REVIEW FOR TEST # 3 - MATH 401/501 - FALL 2010

December 6-10, 2010

Instructor: C. Pereyra

1. Show that any function f with domain the integers $\mathbb{Z}$ will necessarily be continuous at every point on its domain. More generally, show that if $f : X \to \mathbb{R}$, and x_0 is an isolated point of $X \subset \mathbb{R}$, then f is continuous at x_0.

2. For each choice of subsets A of the real numbers, construct a function $f : \mathbb{R} \to \mathbb{R}$ that has discontinuities at every point $x \in A$ and is continuous on its complement $\mathbb{R} \setminus A$. Explain.

   (a) $A = \{x : 0 \leq x \leq 1\} = [0, 1]$.

   (b) $A = \{1/n : 0 < n \in \mathbb{N}\}$.

3. Give an example of a function on $\mathbb{R}$ that has the intermediate value property for every interval (that is it takes on all values between $f(a)$ and $f(b)$ on $a \leq x \leq b$ for all $a < b$), but fails to be continuous at a point. Can such function have a jump discontinuity?

4. Let $X \subset \mathbb{R}$, α and C positive real numbers. Suppose a function $f : X \to \mathbb{R}$ satisfies the following Hölder continuity property:

$$|f(x) - f(y)| \leq C|x - y|^{\alpha}, \quad \text{for all } x, y \in X.$$

   Show that f is uniformly continuous on X.

5. Assume g is defined on an open interval (a, c) and it is known to be uniformly continuous on $(a, b]$ and on $[b, c)$ where $a < b < c$. Prove that g is uniformly continuous on (a, c).

   Show that if f is uniformly continuous on (a, b) and (b, c), for some $b \in (a, c)$, then f is uniformly continuous on (a, c) if and only if f is continuous at b.

6. Let $a < b$ be real numbers, and let $f : [a, b] \to \mathbb{R}$ be a function which is both continuous and one-to-one. Show that f is strictly monotone. (See hint in Exercise 9.8.3 p. 241 2nd ed).

7. Verify the chain rule (see class notes from Tuesday Nov 30 and/or Exercise 10.1.7. p. 256 2nd ed).

8. A function $f : \mathbb{R} \to \mathbb{R}$ satisfies a Lipschitz condition with constant $M > 0$ if for all $x, y \in \mathbb{R}$,

$$|f(x) - f(y)| \leq M|x - y|.$$

   Assume $h, g : \mathbb{R} \to \mathbb{R}$ each satisfy a Lipschitz condition with constant M_1 and M_2 respectively.

   (a) Show that $(h + g)$ satisfies a Lipschitz condition with constant $(M_1 + M_2)$.

   (b) Show that the composition $(h \circ g)$ satisfy a Lipschitz condition. With what constant?

   (c) Show that the product (hg) does not necessarily satisfy a Lipschitz condition. However if both functions are bounded then the product satisfies a Lipschitz condition.

9. Assume known that the derivative of $f(x) = \sin x$ equals $\cos x$, that is, f is differentiable on $\mathbb{R}$ and $f'(x) = \cos x$. Show that $f : [0, \pi/2] \to [0, 1]$ is invertible, and that its inverse $f^{-1} : [0, 1] \to [0, \pi/2]$ is differentiable. Find the derivative of the inverse function.

10. Assume known that the functions $\sin x$ and $\cos x$ are differentiable, and that their derivatives are $\cos x$ and $-\sin x$ respectively. Let $g_a : \mathbb{R} \to \mathbb{R}$ be defined by

$$g_a(x) = \begin{cases} x^a \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

   Find particular (potentially noninteger) values of a so that

   (a) g_a is differentiable on $\mathbb{R}$ but g'_a is unbounded on $[0, 1]$.

   (b) g_a is differentiable on $\mathbb{R}$ with g'_a continuous but not differentiable at zero.

   (c) g_a is differentiable on $\mathbb{R}$ and g'_a differentiable on $\mathbb{R}$, but such that g''_a is not continuous at zero.

The following are 4 multi-step problems. TURN ONE OF THEM IN ON FINAL EXAM DAY 12/16/10, FOR UP TO 5 BONUS POINTS IN THE FINAL EXAM.

1. (L'Hopital's Rule). Show that if $f, g : X \to \mathbb{R}$, $x_0 \in X$ is a limit point of X such that $f(x_0) = g(x_0) = 0$, f, g are differentiable at x_0, and $g'(x_0) \neq 0$, then there is some $\delta > 0$ such that $g(x) \neq 0$ for all $x \in X \cap (x_0 - \delta, x_0 + \delta)$ and
$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \frac{f'(x_0)}{g'(x_0)}.$$
Hint: Use Newton's approximation theorem.
Show that the following version of L'Hopital's Rule is not correct. Under the above hypothesis then,
$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f'(x)}{g'(x)}.$$
Hint: Consider $f(x) = g_2(x)$ (as in exercise 10), and $g(x) = x$ at $x_0 = 0$.

2. Let $f : [a, b] \to [a, b]$, assume there is $0 < c < 1$ such that
$$|f(x) - f(y)| \leq c|x - y| \quad \text{for all } \ x, y \in [a, b].$$
(a) Show that f is uniformly continuous on $[a, b]$.

(b) Pick some point $y_0 \in [a, b]$, and given y_n define inductively $y_{n+1} = f(y_n)$. Show that the sequence $\{y_n\}_{n \geq 0}$ is a Cauchy sequence. Show that there is some $y \in [a, b]$, such that $\lim_{n \to \infty} y_n = y$.

(c) Prove that y is a fixed point, that is, $f(y) = y$.

(d) Finally prove that given any $x \in [a, b]$, then the sequence defined inductively by: $x_0 = x$, $x_{n+1} = f(x_n)$ converges to y as defined in part (b). (That is the function f has a *unique fixed point.*)

3. (Jean's favorite problem in multiplicative version) Let $f : \mathbb{R} \to \mathbb{R}$ that satisfies the multiplicative property $f(x + y) = f(x)f(y)$ for all $x, y \in \mathbb{R}$. Assume f is not identically equal to zero.
   (i) Show that $f(0) = 1$ and that $f(-x) = \frac{1}{f(x)}$ for all $x \in \mathbb{R}$. Show that $f(x) > 0$ for all $x \in \mathbb{R}$.
   (ii) Let $a = f(1)$ (by (i) $a > 0$). Show that $f(n) = a^n$ for all $n \in \mathbb{N}$. Use (i) to show that $f(z) = a^z$ for all $z \in \mathbb{Z}$.
   (iii) Show that $f(r) = a^r$ for all $r \in \mathbb{Q}$.
   (iv) Show that if f is continuous at $x = 0$, then f is continuous at every point in $\mathbb{R}$.
   (v) Assume f is continuous at zero, use (iii) and (iv) to conclude that $f(x) = a^x$ for all $x \in \mathbb{R}$.

4. (Intermediate Value Theorem for Derivatives or Darboux's Theorem). If f is differentiable on $[a, b]$, and if α is a real number in between $f'(a)$ and $f'(b)$ say $f'(a) < \alpha < f'(b)$ (or $f'(b) < \alpha < f'(a)$), then there exists a point $c \in (a, b)$ such that $f'(c) = \alpha$. (**Warning:** you can not assume that f' is continuous even if it is defined on all of $[a, b]$, so you cannot use the IVT for continuous functions. Consider the example discussed in class $f(x) = x^2 \sin(1/x)$ it is differentiable on $[-1, 1]$ but the derivative is not continuous at $x = 0$.) **Hint:** to simplify define a new function $g(x) = f(x) - \alpha x$ on $[a, b]$. This function g is differentiable on $[a, b]$ and show that our hypothesis on f imply that $g'(a) < 0 < g'(b)$ (or $g'(b) < 0 < g'(a)$). Now show that there is a $c \in (a, b)$ such that $g'(c) = 0$. To do the later, show that there exists a point $x \in (a, b)$ such that $g(a) > g(x)$, and a point $y \in (a, b)$ such that $g(b) > g(y)$. Now finish the proof of Darboux's theorem.