FIINAL REVIEW PROBLEMS - MATH 401/501

Review Week April 30-May 2, 2019

Final Exam on Thursday May 9th, 2019, 10am-12noon

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These problems, together with homework problems, the reviews for the exams, and the past exams should give you a good workout in preparation for the final exam. The final exam will address problems in "advanced calculus": sequences, series, limits, continuity, differentiability, and integration. You will need to understand your definitions well and you need to be able to apply those definitions and properties/laws learned in class, together with basic theorems such us: Extreme Value Theorem, Intermediate Value Theorem, Rolle's Theorem, Mean Value Theorem, Inverse Function Theorem, and the Fundamental Theorems of Calculus. Remember to bring your book to the final exam, you will be able to use it, like you did in the midterms.

1. (a) Show that every bounded sequence in $\mathbb{R}$ has at least a convergent subsequence.

   (b) Show that given a non-empty subset A of real numbers the following are equivalent: (i) A is bounded and closed, (ii) every sequence in A has a convergent subsequence converging to a point in A (sets with this property are called compact sets, this equivalence is the Heine Borel theorem in $\mathbb{R}$).

2. Show that if a sequence of real numbers $\{a_n\}_{n\in\mathbb{N}}$ is increasing and bounded then the sequence converges and to its supremum. In other words $\lim_{n\to\infty} a_n = \sup_{n\in\mathbb{N}} a_n$.

3. Show that any function f with domain the integers $\mathbb{Z}$ will necessarily be continuous at every point on its domain. More generally, show that if $f : X \to \mathbb{R}$, and x_0 is an isolated point of $X \subset \mathbb{R}$, then f is continuous at x_0. Note: a point $x_0 \in X$ is isolated if there is a $\delta > 0$ such that $X \cap [x_0 - \delta, x_0 + \delta] = \{x_0\}$, so the only point of X that is δ-close to x_0 is x_0 itself.

4. For each choice of subsets A_i of the real numbers: Is the set bounded or not? Does it have a least upper bound or a greatest lower bound? Find them. Is the set closed or not? Find its closure.

   (a) $A_1 = [0, 1]$, (b) $A_2 = (0, 1]$, (c) $A_3 = \{1/n : n \in \mathbb{N}\backslash\{0\}\}$, (d) $A_4 = \mathbb{Z}$.

5. For each choice of subsets A_i of the real numbers in Exercise 2, construct a function $f_i : \mathbb{R} \to \mathbb{R}$ that has discontinuities at every point $x \in A_i$ and is continuous on its complement $\mathbb{R} \setminus A_i$. Explain.

6. Decide whether the following series are convergent or not, if possible compute the value of the series. For (d) assume known that $\cos x$ is a continuous function on $\mathbb{R}$. Justify your answers.

   (a) $\displaystyle\sum_{n=0}^{\infty}(2x)^n$ for each $x \in \mathbb{R}$, (b) $\displaystyle\sum_{n=0}^{\infty}\frac{3n}{n^2+1}$, (c) $\displaystyle\sum_{n=1}^{\infty}\frac{3n^2}{5^n}$, (d) $\displaystyle\sum_{n=1}^{\infty}\cos(1/n)$.

7. Let $f : [0, 1] \to [0, 1]$ be a continuous function. Show that there exists a real number x in $[0, 1]$ such that $f(x) = x$, a "fixed point" (Exercise 9.7.2 p.241-242 2nd edition).

8. Let $a < b$ be real numbers, and let $f : [a, b] \to \mathbb{R}$ be a function which is both continuous and one-to-one. Show that f is strictly monotone. (See hint in Exercise 9.8.3 p. 241 2nd ed.)

9. Let $f : \mathbb{R} \to \mathbb{R}$ that satisfies the multiplicative property $f(x + y) = f(x)f(y)$ for all $x, y \in \mathbb{R}$. Assume f is not identically equal to zero.

   (a) Show that $f(0) = 1$, $f(x) \neq 0$ for all $x \in \mathbb{R}$, $f(-x) = \frac{1}{f(x)}$, and $f(x) > 0$ for all $x \in \mathbb{R}$.

   (b) Let $a = f(1)$. Show that $f(n) = a^n$ for all $n \in \mathbb{N}$. Use (a) to show that $f(z) = a^z$ for all $z \in \mathbb{Z}$.

   (c) Show that $f(r) = a^r$ for all $r \in \mathbb{Q}$.

   (d) Show that if f is continuous at $x = 0$, then f is continuous on $\mathbb{R}$. Moreover $f(x) = a^x$ for all $x \in \mathbb{R}$.

10. Decide whether the functions $f_i : X_i \to \mathbb{R}$ are bounded or not, continuous or not, uniformly continuous or not, differentiable or not on their domain. For those that are not continuous or differentiable, find the points of continuity and differentiability, and calculate the derivatives at those later points. Assume known that $f(x) = e^x$ is differentiable on $\mathbb{R}$ with derivative $f'(x) = e^x$.

    (a) $f_1(x) = xe^x - e^{x^2}$ with $X_1 = [0, 1]$, (b) $f_2(x) = x^2$ with $X_2 = [1, \infty)$,

    (c) $f_3(x) = 1/x$ with $X_3 = (0, 2]$, (d) $f_4(x) = \sqrt{x}$ with $X_4 = [0, \infty)$.

11. Assume $g : (a, b] \to \mathbb{R}$ is uniformly continuous on the open interval (a, b) show that g is uniformly continuous on $(a, b]$ if and only if g is continuous on $(a, b]$.

12. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function such that f' is bounded. Show that f is uniformly continuous.

13. A function $f : \mathbb{R} \to \mathbb{R}$ satisfies a *Lipschitz condition* with constant $M > 0$ if for all $x, y \in \mathbb{R}$,

$$|f(x) - f(y)| \leq M|x - y|.$$

Assume $h, g : \mathbb{R} \to \mathbb{R}$ each satisfy a Lipschitz condition with constant M_1 and M_2 respectively.

(a) Show that $(h + g)$ satisfies a Lipschitz condition with constant $(M_1 + M_2)$.

(b) Show that the composition $(h \circ g)$ satisfy a Lipschitz condition. With what constant?

(c) Show that the product (hg) does not necessarily satisfy a Lipschitz condition. However if both functions are bounded then the product satisfies a Lipschitz condition.

14. Assume known that the derivative of $f(x) = \sin x$ equals $\cos x$, that is, f is differentiable on $\mathbb{R}$ and $f'(x) = \cos x$. You also can use your knowledge on the trigonometric functions (you know when they are positive or negative, where are the zeros, etc). Show that $f : [0, \pi/2) \to [0, 1)$ is invertible, and that its inverse $f^{-1} : [0, 1) \to [0, \pi/2)$ is differentiable. Find the derivative of the inverse function.

15. As in the previous exercise we know that the function $\sin x$ is differentiable, and you can use known properties such as $\lim_{y\to 0, y\neq 0} \frac{\sin y}{y} = 1$ if need be. Let $G : \mathbb{R} \to \mathbb{R}$ be defined by

$$G(x) = \begin{cases} x^2 \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

Show that G is continuous and differentiable on $\mathbb{R}$ but its derivative G' is not continuous at zero. Find explicitly $G'(x)$ for all $x \in \mathbb{R}$.

16. Give an example of a function on $\mathbb{R}$ that has the intermediate value property for every interval (that is it takes on all values between $f(a)$ and $f(b)$ on $a \leq x \leq b$ for all $a < b$), but fails to be continuous at a point. Can such function have a jump discontinuity?

17. (L'Hopital's Rule).Let $f, g : X \to \mathbb{R}$, $x_0 \in X$ is a limit point of X such that $f(x_0) = g(x_0) = 0$, f, g are differentiable at x_0, and $g'(x_0) \neq 0$.

(i) Show that there is some $\delta > 0$ such that $g(x) \neq 0$ for all $x \in E := X \cap (x_0 - \delta, x_0 + \delta) \setminus \{x_0\}$.
Hint: Use Newton's approximation theorem.

(ii) Show that $\displaystyle\lim_{x\to x_0, x\in E} \frac{f(x)}{g(x)} = \frac{f'(x_0)}{g'(x_0)}$.

(iii) Show that the following version of L'Hopital's Rule is not correct. Under the above hypothesis then,

$$\lim_{x\to x_0, x\in E} \frac{f(x)}{g(x)} = \lim_{x\to x_0, x\in E} \frac{f'(x)}{g'(x)}.$$

Hint: Consider $f(x) = G(x)$ (as in Exercise 15), and $g(x) = x$ at $x_0 = 0$. If both f' and g' were continuous at x_0 then (ii) and (iii) would be equivalent.

18. Let $f : [a, b] \to \mathbb{R}$ be a strictly decreasing function. Show that f is Riemann integrable on $[a, b]$.

19. Calculate the following Riemann integrals or derivatives, justify your steps. Here you may assume known that the functions $\sin(x)$, $\cos(x)$, e^x are all differentiable on $\mathbb{R}$, and you know their derivatives and anti-derivatives.

(a) $\displaystyle\int_1^3 \left(x^4 + 2x - 7 + \frac{1}{x^3}\right) dx$, (b) $\displaystyle\int_0^1 3e^{2x} dx$, (c) $\displaystyle\int_0^{\sqrt{\frac{\pi}{2}}} x \cos(x^2) dx$, (d) $\displaystyle\frac{d}{dx} \int_{-2}^{x^3} \sin(t^2) dt$.

20. (Integral test for series) Let $f : [1, \infty] \to \mathbb{R}$ be a monotone decreasing non-negative function. Then the sum $\sum_{n=1}^{\infty} f(n)$ is convergent if and only if $\sup_{N>0} \int_1^N f(x)dx$ is finite.
Show by constructing counterexamples that if the hypothesis of monotone decreasing is replaced by Riemann integrable on intervals $[1, N]$ for all $N > 0$ then both directions of the if and only if above are false.