**Review - MATH 401/501 - Spring 2009**
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1. Suppose $f, g : \mathbb{R} \to \mathbb{R}$ are uniformly continuous on $\mathbb{R}$.

   (a) Show that the sum $(f + g)$ and the composition $(f \circ g)$ are uniformly continuous on $\mathbb{R}$.

   (b) Show that the product $(fg)$ is not necessarily uniformly continuous. However if both functions are bounded then the product is uniformly continuous.

2. Assume known that the derivative of $f(x) = e^x$ equals $f$, that is, $f$ is differentiable on $\mathbb{R}$ and $f'(x) = e^x$.

   Show that $f : \mathbb{R} \to (0, \infty)$ is invertible, and that its inverse $f^{-1} : (0, \infty) \to \mathbb{R}$ is differentiable. Find the derivative of the inverse function.

3. (20 points) Let $h$ be a differentiable function defined on the interval $[0, 3]$, and assume that $h(0) = 1$, $h(1) = 2$ and $h(3) = 2$.

   (a) Show that there exists a point $d \in [0, 3]$ such that $h(d) = d$.

   (b) Show that there exists a point $c \in (0, 3)$ such that $h'(c) = 1/3$.

   (c) Show that there exists a point $b \in (0, 3)$ such that $h'(b) = 1/4$.

4. (L'Hopital's Rule). Show that if $f, g : X \to \mathbb{R}$, $x_0 \in X$ is a limit point of $X$ such that $f(x_0) = g(x_0) = 0$, $f, g$ are differentiable at $x_0$, and $g'(x_0) \neq 0$, then there is some $\delta > 0$ such that $g(x) \neq 0$ for all $x \in X \cap (x_0 - \delta, x_0 + \delta)$ and

   $$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \frac{f'(x_0)}{g'(x_0)}.$$

   **Hint:** Use Newton's approximation theorem.

   Show that the following version of L'Hopital's Rule is not correct: Under the above hypothesis then,

   $$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f'(x)}{g'(x)}.$$

   **Hint:** Consider $f(x) = x^2 \sin 1/x$ for $x \neq 0$ and $f(0) = 0$, and $g(x) = x$ at $x_0 = 0$. The problem arises when one of the derivatives is not continuous at $x_0$, as we showed to be the case for $g(x)$ at $x_0 = 0$.