Exam Review #2

Math 361/461

Exam date 11/15/2001

Topics Covered: Sections 11-14, 16-19 in the textbook.

- **The real numbers**
  - Ordered fields.
  - Open and closed sets.
  - Interior, boundary and accumulation points.
  - Supremum and infimum. Completeness axiom.
  - Compact sets
    - Heine-Borel Theorem.
    - Bolzano-Weirstrass Theorem.
- **Sequences**
  - Convergence and limits. Properties.
  - Monotone sequences. Bounded monotone sequences are convergent.
  - Cauchy sequences. Cauchy sequences are convergent.
  - Subsequences, lim inf and lim sup.

Practice Problems

1. Given the set $F = \{0, 1, 2\}$. Define the operations $\oplus$ and $\odot$ on F by:

$\oplus$	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

$\odot$	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

It is true that both operations are *associative* and that the sum is *distributive* with respect to the product: $a \odot (b \oplus c) = a \odot b \ \oplus \ a \odot c$. You do not have to check these properties.

(a) Show that the set F with the operations $\oplus$ and $\odot$ is a field.

(b) Show that you can choose a set P of *positive* elements of F such that given any $x \in F$ then one and only one of the following three holds:
(i) $x = 0$, (ii) $x \in P$, (iii) the additive inverse of x, $(-x)$ is in P.

(c) Prove or give a counterexample to the following statement: for any set P that satisfies (b) if $x, y \in P$ then $x \oplus y \in P$ and $x \odot y \in P$.

(d) Is F an ordered field?

2. Let $a_1 = 1$ and $a_{n+1} = \frac{a_n}{2} + \frac{2}{a_n}$. Show that the sequence $(a_n)_{n \geq 2}$ is monotone and bounded, hence it is convergent. Find the limit. **Hint:** Show by induction that $1 \leq a_n \leq 4$. You can actually show directly that $a_n \geq 2$ for $n \geq 2$. Now check that $a_n - a_{n+1} > 0$ for all $n \geq 2$.

3. **(a)** Suppose that the sequences (x_n) and (y_n) converge to the same number. Prove by definition that the sequence $(x_n - y_n)$ converges to zero. Is the converse true?

(b) Show that the sequence $z_n = \sqrt{1+4n^2} - 2n$ converges to zero.

4. Let (x_n) be a Cauchy sequence. Prove that (x_n) converges if and only if at least one of its subsequences converges. (Do not use the fact that a sequence is convergent if and only if it is Cauchy; just use the definitions.)

5. **(a)** Show that if a sequence (x_n) is bounded and decreasing then the limit exists, moreover,

$$\lim_{n\to\infty} x_n = \inf_{n>0} x_n.$$

(b) Prove that if (x_n) is a bounded sequence then,

$$\limsup(x_n) = \inf_{n>0}\left(\sup_{k\geq n} x_k\right).$$

(this is actually true for any sequence).

6. Suppose the sequence (x_n) satisfies the following property:

$$|x_{n+1} - x_n| \leq \frac{1}{2}|x_n - x_{n-1}|, \qquad \forall n = 2, 3, \ldots.$$

Show that (x_n) is convergent. (**Hint:** Show that (x_n) is a Cauchy sequence.)

7. Show that the set of numbers of the form $\frac{k}{2^n}$, where k is an integer and n is a positive integer, is dense in $\mathbb{R}$. (That is, show that given any to real numbers $x < y$ there exists k and $n > 0$ integers such that $x < \frac{k}{2^n} < y$.

8. **(a)** Let A be an open set. Show that if a finite number of points are removed from A, the remaining set is still open. Is the same true if a countable number of points are removed?

(b) Let B be a closed set, x a point in B. Let C be the set B with x removed. Under what conditions is C closed?

9. Show that the set of subsequential limits of a sequence is a closed set.

10. Find the interior of the set S, the set S' of accumulation points of S. Decide whether the set S is open, closed, and/or compact.

(a) $S = \{x = 3n : n \in \mathbb{Z}\}$.

(b) $S = (-\infty, 0] \cup \{1, \sqrt{2}, \pi\}$.

(c) $S = \cap_{n>0}[1 + 1/n, 3]$

11. Prove whether the limit exists or not. Find the limits when possible, including infinite limits.

(a) $\lim_{n\to\infty} \frac{n+1}{2^n}$, **(b)** $\lim_{n\to\infty} \left(3 + \frac{1}{n}\right) \sin\left(\frac{n\pi}{2}\right)$,

(c) $\lim_{n\to\infty} \frac{n}{1+\sqrt{n}}$.

12. Given the sequence (x_n) find $\sup(x_n)$, $\inf(x_n)$, $\limsup(x_n)$ and $\liminf(x_n)$. Justify your answers.

(a) $x_n = \frac{n+(-1)^n(2n+1)}{n}$,

(b) $x_n = 2n + \cos(n\pi)$.