Practice Problems for Exam # 1 - MATH 401/501 - Spring 2016

Instructor: C. Pereyra

1. Show by induction that the statement
$$P(n): \quad 1 + x + x^2 + \cdots + x^n = \frac{1 - x^{n+1}}{1 - x}$$
is true for all natural numbers n, and any fixed rational number $|x| \neq 1$.

2. Given functions $f : X \to Y$, $g : Y \to Z$, show that if $g \circ f$ is surjective (onto) then g must be surjective. Is it true that f must also be surjective? If true prove it, if false present a counterexample.

3. Given a set X and subsets A and B of X. Show that
$$X \setminus (A \cup B) = (X \setminus A) \cap (X \setminus B).$$

4. Let a and b be integer numbers. Show that if $ab = 0$ then $a = 0$ or $b = 0$. (You can use the fact that this is true for natural numbers).

5. Let r and q be rational numbers. Show that if $rq = 0$ then $r = 0$ or $q = 0$. (You can use the fact that this is true for integer numbers).

6. Given a rational number r, and natural numbers n and m. We define $r^0 := 1$ and given the rational number r^n then we define the rational number $r^{n+1} := r^n \times r$.

   (a) Show that
   $$(r^n)^m = r^{n \times m}.$$
   Hint: fix one of the natural numbers and induct on the other.

   (b) Assume now that $r \neq 0$, p and q are integers, and show that $(r^p)^q = r^{p \times q}$. Where we define for a negative integer $p = -n$, $n \in \mathbb{N}$, $r^p = r^{-n} := (r^n)^{-1}$. Useful auxiliary lemma is to show that: $(r^n)^{-1} = (r^{-1})^n$.

7. Let $\epsilon > 0$ be a positive rational number (a "step" or "unit"). Show that given a positive rational number $x \geq 0$, there exists a natural number n (depending both on the step ϵ and on x) such that $|x| < n\epsilon$. In words: given any positive step size we can overcome any fixed rational number with a finite number of steps.

8. Given a rational number x, show directly from the definition of absolute value, that $|-x| = |x|$.

9. In this problem, use any property of absolute value you wish, make sure you state properly the properties you are using.

   (a) Show that if $x, y, z \in \mathbb{Q}$, $|x - y| < 1/2$, then $\quad |xz - yz| < |z|/2$.

   (b) Show that if $w, x, y, z \in \mathbb{Q}$, $|w - x| \leq 1/2$ and $|y - z| \leq 1/2$ then $\quad |(w + y) - (x + z)| \leq 1$. Can you find rational numbers x, w, x, y, z such that the hypothesis are satisfied and $|(w+y)-(x+z)| = 1$?

10. Show that the "reverse triangle inequality" holds for $x, y \in \mathbb{Q}$: $\Big| |x| - |y| \Big| \leq |x - y|$.

The following are additional problems in case you want more. Do not worry about them for the purpose of the exam on Thursday.

- Suppose $f : X \to Y$, and suppose that A, B are subsets of X and C, D are subsets of Y. The *direct image of A under f* is the subset of Y defined by

$$f(A) := \{y \in Y : y = f(x), x \in A\}.$$

  The *inverse image of C under f* is the subset of X defined by

$$f^{-1}(C) := \{x \in X : f(x) \in C\}.$$

  Determine which inclusion relationship must hold for the following pairs of sets:

  (a) $f(A \cap B)$ and $f(A) \cap f(B)$,
  (b) $f^{-1}(C \cap D)$ and $f^{-1}(C) \cap f^{-1}(D)$.

- Given sets A and B, the power set A^B is the collection of all functions $f : B \to A$. Show that given any sets (finite or not) A, B, C then

$$\#\big((A^B)^C\big) = \#\big(A^{B \times C}\big).$$