MIDTERM
MATH 362 - Spring 2002
Instructor: C. Pereyra

There are 4 problems, each worth 20 points. If you have any question, please ask the instructor in charge. Justify all your answers. GOOD LUCK and have a very good Spring Break!

1. Let (X, d) and (Y, ρ) be metric spaces. Assume that (Y, ρ) is a complete metric space. Let $(f_n)_{n>0}$ be a sequence of functions from X into Y which is uniformly Cauchy. Show that the sequence is uniformly convergent.

2. Let (X, d) be a compact metric space, (Y, ρ) a metric space. Let $f : X \to Y$ be continuous and onto (surjective). Prove that a subset A of Y is closed if $f^{-1}(A)$ is closed in X.

3. (a) Show that if the function $f : \mathbb{R} \to \mathbb{R}$ is uniformly continuous then the sequence of functions $f_n(x) = f\left(x + \frac{1}{n}\right)$ is uniformly convergent.

(b) A function $f : \mathbb{R} \to \mathbb{R}$ satisfies a *Lipschitz condition* if there exists a constant $M > 0$ such that for all $x, y \in \mathbb{R}$, $|f(x) - f(y)| \leq M|x - y|$. Show that if f satisfies a Lipschitz condition, then f is uniformly continuous.

4. Consider the space X of sequences $s = (s_n)_{n\in N}$ such that $\sum_{n\in \mathbb{N}} |s_n| < \infty$ (summable sequences or the space ℓ^1). Introduce the distance function between two sequences $s = (s_n)_{n\in \mathbb{N}}$ and $t = (t_n)_{n\in \mathbb{N}}$ in X,

$$d(s, t) = \sum_{n\in N} |s_n - t_n|.$$

Show that (X, d) is a metric space.
(Assume basic facts about convergent series learned in elementary calculus courses.)