Practice Problems for Exam # 1 - MATH 401/501 - Spring 2020

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1. Show by induction that the statement $P(n): \quad 1 + x + x^2 + \cdots + x^n = \dfrac{1 - x^{n+1}}{1 - x}$ is true for all natural numbers n, and any fixed rational number $x \neq 1$.

2. Given functions $f : X \to Y$, $g : Y \to Z$, show that if $g \circ f$ is surjective (onto) then g must be surjective. Is it true that f must also be surjective? If instead of surjective we ask about injective what is the correct statement? And bijective?

3. Given a set X and subsets A and B of X. Show that $A \cup B = (A \setminus B) \cup (A \cap B) \cup (B \setminus A)$.

4. Let a, b and c be integer numbers, $c \neq 0$. Show that $ac = bc$ then $a = b$. (You may assume corresponding result for naturals.)

5. Let r and q be rational numbers. Show that if $rq = 0$ then $r = 0$ or $q = 0$. (You may assume corresponding result for integers.)

6. Given finite sets $A_1, A_2, \ldots, A_n$ their union is a finite set. If the given sets are pairwise disjoint, that is $A_i \cap A_j = \emptyset$ for all $i \neq j$, show that the cardinality of their union is the sum of the cardinalities of the sets:
$$\#(A_1 \cup A_2 \cup \cdots \cup A_n) = \#A_1 + \#A_2 + \cdots + \#A_n.$$
Use the fact that the statement is true for $n = 2$ (Proposition 3.6.14(b) in Cardinal Arithmetic).

7. Given a rational number r, and natural numbers n and m. We define $r^0 := 1$ and given the rational number r^n then we define the rational number $r^{n+1} := r^n \times r$.

   (a) Show that $(r^n)^m = r^{n \times m}$.

   Hint: fix one of the natural numbers and induct on the other.

   (b) Assume now that $r \neq 0$, p and q are integers, and show that $(r^p)^q = r^{p \times q}$. Where we define for a negative integer $p = -n$, $n \in \mathbb{N}$, $r^p = r^{-n} := (r^n)^{-1}$. Useful auxiliary lemma is to show that: $(r^n)^{-1} = (r^{-1})^n$.

8. Let $\epsilon > 0$ be a positive rational number (a "step" or "unit"). Show that given a positive rational number $x > 0$, there exists a natural number n (depending both on the step ϵ and on x) such that $x < n\epsilon$. In words: given any positive step size we can overcome any fixed positive rational number with a finite number of steps.

9. Given a rational number x, show directly from the definition of absolute value, that $|-x| = |x|$.

10. In this problem, use any property of absolute value you wish, make sure you state properly the properties you are using.

    (a) Show that if $x, y, z \in \mathbb{Q}$, $z \neq 0$, and $|x - y| < 1/2$, then $|xz - yz| < |z|/2$.

    (b) Show that if $|x - y| \leq 1/3$, $|z - w| \leq 1/4$ then xz and yw are $(|z|/3 + |x|/4 + 1/12)$-close. Show this is optimal, that is, find x, y, z, w satisfying the hypothesis and such that $|xz - yw| = |z|/3 + |x|/4 + 1/12$.

11. Show that the "reverse triangle inequality" holds for $x, y \in \mathbb{Q}$: $\Big||x| - |y|\Big| \leq |x - y|$.