REVIEW FOR FINAL EXAM - MATH 401/501 - FINAL 2014

Review Week Dec 1-5, 2014

Final Exam on December 9th, 2014 12:30-2:30pm

Instructor: C. Pereyra

1. (a) Show that every bounded sequence in $\mathbb{R}$ has at least a convergent subsequence.

   (b) Show that given non-empty subset A of real numbers the following are equivalent:

   (i) A is bounded and closed,

   (ii) every sequence in A has a convergent subsequence converging to a point in A.

2. Show that any function f with domain the integers $\mathbb{Z}$ will necessarily be continuous at every point on its domain. More generally, show that if $f : X \to \mathbb{R}$, and x_0 is an isolated point of $X \subset \mathbb{R}$, then f is continuous at x_0. Note: a point $x_0 \in X$ is isolated if there is a $\delta > 0$ such that $X \cap [x_0 - \delta, x_0 + \delta] = \{x_0\}$, so the only point of X that is δ-close to x_0 is x_0 itself.

3. For each choice of subsets A_i of the real numbers: Is the set bounded or not? Does it have a least upper bound or a greatest lower bound? Find them. Is the set closed or not? Find its closure.

   (a) $A_1 = [0, 1]$, (b) $A_2 = (0, 1]$, (c) $A_3 = \{1/n : n \in \mathbb{N} \backslash \{0\}\}$, (d) $A_4 = \mathbb{Z}$.

4. For each choice of subsets A_i of the real numbers in Exercise 2, construct a function $f_i : \mathbb{R} \to \mathbb{R}$ that has discontinuities at every point $x \in A_i$ and is continuous on its complement $\mathbb{R} \setminus A_i$. Explain.

5. Let $f : [0, 1] \to [0, 1]$ be a continuous function. Show that there exists a real number x in $[0, 1]$ such that $f(x) = x$, a "fixed point" (Exercise 9.7.2 p.241-242 2nd edition).

6. Let $a < b$ be real numbers, and let $f : [a, b] \to \mathbb{R}$ be a function which is both continuous and one-to-one. Show that f is strictly monotone. (See hint in Exercise 9.8.3 p. 241 2nd ed.)

7. Let $f : \mathbb{R} \to \mathbb{R}$ that satisfies the multiplicative property $f(x + y) = f(x)f(y)$ for all $x, y \in \mathbb{R}$. Assume f is not identically equal to zero.

   (a) Show that $f(0) = 1$, $f(x) \neq 0$ for all $x \in \mathbb{R}$, and $f(-x) = \frac{1}{f(x)}$ for all $x \in \mathbb{R}$. Show that $f(x) > 0$ for all $x \in \mathbb{R}$.

   (b) Let $a = f(1)$ (by (i) $a > 0$). Show that $f(n) = a^n$ for all $n \in \mathbb{N}$. Use (a) to show that $f(z) = a^z$ for all $z \in \mathbb{Z}$.

   (c) Show that $f(r) = a^r$ for all $r \in \mathbb{Q}$.

   (d) Show that if f is continuous at $x = 0$, then f is continuous at every point in $\mathbb{R}$. Moreover $f(x) = a^x$ for all $x \in \mathbb{R}$.

8. Decide whether the functions $f_i : X_i \to \mathbb{R}$ are uniformly continuous or not on their domain.

   (a) $f_1(x) = x^{13} - 8x^5 + 7 + 2^x$ with $X_1 = [-3, 14]$, (b) $f_2(x) = x^2$ with $X_2 = [1, \infty)$,

   (c) $f_3(x) = 1/x$ with $X_3 = (0, 2]$, (d) $f_4(x) = \sqrt{x}$ with $X_4 = [0, \infty)$.

9. Show that if $f : [a, b] \to \mathbb{R}$ is continuous then it is uniformly continuous.

10. (a) Assume g is defined on an open interval (a, c) and it is known to be uniformly continuous on $(a, b]$ and on $[b, c)$ where $a < b < c$. Prove that g is uniformly continuous on (a, c).

    (b) Show that if f is uniformly continuous on (a, b) and (b, c), for some $b \in (a, c)$, then f is uniformly continuous on (a, c) if and only if f is continuous at b.

11. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function such that f' is bounded. Show that f is uniformly continuous.

12. Verify the chain rule (see Exercise 10.1.7. p. 256 2nd ed).

13. A function $f : \mathbb{R} \to \mathbb{R}$ satisfies a *Lipschitz condition* with constant $M > 0$ if for all $x, y \in \mathbb{R}$,

$$|f(x) - f(y)| \leq M|x - y|.$$

Assume $h, g : \mathbb{R} \to \mathbb{R}$ each satisfy a Lipschitz condition with constant M_1 and M_2 respectively.

(a) Show that $(h + g)$ satisfies a Lipschitz condition with constant $(M_1 + M_2)$.

(b) Show that the composition $(h \circ g)$ satisfy a Lipschitz condition. With what constant?

(c) Show that the product (hg) does not necessarily satisfy a Lipschitz condition. However if both functions are bounded then the product satisfies a Lipschitz condition.

14. Assume known that the derivative of $f(x) = \sin x$ equals $\cos x$, that is, f is differentiable on $\mathbb{R}$ and $f'(x) = \cos x$. You also can use your knowledge on the trigonometric functions (you know when they are positive and negative, where are the zeros, etc). Show that $f : [0, \pi/2) \to [0, 1)$ is invertible, and that its inverse $f^{-1} : [0, 1) \to [0, \pi/2)$ is differentiable. Find the derivative of the inverse function.

15. As in the previous exercise we know that the function $\sin x$ is differentiable, and you can use known properties such as $\lim_{x\to 0, x\neq 0} \frac{\sin x}{x} = 1$ if need be. Let $G : \mathbb{R} \to \mathbb{R}$ be defined by

$$G(x) = \begin{cases} x^2 \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

Show that G is differentiable on $\mathbb{R}$ but G' is not continuous at zero.

16. Give an example of a function on $\mathbb{R}$ that has the intermediate value property for every interval (that is it takes on all values between $f(a)$ and $f(b)$ on $a \leq x \leq b$ for all $a < b$), but fails to be continuous at a point. Can such function have a jump discontinuity?

17. (L'Hopital's Rule).Let $f, g : X \to \mathbb{R}$, $x_0 \in X$ is a limit point of X such that $f(x_0) = g(x_0) = 0$, f, g are differentiable at x_0, and $g'(x_0) \neq 0$.

(i) Show that there is some $\delta > 0$ such that $g(x) \neq 0$ for all $x \in X \cap (x_0 - \delta, x_0 + \delta) \setminus \{x_0\}$.
Hint: Use Newton's approximation theorem.

(ii) Show that $\displaystyle \lim_{x\to x_0, x\neq x_0} \frac{f(x)}{g(x)} = \frac{f'(x_0)}{g'(x_0)}$.

(iii) Show that the following version of L'Hopital's Rule is not correct. Under the above hypothesis then,

$$\lim_{x\to x_0, x\neq x_0} \frac{f(x)}{g(x)} = \lim_{x\to x_0, x\neq x_0} \frac{f'(x)}{g'(x)}.$$

Hint: Consider $f(x) = G(x)$ (as in Exercise 15), and $g(x) = x$ at $x_0 = 0$. The problem arises because f' is NOT continuous at $x_0 = 0$. If both f' and g' were continuous at x_0 then (ii) and (iii) are equivalent.

18. (Integral test for series) Let $f : [1, \infty] \to \mathbb{R}$ be a monotone decreasing non-negative function. Then the sum $\sum_{n=1}^{\infty} f(n)$ is convergent if and only if $\sup_{N>0} \int_1^N f(x)dx$ is finite.

Show by constructing counterexamples that if the hypothesis of monotone decreasing is replaced by Riemann integrable on intervals $[1, N]$ for all $N > 0$ then both directions of the if and only if above are false.