# Practice Problems for Exam # 1 - MATH 401/501 - Spring 2014

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1. Show by induction that the statement

$$P(n): \quad 1 + x + x^2 + \cdots + x^n = \frac{1 - x^{n+1}}{1 - x}$$

   is true for all natural numbers $n$, and any fixed rational number $x \neq 1$.

2. Given a set $X$ and subsets $A$ and $B$ of $X$. Show that

$$X \setminus (A \cup B) = (X \setminus A) \cap (X \setminus B).$$

3. Let $r$ and $q$ be rational numbers. Show that if $rq = 0$ then $r = 0$ or $q = 0$.

4. Suppose $f : X \to Y$, and suppose that $A, B$ are subsets of $X$ and $C, D$ are subsets of $Y$. Determine which inclusion relationship must hold for the following pair of sets:

   (a) $f(A \cap B)$ and $f(A) \cap f(B)$,

   (b) $f^{-1}(C \cap D)$ and $f^{-1}(C) \cap f^{-1}(D)$.

   Remember that for $A \subset X$, $f(A) = \{y \in Y : y = f(x), x \in A\}$ (the direct image of $A$ under $f$); and for $C \subset Y$, $f^{-1}(C) = \{x \in X : f(x) \in B\}$ (the inverse image of $B$ under $f$).

5. Show that given any sets (finite or not) $A, B, C$ such that $B \cap C = \emptyset$ then

$$\#(A^B \times A^C) = \#(A^{B \cup C}).$$

   Remember that the set $A^B$ consists of all functions $f : B \to A$.

6. Given a rational number $r$, and natural numbers $n$ and $m$. We define $r^0 := 1$ and given the rational number $r^n$ then we define the rational number $r^{n+1} := r^n \times r$.

   (a) Show that

$$(r^n)^m = r^{n \times m}.$$

   Hint: fix one of the natural numbers and induct on the other.

   (b) Assume now that $r \neq 0$, $p$ and $q$ are integers, and show that $(r^p)^q = r^{p \times q}$. Where we define for a negative integer $p = -n$, $n \in \mathbb{N}$, $r^p = r^{-n} := (r^n)^{-1}$. Useful auxiliary lemma is to show that: $(r^n)^{-1} = (r^{-1})^n$.

7. Let $\epsilon > 0$ be a positive rational number (a "step" or "unit"). Show that given a positive rational number $x \geq 0$, there exists a natural number $n$ (depending both on the step $\epsilon$ and on $x$) such that $|x| < n\epsilon$. In words: given any positive step size we can overcome any fixed rational number with a finite number of steps.

8. Given a rational number $x$, show directly from the definition of absolute value, that $|-x| = |x|$.

9. In this problem, use any property of absolute value you wish, make sure you state properly the properties you are using.

   (a) Show that if $x, y, z \in \mathbb{Q}$, $|x - y| < 1/2$, then $|xz - yz| < |z|/2$.

   (b) Show that if $w, x, y, z \in \mathbb{Q}$, $|w - x| \leq 1/2$ and $|y - z| \leq 1/2$ then $|(w + y) - (x + z)| \leq 1$. Can you find rational numbers $x, w, x, y, z$ such that the hypothesis are satisfied and $|(w+y)-(x+z)| = 1$?

10. Show that the "reverse triangle inequality" holds for $x, y \in \mathbb{Q}$: $\Big||x| - |y|\Big| \leq |x - y|$.