(1) A vector space is a set V and two operations addition and scalar multiplication defined on the set such that

(i) addition and scalar multiplication are closed in V: if $x, y \in V$, $\lambda \in \mathbb{R}$ then $x+y \in V$ and $\lambda x \in V$

(ii) Addition is <u>commutative</u>, <u>associative</u>, there is an <u>additive identity</u> the <u>zero vector</u> $0 \in V$ such that for all $x \in X$ $x+0=0+x=x$; there is an <u>additive inverse</u> to each $x \in V$, denoted $(-x)$ st $x+(-x)=0=(-x)+x$.

(iii) Scalar multiplication is <u>associative</u>, $\lambda, \beta \in \mathbb{R}$ then $(\lambda\beta)x = \lambda(\beta x)$; and $1x = x$ <u>unity</u>

(iv) Addition and scalar multiplication obey distributive laws: $x, y \in X$, $\lambda, \beta \in \mathbb{R}$ then $\lambda(x+y) = \lambda x + \lambda y$ and $(\lambda+\beta)x = \lambda x + \beta x$.

$\mathbb{R}^3$ is a vector space with componentwise addition and scalar multiplication. A subset $V \subseteq \mathbb{R}^3$ will be a subspace iff $0 \in V$, and addition and scalar multiplication are closed in V.

(a) $V = \{(x,y,z) \in \mathbb{R}^3 : x = 3y,\ y = x+z\}$.

Note that $(0,0,0) \in V$ since $0 = 3.0$, $0=0+0$.

If $(x_1,y_1,z_1), (x_2,y_2,z_2) \in V$ then

$(*)$ $x_1 = 3y_1$ $\quad y_1 = x_1+z_1$

$(**)$ $x_2 = 3y_2$ $\quad y_2 = x_2+z_2$

$\Rightarrow \begin{cases} x_1+x_2 = 3y_1+3y_2 = 3(y_1+y_2) \\ y_1+y_2 = (x_1+z_1)+(x_2+z_2) = (x_1+x_2)+(z_1+z_2) \end{cases}$

(Continuation ①(a))

This implies that $(x_1+x_2,\ y_1+y_2,\ z_1+z_2) \in V$ so addition is closed in V.

Also from (*) $\lambda x_1 = \lambda 3 y_1 = 3(\lambda y_1)$

$\lambda y_1 = \lambda(x_1+z_1) = (\lambda x_1) + (\lambda z_1)$

This implies $(\lambda x_1, \lambda y_1, \lambda z_1) \in V$, so scalar multiplication is closed in V.

(b) This time

$$V = \{(x,y,z) \in \mathbb{R}^3 : x = y+z-1\}.$$

Note that $(0,0,0) \notin V$ because

$$0 \neq 0+0-1 = -1$$

Therefore V is not a subspace of $\mathbb{R}^3$.

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② Consider the following subspace of $\mathcal{P}_3(\mathbb{R})$

$$X = \{a + bx + cx^2 + dx^3 : a+b=c\}$$

$(a,b,c,d \in \mathbb{R})$

We are told that X is a subspace (it is easy to check that $0 \in X$, and that addition and scalar multiplication are closed in X like in 1(a)).

Let $p(x)$ be a polynomial in X, then

$p(x) = a + bx + cx^2 + dx^3$, where $\boxed{a+b=c}$

Then
$$p(x) = a + bx + (a+b)x^2 + dx^3$$
collecting all terms with a, b, d we get
$$p(x) = a(1+x^2) + b(x+x^2) + dx^3$$
This shows that every $p(x) \in X$ is a linear combination of three polynomials in $P_3(\mathbb{R})$, namely: $1+x^2$, $x+x^2$, x^3

In other words
$$X = \text{span}\{(1+x^2), (x+x^2), x^3\}$$
This shows that $\dim X \leq 3$

If we can show that these polynomials are linearly independent, then we will show that they are a basis of X and that $\dim X = 3$.

(Note that $1+x^2$, $x+x^2$ and $x^3 \in X$).

Linear independence: suppose there exist scalars a, b, d s t
$$a(1+x^2) + b(x+x^2) + dx^3 = 0$$
$$\Rightarrow a + bx + (a+b)x^2 + dx^3 = 0 \Rightarrow \boxed{a = b = d = 0}$$
$(\Rightarrow a+b = 0)$

Therefore the polynomials are linearly independent, and a basis of X is

$\boxed{\dim X = 3}$

$$\boxed{B = \{(1+x^2), (x+x^2), x^3\}}$$

(3) Decide whether the following are true or false, explain. X is a vector and $\dim X = n \geq 1$.

(a) You can find a collection of $(n+2)$ linear independent vectors in X. False, in an n-dimensional vector space you can find at most n linearly independent vectors.

(b) You can find a collection of $n+1$ vectors that span X. TRUE. Take a basis of X, these are exactly n vectors that span X, and take a linear combination of them, that is different than any of the basis vectors. Now you have $n+1$ vectors that span X.

(c) If V is a subspace of X, then $\dim V \leq n$. TRUE. Subspaces have dimension at most equal to the dimension of the "parent" space.

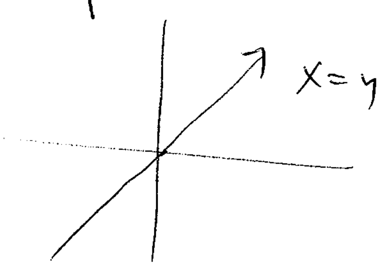
If V had dimension $m > n$ then we could identify m linearly independent vectors in V which will be necessarily linearly independent in X, therefore $\dim X \geq m > n = \dim X$, and this is a contradiction.

(Continuation (3))

(d) If V is a non-trivial subspace of X (That is $V \neq \{0\}$), and $B = \{x_1, x_2, \ldots, x_n\}$ is a basis of X, Then you can select vectors in B to obtain a basis for V. FALSE

Consider $X = \mathbb{R}^2$, $B = \left\{ \binom{1}{0}, \binom{0}{1} \right\}$ The standard basis, and $V = \left\{ \binom{x}{y} \in \mathbb{R}^2 : x = y \right\}$

(V is represented by The diagonal line)



is a one-dimensional subspace of $\mathbb{R}^2$, and any basis will consist of one vector with "equal and non-zero coordinates". Neither $\binom{1}{0}$ nor $\binom{0}{1}$ belong to V, so They cannot be basis elements of V.

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(4) Show That $u, v \in X$ are linearly independent if and only if $(u+v)$ and $(v-u)$ are l.i.

We have to show two statements

($\Rightarrow$) if u, v are l.i Then $(u+v), (v-u)$ are l.i

and

($\Leftarrow$) if $(u+v), (v-u)$ are l.i Then u, v are l.i

(continuation 4)

Proof of ($\Rightarrow$): Hypothesis: U, V are l. i

That means that if $a, b \in \mathbb{R}$ and

(*) $aU + bV = 0$ Then $a = b = 0$.

We want to show that if $c, d \in \mathbb{R}$ and

(**) $c(U+V) + d(V-U) = 0$ | Then $c = d = 0$ we don't know yet.

$\rightarrow (c-d)U + (c+d)V = 0$

let $a = c-d$, $b = c+d$ in (*), l. i. of

U and V imply $\begin{cases} c-d=0, \\ c+d=0 \end{cases} \Rightarrow 2c = 0$ $\boxed{c=0}$ $\boxed{d=c=0}$

We have shown that $(U+V), (V-U)$ are l. i.

Proof of $\Leftarrow$) Hypothesis now is $(U+V), (V-U)$ are l.i.

Suppose that there are $a, b \in \mathbb{R}$ st

$\boxed{aU + bV = 0}$, can we find $c, d \in \mathbb{R}$ so that

replacing $a = c-d$, $b = c+d$ $\Leftarrow$ $c-d = a$ and $c+d = b$?

we get

$(c-d)U + (c+d)V = 0$

$\rightarrow c(U+V) + d(V-U) = 0$

l. i of $(U+V), (V-U) \Rightarrow c = d = 0$

$\Rightarrow \boxed{a = c-d = 0}, \boxed{b = c+d = 0}$ so U, V are l. i.

Sure, solving the system of 2 equations in the two unknowns c, d given a, b, we get

$2c = a+b \Rightarrow \boxed{c = \frac{a+b}{2}}$

$d = c-a \Rightarrow \boxed{d = \frac{b-a}{2}}$

(5) Define $T : P_2(\mathbb{R}) \longmapsto M_{2\times 2}(\mathbb{R})$ for each $p(x) = a + bx + cx^2 \in P_2(\mathbb{R})$ by

$$(*) \quad T(p(x)) = \begin{pmatrix} 2p''(0) & p'(0) + p(0) \\ 0 & p(1) \end{pmatrix}$$

Show that T is a linear transformation.

First I will rewrite $T(p(x))$ in terms of the coefficients a, b, c of $p(x)$: $p'(x) = b + 2cx$, $p''(x) = 2c$

$$(**) \quad T(p(x)) = \begin{pmatrix} 4c & b + a \\ 0 & a + b + c \end{pmatrix}$$

$$\boxed{\begin{aligned} p''(0) &= 2c \\ p'(0) &= b \\ p(0) &= a \\ p(1) &= a + b + c \end{aligned}}$$

We can use either $(*)$ or $(**)$ to show that T is a linear transformation.

I will use $(*)$. Given $p, q \in P_2(\mathbb{R})$, $\lambda \in \mathbb{R}$

$\lambda p(x) + q(x) \in P_2(\mathbb{R})$

linearity properties of differentiation imply that

$$(\lambda p + q)'(x) = \lambda p'(x) + q'(x)$$

$$(\lambda p + q)''(x) = \lambda p''(x) + q''(x)$$

In particular:

$$2(\lambda p + q)''(0) = 2\lambda p''(0) + 2q''(0)$$

$$(\lambda p + q)'(0) + (\lambda p + q)(0) = \lambda p'(0) + q'(0) + \lambda p(0) + q(0) = \lambda(p'(0) + p(0)) + (q'(0) + q(0))$$

$$(\lambda p + q)(1) = \lambda p(1) + q(1)$$

(continuation of 5)

Therefore:

$$T((\lambda p+q)(x)) = \begin{pmatrix} 2\lambda p''(0)+2q''(0) & \lambda(p'(0)+p(0))+(q'(0)+q(0)) \\ 0 & \lambda p(1)+q(1) \end{pmatrix}$$

using matrix addition and scalar mult.

$$= \lambda\begin{pmatrix} 2p''(0) & p'(0)+p(0) \\ 0 & p(1) \end{pmatrix} + \begin{pmatrix} 2q''(0) & q'(0)+q(0) \\ & q(1) \end{pmatrix}$$

$$= \lambda Tp(x) + Tq(x)$$

This shows That T is a linear transformation

Is T one-to-one? NO

Suffices to check whether nullspace of T is trivial or not, $N(T)=\{0\}$?

Suppose $Tp(x)=\begin{pmatrix}0&0\\0&0\end{pmatrix}$, using (**) this means if $p(x)=a+bx+cx^2$ Then

$$\begin{pmatrix} 4c & b+a \\ 0 & a+b+c \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \Rightarrow \begin{matrix} 4c=0 \\ b+a=0 \\ a+b+c=0 \end{matrix}$$

$\Rightarrow c=0,\ b=-a \Rightarrow (a+b+c=0)$

T is not one-to-one, The polynomial $p(x)=a-ax$ is mapped into $\begin{pmatrix}0&0\\0&0\end{pmatrix}$

$T(a-ax)=\begin{pmatrix}0&0\\0&0\end{pmatrix}$, so $a-ax\in N(T)$ so $N(T)\neq\{0\}$.

We learn more

$\boxed{N(T) = \{ p(x) = a - ax = a(1-x) : a \in \mathbb{R} \}}$

is a one-dimensional vector space spanned by the polynomial $(1-x)$.

$\boxed{\dim N(T) = 1}$, $\boxed{N(T) = \text{span}\{(1-x)\}}$

Is T onto? NO simply from dimension considerations. $\dim(M_{2\times2}) = 4$

however $\dim P_2(\mathbb{R}) = 3 = \dim N(T) + \dim R(T)$. There is

$\dim R(T) \leq 3$ in all cases,

no way $R(T) = M_{2\times2}$.

We know more, $\dim N(T) = 1 \Rightarrow \boxed{\dim R(T) = 2}$

We can use (**) to describe the range of T, R(T)

An 2×2 matrix A belongs to $R(T)$ if it is of the form

$$A = \begin{pmatrix} 4c & b+a \\ 0 & a+b+c \end{pmatrix} = T(a+bx+cx^2)$$

$$A = c\begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix} + (a+b)\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$$

This shows

That $\boxed{R(T) = \text{span}\left\{ \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \right\}}$

These must be l.i, hence a basis of R(T).

(6) Let $T: \mathbb{R}^4 \longmapsto \mathbb{R}^4$ be a linear transformation defined by $T\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x_4 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix}$ ("a permutation")

(a) Show that T is one-to-one and onto, hence invertible. Find the inverse transformation.

<u>one-to-one</u>: if $T\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} x_4 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix}$

Then $x_4 = x_1 = x_2 = x_3 = 0 \Rightarrow$ <u>T is 1-1</u>

<u>T is onto</u> $\Leftrightarrow$ Given $\vec{y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} \in \mathbb{R}^4$ we want to find $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$ such that $T\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} x_4 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix}$

clearly
$$y_1 = x_4, \quad y_2 = x_1, \quad y_3 = x_2, \quad y_4 = x_3$$

Let $\vec{x} = \begin{pmatrix} y_2 \\ y_3 \\ y_4 \\ y_1 \end{pmatrix}$ Then

$$T\vec{x} = T\begin{pmatrix} y_2 \\ y_3 \\ y_4 \\ y_1 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \vec{y}$$

This proves directly that T is onto, and also gives a formula for its inverse

$$\boxed{T^{-1}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} y_2 \\ y_3 \\ y_4 \\ y_1 \end{pmatrix}}$$

Let us check that these linear transformations are inverse of each other:

$$T^{-1}\circ T\begin{pmatrix}x_1\\y_2\\x_3\\x_4\end{pmatrix}=T^{-1}\begin{pmatrix}x_4\\x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix}\quad\checkmark.$$

$$T\circ T^{-1}\begin{pmatrix}y_1\\y_2\\y_3\\y_4\end{pmatrix}=T\begin{pmatrix}y_2\\y_3\\y_4\\y_1\end{pmatrix}=\begin{pmatrix}y_1\\y_2\\y_3\\y_4\end{pmatrix}\quad\checkmark.$$

(b) Let α be standard basis in $\mathbb{R}^4$,

$$\beta=\left\{\begin{pmatrix}1\\2\\0\\0\end{pmatrix},\begin{pmatrix}0\\-1\\0\\0\end{pmatrix},\begin{pmatrix}0\\0\\3\\1\end{pmatrix},\begin{pmatrix}1\\0\\0\\1\end{pmatrix}\right\}.$$

Find $[T]_\alpha^\alpha$, $[T^{-1}]_\alpha^\alpha$, $[T]_\beta^\alpha$.

$[T]_\alpha^\alpha = A$, so that $L_A x = Ax = Tx$

Notice that
$$\underset{[Tx]_\alpha}{Tx=\begin{pmatrix}x_4\\x_1\\x_2\\x_3\end{pmatrix}}=\underbrace{\begin{pmatrix}0&0&0&1\\1&0&0&0\\0&1&0&0\\0&0&1&0\end{pmatrix}}_{A}\underset{[x]_\alpha}{\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix}}$$

The columns are exactly
$$T\begin{pmatrix}1\\0\\0\\0\end{pmatrix}=\begin{pmatrix}0\\1\\0\\0\end{pmatrix}$$
$$T\begin{pmatrix}0\\1\\0\\0\end{pmatrix}=\begin{pmatrix}0\\0\\1\\0\end{pmatrix}$$
$$T\begin{pmatrix}0\\0\\1\\0\end{pmatrix}=\begin{pmatrix}0\\0\\0\\1\end{pmatrix}$$
$$T\begin{pmatrix}0\\0\\0\\1\end{pmatrix}=\begin{pmatrix}1\\0\\0\\0\end{pmatrix}$$

$$\boxed{[T]_\alpha^\alpha=\begin{pmatrix}0&0&0&1\\1&0&0&0\\0&1&0&0\\0&0&1&0\end{pmatrix}}$$

Similarly we can get $[T^{-1}]_\alpha^\alpha$

finding 4x4 matrix B st $T^{-1}y = By$

$$\underbrace{T^{-1}y}_{[T^{-1}y]_\alpha} = \begin{pmatrix} y_2 \\ y_3 \\ y_4 \\ y_1 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}}_{B} \underbrace{\begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix}}_{[y]_\alpha}$$

so $\boxed{[T^{-1}]_\alpha^\alpha = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}}$

[You can check that $B = A^{-1}$, by calculating

$AB = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ 4x4 Identity matrix]

Finally to find $[T]_\beta^\alpha$ we need to compute action of T on each vector in β basis

$$T\begin{pmatrix} 1 \\ 2 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 2 \\ 0 \end{pmatrix} = \left[T\begin{pmatrix} 1 \\ 2 \\ 0 \\ 0 \end{pmatrix}\right]_\alpha$$

$$T\begin{pmatrix} 0 \\ -1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -1 \\ 0 \end{pmatrix} = \left[T\begin{pmatrix} 0 \\ -1 \\ 0 \\ 0 \end{pmatrix}\right]_\alpha$$

$$T\begin{pmatrix} 0 \\ 0 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 3 \end{pmatrix} = \left[T\begin{pmatrix} 0 \\ 0 \\ 3 \\ 1 \end{pmatrix}\right]_\alpha$$

$$T\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \left[T\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}\right]_\alpha$$

columns of $[T]_\beta^\alpha$

$$\boxed{[T]_\beta^\alpha = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 2 & -1 & 0 & 0 \\ 0 & 0 & 3 & 0 \end{pmatrix}}$$

7. Find all linear transformations $T: \mathbb{R}^3 \mapsto \mathbb{R}^2$ such that $T(1,2,0)=(3,-1)$, $T(4,-1,1)=(2,2)$.
Is such T one-to-one? Is it onto?

From dimensional perspective such <u>T can never be one-to-one</u>, That would imply $\dim N(T)=0$ since $\dim(\mathbb{R}^3)=3 \Rightarrow \dim R(T)=3$ (Contradiction) but range of T sits in 2-dimensional space $\mathbb{R}^2$, so $\dim R(T) \leq 2 \Rightarrow$ <u>$\dim N(T) \geq 1$</u>

<u>Is such T onto?</u> <u>Yes always</u> because we already know That the vectors $\begin{pmatrix}3\\-1\end{pmatrix}$ and $\begin{pmatrix}2\\2\end{pmatrix}$ are in the range of T, These are clearly linearly independent vectors in $\mathbb{R}^2$, hence a basis,

$$\mathbb{R}^2 = \text{span}\left\{\begin{pmatrix}3\\-1\end{pmatrix}, \begin{pmatrix}2\\2\end{pmatrix}\right\} \subseteq R(T) \subseteq \mathbb{R}^2$$

so $R(T)=\mathbb{R}^2$.

To find all linear transformations with the given property note That The vectors $\begin{pmatrix}1\\2\\0\end{pmatrix}$ and $\begin{pmatrix}4\\-1\\1\end{pmatrix}$ are linearly independent in $\mathbb{R}^3$, 3-dimensional vector space.

(Continuation ④)

T is completely defined by its action on a basis of $\mathbb{R}^3$ so let us complete to a basis in $\mathbb{R}^3$

$$\begin{pmatrix}1\\2\\0\end{pmatrix}, \begin{pmatrix}4\\-1\\1\end{pmatrix}, \begin{pmatrix}a\\b\\c\end{pmatrix},$$

where $\begin{pmatrix}a\\b\\c\end{pmatrix}$ has been chosen to be linearly independent from the other two, that is if $\alpha\begin{pmatrix}1\\2\\0\end{pmatrix}+\beta\begin{pmatrix}4\\-1\\1\end{pmatrix}+\gamma\begin{pmatrix}a\\b\\c\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$ Then $\alpha=\beta=\gamma=0$.

For example, choose $\begin{pmatrix}a\\b\\c\end{pmatrix}=\begin{pmatrix}1\\0\\0\end{pmatrix}$ (in This case $\beta=0, \alpha=0, \gamma=0$).

Fix now The basis of $\mathbb{R}^3$

$$B=\left\{\begin{pmatrix}1\\2\\0\end{pmatrix}, \begin{pmatrix}4\\-1\\1\end{pmatrix}, \begin{pmatrix}1\\0\\0\end{pmatrix}\right\},$$

given $\begin{pmatrix}x\\y\\z\end{pmatrix}$ in $\mathbb{R}^3$

There are coeff α, β, γ

$$\begin{pmatrix}x\\y\\z\end{pmatrix}=\alpha\begin{pmatrix}1\\2\\0\end{pmatrix}+\beta\begin{pmatrix}4\\-1\\1\end{pmatrix}+\gamma\begin{pmatrix}1\\0\\0\end{pmatrix}$$

so $$\begin{bmatrix}x\\y\\z\end{bmatrix}_B=\begin{pmatrix}\alpha\\\beta\\\gamma\end{pmatrix}$$

Then $$T\begin{pmatrix}x\\y\\z\end{pmatrix}=\alpha T\begin{pmatrix}1\\2\\0\end{pmatrix}+\beta T\begin{pmatrix}4\\-1\\1\end{pmatrix}+\gamma T\begin{pmatrix}1\\0\\0\end{pmatrix}$$

all we need is to specify $T\begin{pmatrix}1\\0\\0\end{pmatrix}=\begin{pmatrix}a\\b\end{pmatrix}\in\mathbb{R}^2$

$$\underset{\left[T\begin{pmatrix}x\\y\\z\end{pmatrix}\right]_{\text{standard}}}{\underset{\parallel}{T\begin{pmatrix}x\\y\\z\end{pmatrix}}}=\alpha\begin{pmatrix}3\\-1\end{pmatrix}+\beta\begin{pmatrix}2\\2\end{pmatrix}+\gamma\begin{pmatrix}a\\b\end{pmatrix}=\underset{[T]_B^{\text{standard}}}{\underset{\parallel}{\begin{bmatrix}3&2&a\\-1&2&b\end{bmatrix}}}\underset{\begin{bmatrix}x\\y\\z\end{bmatrix}_B}{\underset{\parallel}{\begin{pmatrix}\alpha\\\beta\\\gamma\end{pmatrix}}}$$

for each pair $\binom{a}{b}$ we obtain a different linear transformation, and those are all! We can parametrize their matrix representation from the ~~standard~~ basis $\mathcal{B} = \left\{ \begin{pmatrix}1\\2\\0\end{pmatrix}, \begin{pmatrix}4\\-1\\1\end{pmatrix}, \begin{pmatrix}1\\0\\0\end{pmatrix} \right\}$ in $\mathbb{R}^3$

to the standard basis $\alpha = \left\{ \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix} \right\}$ in $\mathbb{R}^2$

$$[T_{a,b}]_{\mathcal{B}}^{\alpha = \text{standard}} = \begin{bmatrix} 3 & 2 & a \\ -1 & 2 & b \end{bmatrix}$$

Where $T_{a,b}$ is defined to map

$$T_{a,b}\begin{pmatrix}1\\2\\0\end{pmatrix} = \begin{pmatrix}3\\-1\end{pmatrix}, \quad T_{a,b}\begin{pmatrix}4\\-1\\1\end{pmatrix} = \begin{pmatrix}2\\2\end{pmatrix} \quad T_{a,b}\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}a\\b\end{pmatrix}.$$

---

8) Let $T: X \to X$ be a linear transformation, $\dim X = 3$, $\alpha = \{x_1, x_2, x_3\}$, $\beta = \{y_1, y_2, y_3\}$ bases on X

Let (Note typo α not β) $[T]_{\alpha}^{\beta} = \begin{pmatrix} 1 & -3 & 2 \\ 0 & 1 & -1 \\ 2 & 1 & 0 \end{pmatrix}$

(a) If $x = 2x_1 - x_2 + x_3$ what is Tx?

$$[x]_\alpha = \begin{pmatrix}2\\-1\\1\end{pmatrix}, \quad [Tx]_\beta = [T]_\alpha^\beta [x]_\alpha = \begin{pmatrix} 1 & -3 & 2 \\ 0 & 1 & -1 \\ 2 & 1 & 0 \end{pmatrix}\begin{pmatrix}2\\-1\\1\end{pmatrix} = \begin{pmatrix}7\\-2\\3\end{pmatrix}$$

$$[Tx]_\beta = \begin{pmatrix}7\\-2\\3\end{pmatrix} \Rightarrow \boxed{Tx = 7y_1 - 2y_2 + 3y_3}$$

(continuation (8))

(b) Suppose change of basis matrix from α to β is $A_\alpha^\beta = \begin{pmatrix} 1 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$ check that $A_\beta^\alpha = \begin{pmatrix} 1 & -1 & -1 \\ 0 & -1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$

We know that $A_\beta^\alpha = [A_\alpha^\beta]^{-1}$ so we will check that $A_\alpha^\beta \cdot A_\beta^\alpha \underset{?}{=} I$

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 & -1 \\ 0 & -1 & 0 \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \checkmark.$$

(c) Find $[T]_\alpha^\alpha$, this is the matrix that given $[x]_\alpha$ will produce by left mult. $[Tx]_\alpha$ that is

$$[Tx]_\alpha = [T]_\alpha^\alpha \, [x]_\alpha$$

we know $[T]_\alpha^\beta$ that is

$$[Tx]_\beta = [T]_\alpha^\beta \, [x]_\alpha \quad \text{same input}$$

we also know how to change basis from β to α.

$$A_\beta^\alpha \, [Tx]_\beta = [Tx]_\alpha \; ; \quad \text{so}$$

$$\begin{tikzcd} \overset{[x]_\alpha}{X,\alpha} \arrow[r, "{[T]_\alpha^\beta}"] \arrow[dr, "{[T]_\alpha^\alpha}"'] & \overset{[Tx]_\beta}{X,\beta} \arrow[d, "A_\beta^\alpha"] \\ & X,\alpha \;\; [Tx]_\alpha \end{tikzcd}$$

$$[Tx]_\alpha = A_\beta^\alpha \, [Tx]_\beta = A_\beta^\alpha \left([T]_\alpha^\beta \, [x]_\alpha \right) = \left(A_\beta^\alpha [T]_\alpha^\beta \right) [x]_\alpha$$

So $\boxed{[T]_\alpha^\alpha = A_\beta^\alpha \, [T]_\alpha^\beta}$

Finally we compute the matrix mult.

$$[T]_\alpha^\alpha = \begin{pmatrix} 1 & -1 & -1 \\ 0 & -1 & 0 \\ 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & -3 & 2 \\ 0 & 1 & -1 \\ 2 & -1 & 0 \end{pmatrix} = \boxed{\begin{pmatrix} -1 & -3 & 3 \\ 0 & -1 & 1 \\ 2 & 0 & -1 \end{pmatrix} = [T]_\alpha^\alpha}$$

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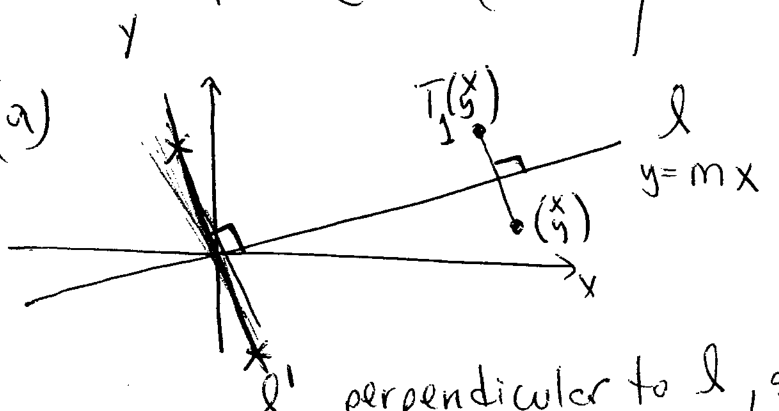
9) In $\mathbb{R}^2$ let l be the line $y = mx$ $m \neq 0$.

Find an expression for $T\begin{pmatrix}x\\y\end{pmatrix}$ where T is the linear transformation given by

(a) T_1 is reflection of $\mathbb{R}^2$ about line l.

(b) T_2 is projection on l along line perpendicular to l (orthogonal projection)

(a)



To find reflection about l of a point $\begin{pmatrix}x\\y\end{pmatrix}$ we draw the line perpendicular l' to l that goes through $\begin{pmatrix}x\\y\end{pmatrix}$ we "walk along this line" until we reach a point at equal distance to l than $\begin{pmatrix}x\\y\end{pmatrix}$ on other side

perpendicular to l, slope is $-\frac{1}{m}$

$y = \left(-\frac{1}{m}\right)x \Rightarrow my = -x$

You can try to use plain geometry to find coordinates of $T_1\begin{pmatrix}x\\y\end{pmatrix}$. Since we are told T_1 is a l. t.

If we can identify two linear independent vectors whose reflections are easy to calculate we can then use linear algebra to find T.

These preferred vectors will be one parallel to l, for example $\begin{pmatrix}1\\m\end{pmatrix}$, the other perpendicular eg $\begin{pmatrix}m\\-1\end{pmatrix}$

They form a basis of $\mathbb{R}^2$ $\beta = \left\{ \begin{pmatrix}1\\m\end{pmatrix}, \begin{pmatrix}m\\-1\end{pmatrix} \right\}$

(Continuation 9)

The vector $\begin{pmatrix}1\\m\end{pmatrix}$ is on ℓ, so its reflection is itself

$$T_1\begin{pmatrix}1\\m\end{pmatrix}=\begin{pmatrix}1\\m\end{pmatrix}$$

On the other hand, the reflection of $\begin{pmatrix}m\\-1\end{pmatrix}$ is obtained by changing the signs of both coordinates

$$T_1\begin{pmatrix}m\\-1\end{pmatrix}=\begin{pmatrix}-m\\1\end{pmatrix}$$

Given any vector $\begin{pmatrix}x\\y\end{pmatrix}\in\mathbb{R}^2$ we can find its coordinates on the $\mathcal{B}$-basis, $\mathcal{B}=\left\{\begin{pmatrix}1\\m\end{pmatrix},\begin{pmatrix}m\\-1\end{pmatrix}\right\}$.

$$(*)\quad \begin{pmatrix}x\\y\end{pmatrix}=a\begin{pmatrix}1\\m\end{pmatrix}+b\begin{pmatrix}m\\-1\end{pmatrix}=\begin{pmatrix}1&m\\m&-1\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix}$$

by solving the system of eqns for a,b in terms of x,y,m or equivalently, inverting $A=\begin{pmatrix}1&m\\m&-1\end{pmatrix}$ $\quad \det A=-1-m^2$

$$A^{-1}=\frac{1}{-m^2-1}\begin{pmatrix}-1&-m\\-m&1\end{pmatrix}=\frac{1}{m^2+1}\begin{pmatrix}1&m\\m&-1\end{pmatrix}$$

Multiplying out we get

$$(**)\quad \begin{pmatrix}a\\b\end{pmatrix}=A^{-1}\begin{pmatrix}x\\y\end{pmatrix}=\frac{1}{m^2+1}\begin{pmatrix}1&m\\m&-1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}\frac{x+my}{m^2+1}\\ \frac{mx-y}{m^2+1}\end{pmatrix}$$

T_1 is a linear transformation so by $(*)$

$$T_1\begin{pmatrix}x\\y\end{pmatrix}=aT_1\begin{pmatrix}1\\m\end{pmatrix}+bT_1\begin{pmatrix}m\\-1\end{pmatrix}=a\begin{pmatrix}1\\m\end{pmatrix}+b\begin{pmatrix}-m\\1\end{pmatrix}$$

$$=\begin{pmatrix}1&-m\\m&1\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix}=\begin{pmatrix}1&-m\\m&1\end{pmatrix}\begin{pmatrix}(x+my)/m^2+1\\(mx-y)/m^2+1\end{pmatrix}$$

$$=\begin{pmatrix}(x+my-m^2x+my)/m^2+1\\(mx+m^2y+mx-y)/m^2+1\end{pmatrix}=\begin{pmatrix}\frac{1-m^2}{m^2+1}x+\frac{2my}{m^2+1}\\ \frac{2mx}{m^2+1}+\frac{(m^2-1)}{m^2+1}y\end{pmatrix}$$

So $\boxed{T_1\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}\frac{1-m^2}{1+m^2}x+\frac{2m}{1+m^2}y\\[4pt]\frac{2m}{1+m^2}x+\frac{m^2-1}{m^2+1}y\end{pmatrix}}$

(b) We can use some basis $B=\left\{\begin{pmatrix}1\\m\end{pmatrix},\begin{pmatrix}m\\-1\end{pmatrix}\right\}$ since now the projection of $\begin{pmatrix}1\\m\end{pmatrix}$ is itself again but the projection of $\begin{pmatrix}m\\-1\end{pmatrix}$ is $\begin{pmatrix}0\\0\end{pmatrix}$,

$T_2\begin{pmatrix}1\\m\end{pmatrix}=\begin{pmatrix}1\\m\end{pmatrix}$, $T_2\begin{pmatrix}m\\-1\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$

Using (*) then

$$T_2\begin{pmatrix}x\\y\end{pmatrix}=a\,T_2\begin{pmatrix}1\\m\end{pmatrix}+b\,T_2\begin{pmatrix}m\\-1\end{pmatrix}=a\begin{pmatrix}1\\m\end{pmatrix}$$

$$=\left(\frac{x+my}{1+m^2}\right)\begin{pmatrix}1\\m\end{pmatrix}=\begin{pmatrix}\frac{x+my}{1+m^2}\\[4pt]\frac{mx+m^2y}{1+m^2}\end{pmatrix}$$

So $T_2\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}\frac{x+my}{1+m^2}\\[4pt]\frac{mx+m^2y}{1+m^2}\end{pmatrix}$

Now notice that $T_2\begin{pmatrix}x\\y\end{pmatrix}$ is half way between $\begin{pmatrix}x\\y\end{pmatrix}$ and its reflection $T_1\begin{pmatrix}x\\y\end{pmatrix}$, so it should be true that $\boxed{T_2\begin{pmatrix}x\\y\end{pmatrix}=\frac{\begin{pmatrix}x\\y\end{pmatrix}+T_1\begin{pmatrix}x\\y\end{pmatrix}}{2}}$ it works check it.