largest limit point" (or "largest subsequential limit") of the sequence, and liminf is the "smallest limit point" (or "smallest subsequential limit") of the sequence.
  - A sequence converges to L iff all its subsequences converge to L iff the unique limit point of the sequence is L.
  - Every bounded sequence has a at least one convergent subsequence or equivalently at least one "limit point" (Bolzano-Weierstrass theorem).
- Be able to identify the $\limsup$ and $\liminf$ of a given sequence. Use this knowledge to conclude that if $\limsup a_n = \liminf a_n = L$ then the sequence $\{a_n\}$ converges AND $\lim_{n\to\infty} a_n = L$.
- Be able to use the squeeze theorem to deduce convergence of the sequence being squeezed.

Series

- Understand that convergence of a series is by definition convergence of the sequence of partial sums.
- Be able to deduce from the theory of sequences basic convergence tests: Cauchy test, divergence test, absolute convergence test, comparison test.

- Be familiar with other useful tests such as: alternating series test, p-test, root test, and ratio test. Be able to use these tests to deduce convergence or divergence of specific series.
- Be able to exploit convergence properties of geometric series: $\sum_{n=0}^{\infty} r^n$ converges to $1/(1-r)$ if $|r|<1$, diverges otherwise.

Limits and continuity of functions $f : E \to \mathbb{R}$, $E \subset \mathbb{R}$

- Know definition of a bounded function: $\exists M > 0$ such that $|f(x)| \le M$ for all $x \in E$.
- Know the equivalent "definitions" of $\lim_{x\to x_0} f(x) = L$. Let $E \subset \mathbb{R}$ and $f : E \to \mathbb{R}$, x_0 is an *adherent point*[1] of E ($x \in \overline{E}$), then $\lim_{x\to x_0, x\in E} f(x) = L$ if and only if
  - (ϵ-δ definition) $\forall \epsilon > 0\ \exists \delta > 0$ such that $|f(x) - L| \le \epsilon\ \forall x \in E$ such that $|x - x_0| \le \delta$.
  - (Sequential definition) For all sequences $\{x_n\}_{n\ge 0}$ in E if $\lim_{n\to\infty} x_n = x_0$ then $\lim_{n\to\infty} f(x_n) = L$.
- Know the equivalent "definitions" of continuity at a point x_0. Let $E \subset \mathbb{R}$ and $f : E \to \mathbb{R}$, $x_0 \in E$, then f is continuous at x_0 if and only if
  - (Limit definition) $\lim_{x\to x_0, x\in E} f(x) = f(x_0)$.
  - (ϵ-δ definition) $\forall \epsilon > 0\ \exists \delta > 0$ such that $|f(x) - f(x_0)| \le \epsilon\ \forall x \in E$ such that $|x - x_0| \le \delta$.
  - (Sequential definition) For all sequences $\{x_n\}_{n\ge 0}$ in E if $\lim_{n\to\infty} x_n = x_0$ then $\lim_{n\to\infty} f(x_n) = f(x_0)$.
- Be able to decide whether a function is bounded or not and whether a function is continuous or not.
- Know that basic functions are continuous such us: constant function ($f(x) = c$), identity function ($f(x) = x$), absolute value function ($f(x) = |x|$ for $x \in \mathbb{R}$), and exponential functions ($f(x) = x^p$ for $x > 0$, and $g(x) = a^x$ for $a > 0$ and $x \in \mathbb{R}$).
- Know the limit laws for functions and be able to prove them and use them to compute limits.
- Know and be able to prove that composition and arithmetic operations preserve continuity. Use these properties to conclude that more complex functions are continuous such us: polynomials ($p(x) = a_0 + a_1 x + \cdots + a_n x^n$), rational functions (quotients of polynomials, wherever the denominator is non-zero), exponentials with continuous base or exponent ($f(x) = a^{p(x)}$ or $g(x) = f(x)^q$ where f is a positive and continuous function and $q \in \mathbb{R}$.
- Use your knowledge of continuous functions to compute limits for example:
  - $\lim_{x\to x_0} x^q = x_0^q$ for $x, x_0 > 0$ and $q \in \mathbb{R}$.
  - $\lim_{x\to x_0} a^x = a^{x_0}$ for $x, x_0 \in \mathbb{R}$ and $a > 0$.

Practice Problems for Midterm #2

1. If the real number x is not rational we say x is "irrational".
   (a) Show that if $p \in \mathbb{Q}$, $p \ne 0$, and x is irrational then px is irrational.
   (b) Show that if $x, y \in \mathbb{R}$ and $x < y$ then there is an irrational number w such that $x < w < y$ (density of the irrational numbers).
2. For each subset A of real numbers decide whether is bounded (above, below or both), find supremum and infimum: (a) $A = \{1, -1/2, 3\}$, (b) $A = \{n/(n+1) :\ n \in \mathbb{N},\ n \ge 1\}$, (c) $A = \{r \in \mathbb{Q} : r < 5\}$.
3. If A and B are nonempty and bounded subsets of $\mathbb{R}$ such that $A \subset B$ show that $\inf(B) \le \inf(A)$.

[1] A point $x_0 \in E \subset \mathbb{R}$ is adherent iff for all $\delta > 0$ there is $x \in E$ such that $|x - x_0| \le \delta$ (in words, we can get arbitrarily close to x_0 with points x in E). The collection of adherent point of E, the closure of E, is denoted $\overline{E}$.

4. Let E be a nonempty and bounded subset of $\mathbb{R}$, let $\lambda \in \mathbb{R}$ and $\lambda > 0$. Define $\lambda E = \{\lambda x : x \in E\}$ a subset of $\mathbb{R}$. Prove that If $\lambda \geq 0$ then $\sup(\lambda E) = \lambda \sup(E)$. What is $\inf(\lambda E)$? What if $\lambda < 0$?

5. Given $\lambda > 0$ and $\{s_n\}_{n\geq 0}$ is a bounded sequence. Show that $\limsup\{\lambda s_n\} = \lambda \limsup\{s_n\}$. What can you say when $\lambda < 0$? (Hint use previous exercise).

6. For each of the following, prove or give a counterexample.

   (a) If $\{x_n\}_{n\geq 0}$ converges to x then $\{|x_n|\}_{n\geq 0}$ converges to $|x|$.

   (b) If $\{|x_n|\}_{n\geq 0}$ is convergent then $\{x_n\}_{n\geq 0}$ is convergent .

7. We say the sequence $\{x_n\}_{n\geq 0}$ diverges to $+\infty$ and we write $\lim\limits_{n\to\infty} x_n = +\infty$ iff for all $M > 0$ there is $N > 0$ such that for all $n \geq N$ we have $x_n \geq M$.

   (a) Write down a definition for a sequence $\{y_n\}_{n\geq 0}$ to diverge to $-\infty$.

   (b) Show that if $x_n \leq z_n$ for all $n \geq 0$ and $\{x_n\}_{n\geq 0}$ diverges to $+\infty$ then $\{z_n\}_{n\geq 0}$ diverges to $+\infty$.

   (c) Let $\{x_n\}_{n\geq 0}$ sequence in $\mathbb{R}$, $x_n > 0$. Show that $\lim\limits_{n\to\infty} x_n = +\infty$ if and only if $\lim_{n\to\infty}(1/x_n) = 0$.

8. The sequence of positive real numbers $\{t_n\}_{n\geq 0}$ converges to t. Decide whether the following sequences are convergent or not. If convergent explain why and identify the limit, if not convergent explain why. Find the lim sup and lim inf of each sequence.

   (a) $a_n = \sqrt{t_n}$, (b) $b_n = 5t_n^3 - t_n^2 + 7$, (c) $c_n = \dfrac{n}{2^n}(-1)^n$, (c) $d_n = n + t_n$.

9. Use squeeze theorem and properties of sine function to show $\lim\limits_{n\to\infty} \dfrac{\sin n}{n} = 0$.

10. Show that the sequence defined by $x_1 = 1$ and $x_{n+1} = \sqrt{1 + x_n}$ for $n \geq 1$ is convergent (hint: show that it is increasing and bounded by 2). Find the limit.

11. Let $x_n = n\sin^2(n\pi/2)$. Find the set S of limit points (subsequencial limits), find $\limsup x_n$ and $\liminf x_n$. (Assume known properties about sine function.)

12. Show that the sequence of partial sums: $S_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$ defined for $n \geq 1$ is not Cauchy (hint: show that $S_{2n} - S_n \geq 1/2$). Conclude that the harmonic series is divergent.

13. A sequence $\{s_n\}_{n\geq 0}$ is contractive if there is a constant r with $0 < r < 1$ such that $|s_{n+2} - s_{n+1}| \leq r|s_{n+1} - s_n|$ for all $n \geq 0$. Show that a contractive sequence is a Cauchy sequence and hence a convergent sequence (hint: recall convergent geometric series).

14. Show that if a series converges absolutely then it converges.

15. Assume that $|a_n| \leq 2b_n + 3^{-n}$ for all $n \geq 0$ and $\displaystyle\sum_{n=0}^{\infty} b_n$ converges. Show that $\displaystyle\sum_{n=0}^{\infty} a_n$ converges.

16. Determine for each $x \in \mathbb{R}$ whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{2^n x^n}{n}$ is convergent or divergent.

17. Let $a_n > 0$ for all $n \geq 1$. Show that $\liminf a_n^{1/n} \geq \liminf \frac{a_{n+1}}{a_n}$.

18. Let $E \subset \mathbb{R}$, $f, g : E \to \mathbb{R}$ be functions, x_0 an adherent point of E. Assume f has limit L at x_0 in E and g has limit M at x_0 in E. Show that $\lim\limits_{x\to x_0, x\in E} f(x)g(x) = LM$. Deduce that the product of two continuous functions at x_0 is continuous at x_0.

19. Show that the function $f : \mathbb{R} \to \mathbb{R}$ defined to be $f(x) = 0$ if $x \in \mathbb{Q}$ and $f(x) = -1$ if $x \notin \mathbb{Q}$ is nowhere continuous.

20. Let $p \in \mathbb{R}$. Show that the function $f : (0, \infty) \to \mathbb{R}$ given by $f(x) = x^p$ is continuous on $(0, \infty)$. Hint: use that whenever $a_n > 0$ and $\lim\limits_{n\to\infty} a_n = 1$ then $\lim\limits_{n\to\infty} (a_n)^p = 1$.

21. Show that the function $f(x) = |x|$ is continuous on $\mathbb{R}$.

22. Study the continuity properties of the function $f : [-1, 1] \to \mathbb{R}$ given by $f(x) = \begin{cases} x^2 & \text{if } -1 \leq x < 0 \\ x + 1 & \text{if } 0 \leq x \leq 1. \end{cases}$