

EXAM # 3
MATH 361/461 - Fall 2001
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Name:

This is a take home test. You can consult your textbook, classnotes and other books, you can discuss the exam with your classmates or somebody else, but I would like that what you write down reflects what you honestly understand. Please make appropriate references to the books that you use in solving this exam. If you have any questions please e-mail me at:

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I will answer collectively if necessary.

There are 5 problems worth a total of 100 points and two bonus problem worth 10 points each.

Read carefully all the problems. Justify all your answers.

Good luck!!!!

Problem	grade
# 1	
# 2	
# 3	
# 4	
# 5	
Bonus	
TOTAL	

Problem 1 (30 points): A function is said to be *Lipschitz*, if there exists $M > 0$, such that for every x, y in the domain of the function,

$$|f(x) - f(y)| \leq M|x - y|.$$

(a) Show that if f and g are bounded Lipschitz functions on \mathbb{R} then their product $f.g$ is also a Lipschitz function. Give a counterexample to show that it is necessary to assume boundedness.

(b) Show that Lipschitz functions are uniformly continuous.

Problem 2: Suppose that $f : [a, b] \rightarrow [a, b]$ is continuous.

- (a) Prove that f has a *fixed point*. that is, prove that there exists $c \in [a, b]$ such that $f(c) = c$.
- (b) Can you devise an algorithm to find a fixed point of f ?

Problem 3: Given the function

$$f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x^2}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

- (a) Show that f is uniformly continuous on the closed interval $[-1, 1]$.
- (b) Show that f is differentiable on $[-1, 1]$. Find explicitly $f'(x)$.
- (c) Is $f'(x)$ continuous at $x = 0$? Is $f'(x)$ bounded on $[-1, 1]$?

Note: In this problem assume that you can compute the derivative in the usual fashion for $x \neq 0$, that is by the usual laws of differential calculus learned in Math 162.

Problem 4: Let f be a function defined on an open interval. Let (x_n) be a sequence of points in the domain of f . Let $y_n = f(x_n)$ be the sequence of image points.

(a) Assume that f is a bounded function. Show that the sequence of images (y_n) must have a convergent subsequence. Give a counterexample to show that it is necessary to assume boundedness.

(b) Assume that f is a uniformly continuous function, and that (x_n) is a Cauchy sequence. Show that the sequence of images (y_n) is Cauchy.

(c) If f is a continuous function on \mathbb{R} , is it necessarily true that

$$f(\liminf_{n \rightarrow \infty} x_n) = \liminf_{n \rightarrow \infty} f(x_n) \quad ?$$

Problem 5: Let x be a real number in $[0, 1]$ with the *ternary expansion* $\langle a_n \rangle$ ¹.

Let $N = \infty$ if none of the a_n are 1, and otherwise let N be the smallest value of n such that $a_n = 1$. Let $b_n = \frac{a_n}{2}$, for $n < N$, and $b_N = 1$.

(a) Show that $\sum_{n=1}^N \frac{b_n}{2^n}$ is independent of the ternary expansion of x (if x has two expansions).

Define the function f by setting

$$f(x) = \sum_{n=1}^N \frac{b_n}{2^n}$$

(b) Show that f is constant on each interval contained in the complement of the Cantor set, and that f maps the Cantor set onto the interval $[0, 1]$. (remember exercise 11.14).

(c) Show that f is an increasing function on $[0, 1]$. Show that f is continuous on $[0, 1]$ (**hint:** the function $g(x) = f(x) + x$ is strictly increasing on $[0, 1]$ and $f([0, 1]) = [0, 2]$, now use exercise 22.13(b)).

Note: This function is called the *Cantor ternary function*.

¹That is: $x = \sum_{n=1}^{\infty} \frac{a_n}{3^n}$, $a_n \in \{0, 1, 2\}$. This expansion is unique, except in the case that $x = \frac{k}{3^n}$, $n, k \in \mathbf{N}$ and

$0 \leq k \leq 3^n - 1$ when there are exactly two such sequences, for example $x = \frac{1}{3} = \sum_{n=2}^{\infty} \frac{2}{3^n}$.

Bonus 1:

(a) Find a bounded function defined on the real numbers that assumes its sup and inf on every closed interval and yet it is not continuous.

(b) Find a continuous function with domain \mathbb{R} such that the inverse image of a compact set is not compact.

Bonus 2:

Given a closed set A , find a continuous function which is equal to zero on A , and different than zero outside of A (that is on A^c).