REVIEW FOR TEST # 3 - MATH 401/501 - FALL 2006

December 5-7, 2006

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1. Let $X \subset \mathbb{R}$, α and C positive real numbers. Suppose a function $f : X \to \mathbb{R}$ satisfies the following Hölder continuity property:
$$|f(x) - f(y)| \leq C|x-y|^{\alpha}, \quad \text{for all } \ x, y \in X.$$
Show that f is uniformly continuous on X.

2. (Squeeze Theorem). Let f, g and h satisfy $f(x) \leq g(x) \leq h(x)$ for all x in some common domain $X \subset \mathbb{R}$. If $\lim_{x\to x_0} f(x) = L = \lim_{x\to x_0} h(x)$ at some adherent point x_0 of X, show that $\lim_{x\to x_0} g(x)$ exists and is L. (You might want to use the squezze theorem for sequences and the sequential characterization of limits of functions).

3. Show that any function f with domain the integers $\mathbb{Z}$ will necessarily be continuous at every point on its domain. More generally, show that if $f : X \to \mathbb{R}$, and x_0 is an isolated point of $X \subset \mathbb{R}$, then f is continuous at x_0.

4. Let $f : \mathbb{R} \to \mathbb{R}$ that satisfies the additive property $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$.
   (i) Show that $f(0) = 0$ and that $f(-x) = -f(x)$ for all $x \in \mathbb{R}$.
   (ii) Show that if f is continuous at $x = 0$, then f is continuous at every point in $\mathbb{R}$.
   (iii) Let $k = f(1)$. Show that $f(n) = kn$ for all $n \in \mathbb{N}$, and then prove that $f(z) = kz$ for all $z \in \mathbb{Z}$. Now show that $f(r) = kr$ for all $r \in \mathbb{Q}$.
   (iv) Assume f is continuous at zero, use (ii) and (iii) to conclude that $f(x) = kx$ for all $x \in \mathbb{R}$. Thus an additive function that is continuous at $x = 0$ must necessarily be a linear function through the origin.

5. For each choice of subsets A of the real numbers, construct a function $f : \mathbb{R} \to \mathbb{R}$ that has discontinuities at every point $x \in A$ and is continuous on its complement $\mathbb{R} \setminus A$.
   (a) $A = \mathbb{Z}$.
   (b) $A = \{x : 0 < x < 1\} = (0, 1)$.
   (c) $A = \{x : 0 < x \leq 1\} = (0, 1]$.
   (d) $A = \{1/n : 0 < n \in \mathbb{N}\}$.

6. Assume g is defined on an open interval (a, c) and it is known to be uniformly continuous on $(a, b]$ and on $[b, c)$ where $a < b < c$. Prove that g is uniformly continuous on (a, c).

   Show that if f is uniformly continuous on (a, b) and (b, c), for some $b \in (a, c)$, then f is uniformly continuous on (a, c) if and only if f is continuous at b.

7. Assume $f, g : \mathbb{R} \to \mathbb{R}$ are uniformly continuous on $\mathbb{R}$.

   (a) Show that $(f + g)$ is uniformly continuous on $\mathbb{R}$.

   (b) Show that the composition $(f \circ g)$ is uniformly continuous.

   (c) Show that the product (fg) is not necessarily uniformly continuous unless one of the functions is bounded.

8. Is it true that if f is continuous on $\mathbb{R}$ then

$$f(\limsup_{n\to\infty} x_n) = \limsup_{n\to\infty} f(x_n)?$$

9. Verify the quotient rule for differentiation.

10. (Intermediate Value Theorem for Derivatives or Darboux's Theorem). If f is differentiable on $[a, b]$, and if α is a real number in between $f'(a)$ and $f'(b)$ say $f'(a) < \alpha < f'(b)$ (or $f'(b) < \alpha < f'(a)$), then there exists a point $c \in (a, b)$ such that $f'(c) = \alpha$. (**Warning:** you can not assume that f' is continuous even if it is defined on all of $[a, b]$, so you cannot use the IVT for continuous functions. Consider the example discussed in class $f(x) = x^2 \sin(1/x)$ it is differentiable on $[-1, 1]$ but the derivative is not continuous at $x = 0$.) **Hint:** to simplify define a new function $g(x) = f(x) - \alpha x$ on $[a, b]$. This function g is differentiable on $[a, b]$ and show that our hypothesis on f imply that $g'(a) < 0 < g'(b)$ (or $g'(b) < 0 < g'(a)$). Now show that there is a $c \in (a, b)$ such that $g'(c) = 0$. To do the later, show that there exists a point $x \in (a, b)$ such that $g(a) > g(x)$, and a point $y \in (a, b)$ such that $g(b) < g(y)$. Now finish the proof of Darboux's theorem.

11. Assume known that the functions $\sin x$ and $\cos x$ are differentiable, and that their derivatives are $\cos x$ and $-\sin x$ respectively. Let $g_a : \mathbb{R} \to \mathbb{R}$ be defined by

$$g_a(x) = \begin{cases} x^a \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

Find particular (potentially noninteger) values of a so that

(a) g_a is differentiable on $\mathbb{R}$ but g_a' is unbounded on $[0, 1]$.

(b) g_a is differentiable on $\mathbb{R}$ with g_a' continuous but not differentiable at zero.

(c) g_a is differentiable on $\mathbb{R}$ and g_a' differentiable on $\mathbb{R}$, but such that g_a'' is not continuous at zero.

12. Give an example of a function on $\mathbb{R}$ that has the intermediate value property for every interval (that is it takes on all values between $f(a)$ and $f(b)$ on $a \leq x \leq b$ for all $a < b$), but fails to be continuous at a point. Can such function have a jump discontinuity?

13. (L'Hopital's Rule). Show that if $f, g : X \to \mathbb{R}$, $x_0 \in X$ is a limit point of X such that $f(x_0) = g(x_0)$, f, g are differentiable at x_0, and $g'(x_0) \neq 0$, then there is some $\delta > 0$ such that $g(x) \neq 0$ for all $x \in X \cap (x_0 - \delta, x_0 + \delta)$ and

$$\lim_{x\to x_0} \frac{f(x)}{g(x)} = \frac{f'(x_0)}{g'(x_0)}.$$

Hint: Use Newton's approximation theorem.

Show that the following version of L'Hopital's Rule is not correct. Under the above hypothesis then,

$$\lim_{x\to x_0} \frac{f(x)}{g(x)} = \lim_{x\to x_0} \frac{f'(x)}{g'(x)}.$$

Hint: Consider $f(x) = g_2(x)$ (as in exercise 11), and $g(x) = x$ at $x_0 = 0$.