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Review 2 **Math 321: Linear Algebra** **Spring 2010**

This is the review for the final exam that will take place on Tuesday May 11th, 2010. The main topics are included in Chapters 3-6 of our book, specifically Sections 3.1-3.4, 4.4, 5.1-5.2, 6.1-6.2, and lightly on 6.5. Here is a list of the most important points, followed by some sample problems. When I say you should know how to calculate or decide something, I mean examples comparable to the ones you have been doing in your homework.

Elementary Matrix Operations

- **Elementary Matrices**
  - Must know the basic *elementary row operations*: interchange rows, multiply a row by an scalar, and add to a row the scalar multiple of another row.
  - Must know the corresponding *elementary matrices* constructed applying an elementary row operation to the identity matrix. These elementary matrices are invertible, and the inverse is an elementary matrix of the same type.
  - Must know that the basic row operations can be achieved by multiplying on the left a given matrix times the corresponding elementary matrix.
  - Must know that corresponding *elementary column operations* and their elementary matrices can be obtained taking transposes of the corresponding row matrices.
  - Must be able to use a sequence of elementary row operations to go from a given matrix A to its *row reduced echelon form* $\tilde{A}$. Must understand that in the language of matrices this is equivalent to successively multiplying on the left by the corresponding sequence of elementary matrices, and this process can be reversed.
- **Rank of a Matrix**
  - Must know what the *column space* and *row space* of a matrix are. If A is an $m \times n$ matrix, its column space is the span of its column vectors, and hence it is a subspace of $\mathbb{R}^m$ of dimension less than or equal to n. Similarly, its row space is the span of its row vectors, and hence it is a subspace of $\mathbb{R}^n$ of dimension less than or equal to m.
  - Must know that the *rank of a matrix* is the dimension of its column space. It coincides with the rank of the linear transformation associated to the matrix.
  - Must know that the rank of a matrix does not change when the matrix is multiplied on the right or on the left by an appropriate invertible matrix. In particular when multiplied by an elementary row or column matrix.
  - Must know that the $\text{rank}(A) = \text{rank}(A^t)$, therefore the dimension of the column space of A coincides with the dimension of its row space.
  - Must be able to calculate the rank of a matrix either by inspection of its column or row spaces, or by reducing to row reduced echelon form and counting the number of non-zero pivots.
  - Must be able to read from the row reduced echelon form $\tilde{A}$ of a matrix A its column space: columns of $\tilde{A}$ containing the pivots are linearly independent (they are standard basis vectors), other columns are linear combinations of those and the coefficients are the entries of the given column, corresponding columns of A are l.i. and the other columns are linear combinations of those with the same coefficients.
  - Must know that the null-space of a matrix does not change under row operations (it will change under column operations though). Must be able to read the null-space of A from its row reduced echelon form (this is instrumental for solving linear systems of equations).

- **Inverse of a Matrix**
  - Must know that an invertible matrix is a product of elementary operations that can be all chosen to be row operations, and the row operations can be discovered while reducing the square matrix to its row reduced echelon form (which will be the identity, if not, the matrix is not invertible).
  - Must be able to compute the inverse of a matrix A by reducing its augmented matrix $(A|I)$ to the row reduced echelon form $(I|A^{-1})$.

Solving Systems of linear Equations

- Must know that a system of m linear equations on n unknowns can be encoded very nicely using matrix notation: $Ax = b$, where the A is the $m \times n$ matrix whose entries are the coefficients in the equations, the unknown is a vector x in $\mathbb{R}^n$, and $b \in R^m$ is the vector whose entries are the right-hand-sides of the equations.
- When $b = 0$ vector the system is called *homogeneous*, otherwise is called *non-homogeneous*.
- Homogeneous systems always have the zero solution: $x = 0$ vector. They may have multiple solutions. *The set of solutions of a homogeneous system is the nullspace of the linear transformation induced by the matrix A.* The dimension of the solution subspace is $n - rank(A)$. In particular if the homogeneous system has more variables than equations ($m < n$) it will have non-trivial solutions ($rankA \leq m$).
- Must know how to find the solution subspace of a homogeneous system of linear equations, that is must know how to find the nullspace of the matrix A whose entries are the coefficients of the equations.
- A non-homogeneous system (when $b \neq 0$ vector), may or may not have solutions. If it does, then the set of solutions can be found by identifying a particular solution x_P to the system and identifying all the solutions x_H of the corresponding homogeneous system, all solutions to the system have the form $x_0 = x_P + x_H$.
- Must know that by reducing the augmented matrix $(A|b)$ to its row reduced echelon form one can decide whether the system has solutions or not. If it does, one can read the particular solution and the solutions to the homogeneous system from it.

Determinants

- Must know how to compute determinants of 1×1 and 2×2. Must know how to calculate determinants of 3×3 or $n \times n$ matrices using appropriate minors (matrices found by deleting one row and one column). It is useful to know that the determinant of an upper triangular matrix is the product of its diagonal entries.
- Important properties of determinants that must be known:
  - $\det(AB) = \det(A)\det(B)$.
  - An $n \times n$ matrix A is invertible if and only if $\det(A) \neq 0$. In that case $\det(A^{-1}) = (\det(A))^{-1}$.
  - Similar matrices have the same determinant.
  - For any $n \times n$ matrix A, $\det(A) = \det(A^t)$.

Eigenvalues and Eigenvectors

- **Eigenvectors, eigenvalues, eigenspaces**
  - Must know the basic equation: $Tv = \lambda v$, linking/defining an *eigenvector* v of a linear operator $T : V \to V$, and its corresponding *eigenvalue* λ. Recall that the equation must hold for a non-zero vector v. *Eigenvectors are preferred directions for the transformation.*

- Must know the basic equation: $Ax = \lambda x$, linking/defining an *eigenvector* $x \neq 0$ of an $n \times n$ matrix A, and its corresponding *eigenvalue* λ.
- If T is a linear operator from n-dimensional vector space V into itself, and β is a basis of V, and A is the matrix representation of T in the basis β, $A = [T]_\beta$, then:
  * λ is an eigenvalue of T if and only if it is an eigenvalue of A.
  * A non-zero vector v is an eigenvector of T if and only if $x = [v]_\beta$ is an eigenvector of A (with the same eigenvalue).

  To calculate eigenvectors and eigenvalues of a linear transformation suffices to calculate eigenvectors and eigenvalues of its matrix representation in a basis, and use the eigenvectors of the matrix as coefficients to reconstruct with the given basis the eigenvectors in the original vector space.
- The *eigenspace* associated to the eigenvalue λ of the linear transformation T is the collection of all eigenvectors of T with eigenvalue λ, including the zero vector. Similarly for matrices.
- Must know how to calculate eigenvalues, eigenvectors and eigenspaces of an $n \times n$ matrix A.
  * λ is an eigenvalue of A if and only if it is a root of the nth-degree *characteristic polynomial of* A, $p_A(\lambda) := \det(A - \lambda I) = 0$. If the characteristic polynomial splits in $\mathbb{R}$ (it always does in $\mathbb{C}$ by the fundamental theorem of algebra) that is:

    $$p_A(\lambda) = c(\lambda - \lambda_1)^{n_1}(\lambda - \lambda_2)^{n_2} \ldots (\lambda - \lambda_k)^{n_k},$$

    then the eigenvalues of A are precisely the different numbers $\lambda_1, \lambda_2, ..., \lambda_k$, and the power n_j is called the *algebraic multiplicity* of the eigenvalue λ_j. Note that $n_1 + n_2 + \ldots n_k = n$, the degree of p_A.
  * The *eigenspace* E_λ associated to the eigenvalue λ of A is the nullspace of $(A - \lambda I)$.
  * The dimension of the eigenspace E_λ is at least one, at most the algebraic multiplicity of λ.
  * Eigenspaces *don't talk to each other*, that is their intersection only contains the zero vector.
  * If the characteristic polynomial of an $n \times n$ matrix A splits and all the roots are simple (that is their algebraic multiplicity is one), then there are n different eigenvalues, each one of them contributes an eigenvector, and they are necessarily linearly independent.

- **Diagonalizable linear operators** A linear operator T from n-dimensional vector space V into itself is *diagonalizable* if one can find a basis β of eigenvectors, and necessarily the matrix representation of T in that basis is a diagonal matrix with corresponding eigenvalues on the diagonal entries.

- **Diagonalizable matrices** An $n \times n$ matrix A is *diagonalizable* if it is similar to a diagonal matrix D, that is there is an invertible matrix Q such that $A = QDQ^{-1}$. This happens if and only if one can find a basis β of eigenvectors of A, in that case Q is the matrix whose columns are the vectors in the basis β (this is the change of basis matrix from β to standard basis), and the matrix D is the diagonal matrix with corresponding eigenvalues on the diagonal entries.
  - Diagonal matrices allow for trivial computation of powers, polynomials or exponential of matrices, just do the operation to each diagonal entry. Using this principle one can check that a diagonal matrix satisfies the equation $p(D) = 0$ where $p(\lambda)$ is its characteristic polynomial. This is true for all $n \times n$ matrices A, and is the famous Cayley-Hamilton theorem.
  - Diagonalizable matrices are the next best thing: $A^k = QD^kQ^{-1}$, $e^A = Qe^DQ^{-1}$.
  - Diagonal matrices always commute. Two matrices that share a basis of eigenvectors will also commute because in that case they share the similarity matrix Q.
  - Not all matrices are diagonalizable, it is good to remember a simple 2×2 example: $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

    In this case the only eigenvalue is 0, it has algebraic multiplicity 2, but the dimension of the eigenspace corresponding to eigenvalue zero is just 1.

- **Applications**
  - Solving *systems of linear differential equations*, in the case the matrix of coefficients is diagonalizable, then one can decouple the system.
  - How do internet search engines work? Most algorithms use an eigenvector with the largest eigenvalue of a matrix based on the incidence matrix determined by the oriented graph whose nodes are the sites interesting for a given search and the arrows are the hyperlinks.

Orthogonalization

- **Inner product vector spaces**
  - Must know definition of an *inner product vector space*, and must be able to decide when a candidate for an inner product is in fact one.
  - Must know the standard inner product in $\mathbb{R}^n$ and in $\mathbb{C}^n$.
  - Must know the *induced norm* by a given inner product: $\|x\| := \sqrt{\langle x, x\rangle}$.
  - Must know that in any inner product vector space the Cauchy-Schwarz inequality holds: $|\langle x, y\rangle| \leq \|x\|\,\|y\|$. In the case of $\mathbb{R}^2$ with the standard inner product this is the geometric fact that given two non-zero vectors $x, y \in \mathbb{R}^2$, the cosine of the angle θ between them is determined by $\cos\theta = \frac{\langle x,y\rangle}{\|x\|\,\|y\|}$, and that $|\cos\theta| \leq 1$.
- **Orthogonality**
  - Must know that two vectors x, y in an inner product vector space V are *orthogonal* if and only if their inner product vanishes: $\langle x, y\rangle = 0$
  - Must know that a *set of vectors is orthogonal* if every pair is orthogonal, and that *orthogonal sets are linearly independent.*
  - Must know that every linearly independent finite set can be orthogonalized via the *Gram-Schmidt orthogonalization process*, in a way that the span of the first k vectors of the linearly independent set coincides with the span of the first k orthogonal vectors produced by the process.
  - Must be able to carry on the G-S process. Given a linearly independent vectors $\{w_1, w_2, \ldots, w_n\}$, then can construct recursively the orthogonal vectors $\{v_1, v_2, \ldots, v_n\}$ as follows:

$$v_1 := w_1, \qquad v_j := w_j - \frac{\langle w_j, v_1\rangle}{\|v_1\|^2}v_1 - \cdots - \frac{\langle w_j, v_{j-1}\rangle}{\|v_{j-1}\|^2}v_{j-1}$$

- **Orthonormal bases**
  - An *orthonormal basis* is a basis of vectors that are orthogonal and have norm one, that can be summarized by the equation: $\langle x_i, x_j\rangle = \delta_{i,j} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$
  - Every basis can be orthogonalized via the Gram-Schmidt Process. And it can then be normalized simply by dividing a non-zero vector by its norm: $x/\|x\|$.
  - Given an orthonormal basis $\beta = \{x_1, \ldots, x_n\}$ of an n-dimensional inner product vector space V, and a vector $x \in V$, then one can find the coefficients of x in the basis β simply calculating the inner products of x and corresponding vectors in the basis, that is:

$$x = \langle x, x_1\rangle x_1 + \langle x, x_2\rangle x_2 + \cdots + \langle x, x_n\rangle x_n.$$

- **Orthogonal/Unitary and Adjoint Matrices**
  - An $n \times n$ matrix A is *orthogonal* ($\mathbb{R}$)*/unitary* ($\mathbb{C}$) if and only if its columns form an orthonormal basis.
  - Given an $n \times m$ matrix with real or complex entries A, then its *adjoint* A^* is its complex conjugate transpose $\overline{A^t}$ (if the entries are real then it is simply the transpose).
  - *The adjoint travels through the inner product*, more precisely:

    $$\langle Ax, y\rangle_{\mathbb{R}^n} = \langle x, A^* y\rangle_{\mathbb{R}^m}.$$

    In general one uses this property to define the adjoint of a linear transformation on an inner product vector space.
  - A is orthogonal/unitary if and only if its inverse equals its adjoint: $A^{-1} = A^*$, that is $AA^* = A^*A = I$. Notice that ijth-entry of A^*A is the inner product of the jth and ith columns.
  - Orthogonal matrices preserve inner products and the induced norm, in other words the linear transformation associated to them are *isometries*, that is

    $$\langle Ax, Ay\rangle = \langle x, y\rangle, \qquad \|Ax\| = \|x\|.$$

- **Orthogonal projections**
  - Given an inner product vector space V, a finite dimensional subspace W, and a vector $x \in V$ there is a unique vector on W, the *orthogonal projection of x onto W*, denoted by $\mathrm{Proj}_W x$, such that $x - \mathrm{Proj}_W x$ is orthogonal to $\mathrm{Proj}_W x$.
  - *The orthogonal projection $Proj_W x$ minimizes the distance from x to W*, in the sense that

    $$\|x - \mathrm{Proj}_W x\| \leq \|x - w\|, \qquad \text{for all} \quad w \in W.$$

  - Let $\beta = \{x_1, \ldots, x_n\}$ be an orthonormal basis of the n-dimensional subspace W, then

    $$\mathrm{Proj}_W x = \langle x, x_1\rangle x_1 + \langle x, x_2\rangle x_2 + \cdots + \langle x, x_n\rangle x_n.$$

  - In this language, the Gram-Schmidt process can be reprased as: $v_j = w_j - \mathrm{Proj}_{V_{j-1}} w_j$ where V_{j-1} is the span of the orthonormalized vectors $\left\{\frac{v_1}{\|v_1\|}, \frac{v_2}{\|v_2\|}, \ldots, \frac{v_{j-1}}{\|v_{j-1}\|}\right\}$, which coincides with the span of $\{w_1, w_2, \ldots, w_{j-1}\}$.
- **Application** *Least squares approximations.*

Least Squares Approximation and the following we discussed briefly the last week of classes, and I have included them for completeness. They will not be evaluated in the final exam. Nor will I include applications to solving systems of linear differential equations or to do searches in internet.

Matrix Zoo

The matrix zoo includes matrices that we have already encountered, such as: orthogonal/unitary matrices, similar matrices, symmetric matrices, as well as other species like self-adjoint and normal matrices that we describe here.

- **Self-adjoint Matrices**
  - An $n \times n$ matrix A is *self-adjoint* if it equals to its adjoint: $A = A^*$.
  - At the level of the inner product what it means is that $\langle Ax, y\rangle = \langle x, Ay\rangle$.
  - If an $n \times n$ matrix A is self-adjoint then all its eigenvalues are real.

- *An $n \times n$ matrix A over $\mathbb{R}$ is self-adjoint if and only if there exists an orthonormal basis of eigenvectors.*

- **Normal Matrices**
  - An $n \times n$ matrix A is *normal* if $AA^* = A^*A$.
  - At the level of the inner product this means that $\langle Ax, Ay\rangle = \langle A^*x, A^*y\rangle$.
  - Orthogonal/unitary matrices are normal ($AA^* = A^*A = I$).
  - Self-adjoint matrices are normal ($AA^* = A^*A = A^2$).
  - *An $n \times n$ matrix A over $\mathbb{C}$ is normal if and only if there exists a basis of eigenvectors, that is if A is diagonalizable.*

What's next? Further reading

You now have the background to continue reading on your own. important topics that you can find in our book:

- Dual Spaces (Section 2.6)
- Spectral theorem (Section 6.6)
- Singular Value Decomposition (Section 6.7)
- Quadratic and bilinear forms (Section 6.8)
- Jordan canonical forms (Chapter 7- Sections 7.1-7.2)

After this class, depending on your interest you can go to Math 514/464 (Applied Matrix Theory) where you will revisit many of the concepts but with a much more numerical/applied point of view. You may be interested in applications to analysis/geometry, first make sure you do Math 401 (advanced calculus), then you will be ready for Math 402 (multivariable calculus). If you are interested in differential equations and you have already done basic ODE (Math 316) and PDEs (Math 312), you should now be in good shape to take the next level of ODEs (Math 462/512), and PDEs (Math 463/513).

Sample problems

1. Let A and B be invertible $n \times n$ matrices. Show that there is a sequence of elementary row operations which transform A to B. (Hint: use that there is a sequence of row operations which transforms A to the identity matrix).

2. For the following matrix use elementary row operations to find its inverse,

$$A = \begin{pmatrix} 1 & -1 & 2 \\ 3 & 2 & 1 \\ 0 & 1 & -2 \end{pmatrix}.$$

3. Let the reduced row echelon form of A be

$$\begin{pmatrix} 1 & 0 & 2 & 0 & -2 \\ 0 & 1 & -5 & 0 & -3 \\ 0 & 0 & 0 & 1 & 6 \end{pmatrix}.$$

Determine A if the first, second, and fourth columns of A are

$$\begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}, \quad \text{and} \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}.$$

4. Find all solutions the following system of linear equations

$$\begin{aligned} 2x_1 - 2x_2 - x_3 + 6x_4 - 2x_5 &= 1 \\ x_1 - x_2 + x_3 + 2x_4 - x_5 &= 2 \\ 4x_1 - 4x_2 + 5x_3 + 7x_4 - x_5 &= 6 \end{aligned}$$

Identify a particular solution, and all the solutions of the corresponding homogeneous system.

5. Find the characteristic polynomials of the following $n \times n$ matrices:

$$\begin{pmatrix} a_1 & a_2 & a_3 & \dots & a_{n-1} & a_n \\ 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 1 & 0 \end{pmatrix}.$$

6. Here we are working with scalars the complex numbers $\mathbb{C}$. Let θ be a real number, and let A be the 2×2 rotation matrix

$$A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

- Show that A has eigenvalues $e^{i\theta}$ and $e^{-i\theta}$ (you may use Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$). What are the corresponding eigenvectors?
- Write $A = QDQ^{-1}$ for some invertible matrix Q and diagonal matrix D (note that Q and D have complex entries, also there are several possible answers to this question, you only need to give one of them).
- Let $n \geq 1$ be an integer. Prove that

$$A^n = \begin{pmatrix} \cos n\theta & -\sin n\theta \\ \sin n\theta & \cos n\theta \end{pmatrix}.$$

  (You may find the formulae $e^{in\theta} = \cos n\theta + i\sin n\theta$ and $(e^{-i\theta})^n = e^{-in\theta} = \cos n\theta - i\sin n\theta$ useful).
- Can you give a geometric itnerpretation of item (c)?

7. Show that if $n \times n$ matrices A and B are similar, that is there exists an invertible $n \times n$ matrix Q such that $A = Q^{-1}BQ$, then they have the same characteristic polynomial.

8. Let A be the following 3×3 matrix,

$$\begin{pmatrix} 3 & 0 & 0 \\ 1 & -1 & 0 \\ 2 & 1 & 1 \end{pmatrix}.$$

- Is A invertible? If so, find A^{-1}.
- Verify that $A^3 - 3A^2 - A + 3I = 0$. Is there anything special about this polynomial $p(\lambda) = \lambda^3 - 3\lambda^2 - \lambda + 3 = (\lambda^2 - 1)(\lambda - 3)$?

9. Consider the vector space $\mathcal{P}_3$ of polynomials of degree at most 3. Let $f, g \in \mathcal{P}_3$, define

$$\langle f, g \rangle = \int_1^1 f(t)g(t)\, dt.$$

(a) Check that this defined an inner product.

(b) Use the Gram-Schmidt process to orthonormalize the standard basis polynomials: 1, x, x^2, x^3.

(c) Given a polynomial $p(x) = 2 - 3x^2 + x^3$, find its coefficients in the orthonormal basis constructed in item (b), verify that you can reconstruct $p(x)$ with those coefficients and the corresponding basis.

10. Find the orthogonal projection of the vector $(1, 2, 3, 4, 5)$ onto the subspace of $\mathbb{R}^5$ spanned by the vectors $\{(1, 0, 1, 0, 1), (1, 1, 0, 0, 0), (0, 1, 0, 1, 0)\}$.

11. Show that an $n \times n$ matrix that is both upper triangular and unitary must be a diagonal matrix.

12. Characterize all orthogonal 2×2 matrices (scalars in $\mathbb{R}$).