Exam Review #1

Math 361/461

Exam date 9/27/2001

Topics Covered: Sections 1-8 and 10 in the textbook.

- **Logic and Proof**
  - Logical connectives.
  - Quantifiers.
  - Techniques of proof.
    - Direct proofs.
    - Contrapositive proofs.
    - Proofs by contradiction.
    - Inductive proofs (principle of mathematical induction).
- **Set Theory**
  - Basic concepts: elements, subsets, empty set.
  - Basic set operations: union, intersection, complement. difference, symmetric difference.
  - Set operations over arbitrary collection of sets.
  - Ordered pairs.
  - Cartesian products.
  - Relations.
  - Equivalence relations: equivalence classes, partitions.
- **Functions**
  - Domain and range.
  - Injective, surjective and bijective functions.
  - Functions acting on sets: Image and pre-image.
  - Composition of functions.
  - Inverse functions.
- **Cardinal numbers**
  - Equinumerous sets.
  - Finite and infinite sets.
  - Countable (finite and denumerable) and uncountable sets.

Practice Problems

1. A sorority has a rule for new members: each must always tell the truth or always lie. They know who does which. If I meet three of them on the street and they make the statements below, which ones (if any) should I believe?

- A says: *All three of us are liars.*
- B says: *Exactly two of us are liars.*
- C says: *The other two are liars.*

2. Show that every finite list of real numbers contains a number at least as large as its average.

3. Let U be a universal set, A and B subsets of U. Prove the second Morgan Law:

$$(A \bigcup B)^c = A^c \bigcap B^c.$$

4. Given three sets A, B, and C. Prove that

$$(A \bigcup B) \backslash C \subset [A \backslash (B \bigcup C)] \bigcup [B \backslash (A \bigcap C)].$$

Prove that equality does not need to hold.

5. Find $\bigcap_{A \in \mathcal{F}} A$ and $\bigcup_{A \in \mathcal{F}} A$ for the collection of intervals

$$\mathcal{F} = \left\{ \left[1 - \frac{1}{n}, 1 + \frac{1}{n} \right] : n \in \mathbb{N} \right\}.$$

Justify your answers!

6. Consider the relation defined on $\mathbb{R} \times \mathbb{R}$ by:

$$(x_1, y_1) R (x_2, y_2) \Leftrightarrow x_1^2 + y_1^2 = x_2^2 + y_2^2.$$

Show that this is an equivalence relation. Describe the equivalence classes associated to R.

7. Given a function $f : A \to B$, subsets S, R of A, and subsets X, Y of B. Determine which inclusions hold for the following pairs of sets:

- $f(S \bigcap R)$ and $f(S) \bigcap f(R)$.
- $f^{-1}(X \bigcap Y)$ and $f^{-1}(X) \bigcap f^{-1}(Y)$.

8. Suppose f and g are surjective functions from $\mathbb{Z}$ into $\mathbb{Z}$, and suppose that $h = f \cdot g$ is the pintwise product of f and g (not the composition!). Must h also be surjective? Give a proof or a counterexample.

9. Suppose $f : \mathbb{R} \to \mathbb{R}$ satisfies $f(xy) = xf(y) + yf(x)$ for all $x, y \in \mathbb{R}$. Prove that $f(1) = 0$ and that $f(u^n) = nu^{n-1} f(u)$ for all $n \in \mathbb{N}$, and for all $u \in \mathbb{R}$.

10. Show that if the sets A and B are denumerable (infinite countable) then so is their cartesian product.

11. Show that every polynomial of degree d has at most d zeros. **Hint:** Use induction on the degree of the polynomial.

12. Prove that for all natural numbers n, 6 divides $n^3 - n$.