THIRD REVIEW - MATH 401/501
Review Week December 6-8, 2016
Final Exam on Tuesday December 13th, 2016 7:30-9:30pm

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These third review problems, together with: homework problems from Chapters 6-7, 9-11, the second review problems, and the second exam, should give you a good workout in preparation for the final exam. The final exam will address problems in calculus: sequences, series, limits, continuity, differentiability, and minimally integration. You will need to understand your definitions and you need to be able to apply those definitions together with basic theorems such as: Monotone Bounded Sequence Convergence Theorem, Extreme Value Theorem, Intermediate Value Theorem, Rolle's Theorem, Mean Value Theorem, Inverse Function Theorem, and the Fundamental Theorems of Calculus; properties such as: limit laws, series laws, tests for series, squeeze tests, continuity laws, and differentiation laws. Problems we don't have time to solve during the review week, I will send you solutions or references on Friday.

1. (a) Show that every bounded sequence in $\mathbb{R}$ has at least a convergent subsequence.
   (Bolzano-Weiertrass Theorem, Theorem 6.6.8).
   (b) Show that given a non-empty subset X of real numbers the following are equivalent:
      (i) X is bounded and closed,
      (ii) every sequence in X has a convergent subsequence converging to a point in $L \in X$.
   (Sets that have property (ii) are called compact sets, the above result is the Heine-Borel Theorem (Theorem 9.1.4) and it states that in $\mathbb{R}$ the compact sets are the closed and bounded sets.)
2. Show that any function f with domain the integers $\mathbb{Z}$ will necessarily be continuous at every point on its domain. More generally, show that if $f : X \to \mathbb{R}$, and x_0 is an isolated point of $X \subset \mathbb{R}$, then f is continuous at x_0. Note: a point $x_0 \in X$ is isolated if there is a $\delta > 0$ such that $X \cap [x_0 - \delta, x_0 + \delta] = \{x_0\}$, so the only point of X that is δ-close to x_0 is x_0 itself.
3. For each choice of subsets A_i of the real numbers: Is the set bounded or not? Does it have a least upper bound or a greatest lower bound? Find them. Is the set closed or not? Find its closure.
   (a) $A_1 = [0, 1]$, (b) $A_2 = (0, 1]$, (c) $A_3 = \{1/n : n \in \mathbb{N}\backslash\{0\}\}$, (d) $A_4 = \mathbb{Z}$.
4. For each choice of subsets A_i of the real numbers in Exercise 2, construct a function $f_i : \mathbb{R} \to \mathbb{R}$ that has discontinuities at every point $x \in A_i$ and is continuous on its complement $\mathbb{R}\backslash A_i$. Explain. Are your examples bounded functions? If not, can you modify your examples to make them bounded functions?
5. Show that if $f : [a, b] \to \mathbb{R}$ is a continuous function then f is a bounded function.
6. Show that if $f : [a, b] \to \mathbb{R}$ is a continuous function then there is a point $x_{max} \in [a, b]$ such that $f(x) \leq f(x_{max})$ for all $x \in [a, b]$.
7. Show that if $f : X \to \mathbb{R}$ is differentiable at a limit point $x_0 \in X$, then f is continuous at x_0.
8. Let $f : [0, 1] \to [0, 1]$ be a continuous function. Show that there exists a real number x in $[0, 1]$ such that $f(x) = x$, a "fixed point" (Exercise 9.7.2 p.241-242 2nd edition).
9. Show Rolle's Theorem: given function $g : [a, b] \to \mathbb{R}$ continuous on $[a, b]$, differentiable on (a, b) such that $g(a) = g(b)$ show there is a point $c \in (a, b)$ such that $g'(c) = 0$.
10. Let $a < b$ be real numbers, and let $f : [a, b] \to \mathbb{R}$ be a function which is both continuous and one-to-one. Show that f is strictly monotone. (See hint in Exercise 9.8.3 p. 241 2nd ed.)
11. Let $f : \mathbb{R} \to \mathbb{R}$ that satisfies the multiplicative property $f(x + y) = f(x)f(y)$ for all $x, y \in \mathbb{R}$. Assume f is not identically equal to zero.
    (a) Show that $f(0) = 1$, $f(x) \neq 0$ for all $x \in \mathbb{R}$, and $f(-x) = \frac{1}{f(x)}$ for all $x \in \mathbb{R}$. Show that $f(x) > 0$ for all $x \in \mathbb{R}$.

(b) Let $a = f(1)$ (by (i) $a > 0$). Show that $f(n) = a^n$ for all $n \in \mathbb{N}$. Use (a) to show that $f(z) = a^z$ for all $z \in \mathbb{Z}$.

(c) Show that $f(r) = a^r$ for all $r \in \mathbb{Q}$.

(d) Show that if f is continuous at $x = 0$, then f is continuous at every point in $\mathbb{R}$. Moreover $f(x) = a^x$ for all $x \in \mathbb{R}$.

12. Decide whether the functions $f_i : X_i \to \mathbb{R}$ are bounded, continuous, uniformly continuous, differentiable, or not on their domain.

    (a) $f_1(x) = x^{13} - 8x^5 + 7 + 2^x$ with $X_1 = [-3, 14]$, (b) $f_2(x) = x^2$ with $X_2 = [1, \infty)$,

    (c) $f_3(x) = 1/x$ with $X_3 = (0, 2]$, (d) $f_4(x) = \sqrt{x}$ with $X_4 = [0, \infty)$.

13. Show that if $f : [a, b] \to \mathbb{R}$ is continuous then it is uniformly continuous.

14. Assume $g : (a, b] \to \mathbb{R}$ is uniformly continuous on the open interval (a, b) show that g is uniformly continuous on $(a, b]$ if and only if g is continuous on $(a, b]$.

15. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function such that f' is bounded. Show that f is uniformly continuous.

16. Verify the chain rule (see Exercise 10.1.7. p. 256 2nd ed).

17. A function $f : \mathbb{R} \to \mathbb{R}$ satisfies a *Lipschitz condition* with constant $M > 0$ if for all $x, y \in \mathbb{R}$,

$$|f(x) - f(y)| \leq M|x - y|.$$

    Assume $h, g : \mathbb{R} \to \mathbb{R}$ each satisfy a Lipschitz condition with constant M_1 and M_2 respectively.

    (a) Show that $(h + g)$ satisfies a Lipschitz condition with constant $(M_1 + M_2)$.

    (b) Show that the composition $(h \circ g)$ satisfy a Lipschitz condition. With what constant?

    (c) Show that the product (hg) does not necessarily satisfy a Lipschitz condition. However if both functions are bounded then the product satisfies a Lipschitz condition.

18. Assume known that the derivative of $f(x) = \sin x$ equals $\cos x$, that is, f is differentiable on $\mathbb{R}$ and $f'(x) = \cos x$. You also can use your knowledge on the trigonometric functions (you know when they are positive and negative, where are the zeros, etc). Show that $f : [0, \pi/2) \to [0, 1)$ is invertible, and that its inverse $f^{-1} : [0, 1) \to [0, \pi/2)$ is differentiable. Find the derivative of the inverse function.

19. As in the previous exercise we know that the function $\sin x$ is differentiable, and you can use known properties such as $\lim_{x \to 0, x \neq 0} \frac{\sin x}{x} = 1$ if need be. Let $G : \mathbb{R} \to \mathbb{R}$ be defined by

$$G(x) = \begin{cases} x^2 \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

    Show that G is continuous and differentiable on $\mathbb{R}$ but its derivative G' is not continuous at zero. Find explicitely $G'(x)$.

20. Give an example of a function on $\mathbb{R}$ that has the intermediate value property for every interval (that is it takes on all values between $f(a)$ and $f(b)$ on $a \leq x \leq b$ for all $a < b$), but fails to be continuous at a point. Can such function have a jump discontinuity?

21. Calculate the following Riemann integrals, justify your steps. Here you may assume known that the functions $\sin(x)$, $\cos(x)$, e^x are all differentiable on $\mathbb{R}$, and you know their derivatives and anti-derivatives.

    (a) $\displaystyle\int_1^3 \left(x^4 + 2x - 7 + \frac{1}{x^3}\right) dx$, (b) $\displaystyle\int_0^1 3e^{2x} dx$, (c) $\displaystyle\int_0^{\sqrt{\pi/2}} x \cos x^2 dx$.

22. (Integral test for series) Let $f : [1, \infty] \to \mathbb{R}$ be a monotone decreasing non-negative function. Then the sum $\sum_{n=1}^{\infty} f(n)$ is convergent if and only if $\sup_{N>0} \int_1^N f(x) dx$ is finite.

    Show by constructing counterexamples that if the hypothesis of monotone decreasing is replaced by Riemann integrable on intervals $[1, N]$ for all $N > 0$ then both directions of the if and only if above are false.