Practice Problems for Exam # 1 - MATH 401/501 - Fall 2010

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1. Show by induction that the statement
$$P(n): \quad 1 + x + x^2 + \cdots + x^n = \frac{1 - x^{n+1}}{1 - x}$$
is true for all natural numbers n, and any fixed rational number $x \neq 1$.

2. Given functions $f : X \to Y$, $g : Y \to Z$, show that if $g \circ f$ is surjective (onto) then g must be surjective. Is it true that f must also be surjective?

3. Given a set X and subsets A and B of X. Show that
$$X \setminus (A \cup B) = (X \setminus A) \cap (X \setminus B).$$

4. Suppose $f : X \to Y$, and suppose that A, B are subsets of X and C, D are subsets of Y. Determine which inclusion relationship must hold for the following pair of sets:

   (a) $f(A \cap B)$ and $f(A) \cap f(B)$,

   (b) $f^{-1}(C \cap D)$ and $f^{-1}(C) \cap f^{-1}(D)$.

   Remember that for $A \subset X$, $f(A) = \{y \in Y : y = f(x), x \in A\}$ (the direct image of A under f); and for $C \subset Y$, $f^{-1}(C) = \{x \in X : f(x) \in B\}$ (the inverse image of B under f).

5. Given a rational number r, and natural numbers n and m. We define $r^0 := 1$ and given the rational number r^n then we define the rational number $r^{n+1} := r^n \times r$.

   (a) Show that
$$(r^n)^m = r^{n \times m}.$$
   Hint: fix one of the natural numbers and induct on the other.

   (b) Assume now that $r \neq 0$, p and q are integers, and show that $(r^p)^q = r^{p \times q}$. Where we define for a negative integer $p = -n$, $n \in \mathbb{N}$, $r^p = r^{-n} := (r^n)^{-1}$. Useful auxiliary lemma is to show that: $(r^n)^{-1} = (r^{-1})^n$.

6. Show that given a non-negative rational number $x \geq 0$, there exists a natural number n such that
$$n \leq x < n + 1.$$

7. Show that given any sets (finite or not) A, B, C such that $B \cap C = \emptyset$ then
$$\#(A^B \times A^C) = \#(A^{B \cup C}).$$
   Remember that the set A^B consists of all functions $f : B \to A$.