Review and Practice Problems for Exam # 2 - MATH 401/501 - Spring 2016

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Real numbers

- Understand that the real numbers are the "completion" of the rational numbers. They inherit algebraic and order properties from rationals.
- Real numbers are closed under addition, multiplication, negation, subtraction and division by non-zero real numbers. You are free to use usual arithmetic properties (commutative and associative properties of addition and multiplication, distributive property, etc).
- Real numbers have an order, and obey a trichotomy if x, y are real numbers then exactly one of the following holds: $x = y$, $x < y$ or $x > y$.
- Should know and be able to use
  - (i) the definition of absolute value of a real number,
  - (ii) the triangle inequality (and "reverse" triangle inequality).
- Understand Archimedean properties and their implications: interspersing of integers by $\mathbb{R}$, density of rationals and irrationals.
- Understand the meaning of upper and lower bounds for a set, and the meaning of the supremum (least upper bound or l.u.b.) and infimum (greatest lower bound or g.l.b.) of a set of real numbers.
- Be able to show that a given number is the supremum (infimum) of a set by showing that
  - (i) it is an upper (lower) bound for the set,
  - (ii) it is the smallest upper (largest lower) bound.
- Appreciate the Least Upper Bound and Greatest Lower Bound properties of real numbers: every non-empty and bounded set of real numbers has a unique supremum and a unique infimum.

Sequences of real numbers

- Know the definition of bounded sequences, bounded above sequences and bounded below sequences. More precisely a sequence $\{x_n\}_{n\geq 0}$ is bounded (respectively bounded above or bounded below) iff there is $M > 0$ such that for all $n \geq 0$ we have $|x_n| \leq M$ (respectively $x_n \leq M$ or $M \leq x_n$).
- Know the ϵ, N definition of Cauchy sequences and of convergent sequences in $\mathbb{R}$ to a limit $L \in \mathbb{R}$. More precisely, A sequence $\{x_n\}_{n\geq 0}$ of real numbers
  - is Cauchy iff given $\epsilon > 0$ there is $N > 0$ such that for all $n, m \geq N$ then $|x_n - x_m| \leq \epsilon$,
  - converges to L iff given $\epsilon > 0$ there is $N > 0$ such that for all $n \geq N$ then $|x_n - L| \leq \epsilon$.
- Be able to show that limits are unique (that is if a sequence converges it converges to a unique limit).
- Be able to prove or disprove that a given sequence converges or is Cauchy by using the "ϵ, N definition". E.g. $a_n = 1/n$, $b_n = 2^{-n}$.
- Be able to show that a convergent sequence is a Cauchy sequence.
- Be able to show and use that Cauchy sequences (and hence convergent sequences) are bounded sequences. However not all bounded sequences are convergent, e.g . $b_n = (-1)^n$ for all $n \geq 0$.
- Be able to show that the sum/product of two Cauchy sequences (or two convergent sequences) is a Cauchy sequence (a convergent sequence and convergent to the sum/product of the limits of the given convergent sequences "limit laws").
- Understand that if a Cauchy (convergent) sequence is bounded away from zero then the sequence of reciprocals is Cauchy (hence convergent and to the reciprocal of the limit which is necessarily non-zero, another "limit law").

- Be able to prove or disprove that a given sequence converges by appealing to additive/multiplicative/reciprocal properties of limits (limit laws), and using known basic limits.
- Know and be able to use the Monotone Bounded Sequence Convergence Theorem:
  (i) an increasing and bounded above sequence is convergent and to the sequence's supremum,
  (ii) a decreasing and bounded below sequence is convergent and to the sequence's infimum.
- You should know and use some basic limits :
  - $\lim_{n\to\infty} x^n = 0$ if $|x| < 1$, is 1 if $x = 1$, and does not exist if $x = -1$ or $|x| > 1$;
  - $\lim_{n\to\infty} x^{1/n} = 1$ if $x > 0$;
  - $\lim_{n\to\infty} 1/n^{1/k} = 0$ for all integers $k \geq 1$.
  - $\lim_{n\to\infty} n^{1/n} = 1$.
- Appreciate the deep fact that Cauchy sequences are convergent sequences in $\mathbb{R}$ (completeness of the real numbers) .

Limit points, limsup, liminf

- Appreciate the definition of "limit points" of a sequence as the collection of "subsequencial limits" (the limits of convergent subsequences of the sequence).
- Know that c is a limit point for a sequence $\{x_n\}$ if for all $\epsilon > 0$ there are "infinitely many" terms of the sequence in the interval $[c-\epsilon, c+\epsilon]$. More precisely, for all $\epsilon, N > 0$ there is an $n_N \geq N$ such that $|x_{n_N} - c| \leq \epsilon$ (necessarily the set of labels $\{n_N\}_{N\geq 0}$ is an infinite set!).
- Be able to identify the "limit points" (or "subsequencial limits") of a concrete sequence e.g: $a_n = 3$ for all $n \geq 0$, $b_n = (-1)^n$ for all $n \geq 0$, $c_n = (-1)^n n$ for all $n \geq 0$.
- Know that bounded sequences in $\mathbb{R}$ have limit superior/inferior in $\mathbb{R}$, defined as $\limsup\{x_n\} := \lim_{N\to\infty} \sup_{n\geq N} x_n$ and $\liminf\{x_n\} := \lim_{N\to\infty} \inf_{n\geq N} x_n$.
- Be aware of the epsilon characterization of limsup (similarly liminf): for all $\epsilon > 0$
  (i) Finitely many terms of the sequence $\{x_n\}$ are larger than $\limsup\{x_n\} + \epsilon$. More precisely there is $N > 0$ such that for all $n \geq N$ we have $x_n \leq \limsup\{x_n\} + \epsilon$.
  (ii) Infinitely many terms of the sequence $\{x_n\}$ are in between $\limsup\{x_n\} - \epsilon$ and $\limsup\{x_n\} + \epsilon$.

  And its consequences:
  - Limsup and liminf are limit points (subsequencial limits) of the sequence.
  - A sequence of real numbers converges if and only if the limsup and the liminf coincide.
  - The limsup is the "largest limit point" (or "largest subsequential limit") of the sequence, and liminf is the "smallest limit point" (or "smallest subsequential limit") of the sequence.
  - A sequence converges to L iff all its subsequences converge to L iff the unique limit point of the sequence is L.
  - Bolzano-Weierstrass theorem: every bounded sequence has a at least one convergent subsequence or equivalently at least one "limit point".
- Be able to identify the lim sup and lim inf of a given sequence. Use this knowledge to conclude that if $\limsup a_n = \liminf a_n = L$ then the sequence $\{a_n\}$ converges AND $\lim_{n\to\infty} a_n = L$.
- Be able to use the squeeze theorem to deduce convergence of the sequence being squeezed.

Series

- Understand that convergence of a series is by definition convergence of the sequence of partial sums. Be able to deduce from the theory of sequences basic convergence tests: Cauchy test, divergence test, comparison test.
- Be able to understand and exploit convergence properties of geometric series: $\sum_{n\geq 0} r^n$ converges to $1/(1-r)$ if $|r| < 1$, diverges otherwise. Appreciate how to use them to prove the root and ratio test.

PRACTICE PROBLEMS FOR MIDTERM #2

1. If the real number x is not rational we say x is "irrational".
   (a) Show that if $p \in \mathbb{Q}$, $p \neq 0$, and x is irrational then px is irrational.
   (b) Show that if $x, y \in \mathbb{R}$ and $x < y$ then there is an irrational number w such that $x < w < y$ (density of the irrational numbers).
2. For each subset A of real numbers decide whether is bounded (above, below or both), find supremum and infimum: (a) $A = \{1, -1/2, 3\}$, (b) $A = \{n/(n+1) : n \in \mathbb{N}, \ n \geq 1\}$, (c) $A = \{r \in \mathbb{Q} : r < 5\}$.
3. Let E be a nonempty and bounded subset of $\mathbb{R}$, let $\lambda \in \mathbb{R}$ and $\lambda < 0$. Define $\lambda E = \{\lambda x : x \in E\}$ a subset of $\mathbb{R}$. Prove that $\inf(\lambda E) = \lambda \sup(E)$. What is $\sup(\lambda E)$?
4. If A and B are nonempty and bounded subsets of $\mathbb{R}$ such that $A \subset B$ show that $\inf(B) \leq \inf(A)$.
5. For each of the following, prove or give a counterexample.
   (a) If $\{x_n\}_{n\geq 0}$ converges to x then $\{|x_n|\}_{n\geq 0}$ converges to $|x|$.
   (b) If $\{|x_n|\}_{n\geq 0}$ is convergent then $\{x_n\}_{n\geq 0}$ is convergent .
6. We say the sequence $\{x_n\}_{n\geq 0}$ diverges to $+\infty$ and we write $\lim_{n\to\infty} x_n = +\infty$ iff for all $M > 0$ there is $N > 0$ such that for all $n \geq N$ we have $x_n \geq M$.
   (a) Write down a definition for a sequence $\{y_n\}_{n\geq 0}$ to diverge to $-\infty$.
   (b) Show that if $x_n \leq z_n$ for all $n \geq 0$ and $\{x_n\}$diverges to $+\infty$ then $\{z_n\}$ diverges to $+\infty$.
   (c) Let $\{x_n\}$ be a sequence of positive real numbers. Show that $\lim_{n\to\infty} x_n = +\infty$ if and only if $\lim_{n\to\infty}(1/x_n) = 0$.
7. The sequence of positive real numbers $\{t_n\}_{n\geq 0}$ converges to t. Decide whether the following sequences are convergent or not. If convergent explain why and identify the limit, if not convergent explain why.
   (a) $a_n = \sqrt{t_n}$, (b) $b_n = 5t_n^3 - t_n^2 + 7$, (c) $c_n = \dfrac{n}{2^n}(-1)^n$, (c) $d_n = n + t_n$.
8. Show that the sequence defined by $x_1 = 1$ and $x_{n+1} = \sqrt{1 + x_n}$ for $n \geq 1$ is convergent (hint: show that it is increasing and bounded by 2). Find the limit.
9. Let $x_n = n \sin^2(n\pi/2)$. Find the set S of limit points (subsequencial limits), find $\limsup x_n$ and $\liminf x_n$. (Assume known properties about sine function.)
10. Show that the sequence of partial sums: $S_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$ defined for $n \geq 1$ is not convergent (hint: show is not Cauchy by showing that $S_{2n} - S_n \geq 1/2$). Conclude that the harmonic series is divergent.
11. A sequence $\{s_n\}$ is contractive if there is a constant r with $0 < r < 1$ such that $|s_{n+2} - s_{n+1}| \leq r|s_{n+1} - s_n|$ for all $n \geq 0$. Show that a contractive sequence is a Cauchy sequence and hence convergent sequence. (Recall convergent geometric series: $\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$ when $|r| < 1$).
12. Given $r > 0$ and $\{s_n\}_{n\geq 0}$ is a bounded sequence. Show that $\limsup r s_n = r \limsup s_n$. What can you say when $r < 0$?
13. Use the Cauchy test for series to show that if a series $\sum_{n=0}^{\infty} a_n$ is convergent then $\lim_{n\to\infty} a_n = 0$.
14. Show that if $0 \leq a_n \leq b_n$ for all $n \geq 0$ then if $\sum_{n=0}^{\infty} b_n$ converges then so does $\sum_{n=0}^{\infty} a_n$ (comparison test for series).
15. Show that if a series converges absolutely (that is $\sum_{n=0}^{\infty} |a_n|$ converges) then it converges (that is $\sum_{n=0}^{\infty} a_n$ converges).

16. Exercise 6.4.1 (limits are limit points).
17. Exercises 6.4.5 (squeeze test using comparison principle).
18. Exercise 6.5.3 (limit of n-th root of $x > 0$ as n goes to infinity is one).
19. Exercise 6.6.2 (create two different sequences so that each is a subsequence of the other).
20. Given the sequence 1, -1, -1/2, 1, 1/2, 1/3, -1, -1/2, -1/3, -1/4, 1, 1/2, 1/3, 1/4, 1/5, -1, -1/2, -1/3, -14, -1/5, -1/6, 1, 1/2,... find its supremum, its infimum, its limsup, its liming and all its limit points. Write a short justification for each one of them.
21. Exercise 7.1.4 (binomial formula).
22. Exercise 7.2.2 (Cauchy test for series).
23. Exercise 7.2.1. (decide wether a series converges or not).
24. Exercise 7.3.2 (geometric series).
25. Exercise 7.5.2 (show that a particular series is convergent).