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Review 1 **Math 321: Linear Algebra** **Spring 2010**

This is a review for Midterm 1 that will be on Thursday March 11th, 2010. The main topics are included in Chapters 1-2 of our book, specifically Sections 1.1-1.6 and 2.1-2.5. Here is a list of the most important points, followed by some sample problems. When I say you should know how to calculate or decide something, I mean examples comparable to the ones you have been doing in your homework.

- **Vector Spaces**
  - Should know the following concepts, and be able to decide whether a given object is one of them:
    - What is a *vector space*, and a *subspace*?
    - What is a *linearly independent* set of vectors?
    - What is a *spanning* set of vectors?
    - What is a *basis* for a vector space? A linearly independent and spanning set of vectors.
  - Must know that:
    - A set of linearly independent vectors can always be completed to a basis.
    - A set of spanning vectors can always be purged or trimmed down to a basis.
  - Should be able to describe/identify subspaces by:
    - putting constraints on elements of a vector space,
    - identifying a set of spanning vectors.
  - Should know vector spaces can be *finite dimensional*, and in that case calculate the *dimension* (number of elements in a basis).
  - Must know basic vector spaces such as $\mathbb{R}^n$, $M_{m\times n}$ ($m \times n$ matrices), $\mathcal{P}_n$ (polynomials of degree less than or equal to n). Must know standard bases in those spaces.
  - Given a basis $\beta = \{x_1, x_2, \ldots, x_n\}$ of the n-dimensional vector space X, and a vector $x \in X$, must be able to find the *unique coefficients* $(a_1, a_2, \ldots, a_n)$ such that x is a linear combination of the elements of the basis, $x = a_1x_1 + a_2x_2 + \cdots + a_nx_n$. We arrange the coefficients of x in the basis β in a column vector that we denote $[x]_\beta$. Given the coefficients and the basis we can recover the vector x by writing the appropriate linear combination.

- **Matrices**
  - Must know how to add $n \times m$ matrices, and how to multiply by a scalar an $n \times m$ matrix (entry by entry).
  - Must know that given an $n \times m$ matrix A, its *transpose* is the $m \times n$ matrix A^t found by interchanging rows and columns. A matrix A is *symmetric* if $A = A^t$.
  - Must know how to multiply A, an $n\times m$ matrix, times B, an $m\times k$ matrix, to obtain $A\,B$ an $n\times k$ matrix. Must know that matrix multiplication is *associative* but not *commutative* (order matters). The following formula may be useful: $(AB)^t = B^tA^t$ (check that dimensions are correct).
  - Must know that A is an *invertible* $n \times n$ *matrix* iff there is another $n \times n$ matrix A^{-1} (which is unique) such that $A\,A^{-1} = A^{-1}\,A = I_n$, where I_n is the identity matrix (i.e. the matrix that has 1 in the diagonal entries, 0 everywhere else). Not all matrices are invertible, when $n = 2$ then there is a convenient formula, provided $\det A = ad - bc \neq 0$ (if $\det A = 0$ then A is not invertible),

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad A^{-1} = \frac{1}{\det A}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

- **Linear Transformations**
  - Must know what is a *linear transformation* from a vector space X into a vector space Y, and how to decide if a map between vector spaces is linear or not.
  - Must know what are and how to calculate the associated subspaces to a linear transformation T: *Nullspace* $N(T)$ and *Range* $R(T)$.
  - Must know what are *one-to-one* (or injective), *onto* (or surjective), *invertible* or *bijective* (1-1 and onto) linear transformations, and how to decide whether a given linear transformation is any of the above.
    * T is one-to-one iff the nullspace is trivial $N(T) = \{0\}$ iff $\dim(N(T)) = 0$.
    * T is onto iff its range is the whole target space Y.
  - Must know that two vector spaces X and Y are *isomorphic*, $X \sim Y$, iff there is an invertible linear transformation that maps one space into the other (a so-called *isomorphism*). Must be able to decide when two vector spaces are isomorphic.
  - We will concentrate on linear transformations T defined on finite dimensional vector space X. Must know the *dimension theorem* and its consequences: $\dim(X) = \dim(N(T)) + \dim(R(T))$.
    * T is onto iff $\dim(Y) = n = \dim(X)$
    * T is one-to-one iff T is onto iff T is bijective or invertible.
    * In particular if $\dim(X) = \dim(Y) = \dim(Z) = n$, and $T : X \to Y$, $U : Y \to Z$, then $U \circ T$ is invertible if and only if U and T are invertible. Necessarily in that case $(U \circ T)^{-1} = T^{-1} \circ U^{-1}$.
  - Must know that *all the information about a linear transformation is contained in the images under the transformation of the elements of a basis of the domain.*
  - Must know that if T is an invertible linear transformation from X to Y, and $\{x_1, x_2, \ldots, x_n\}$ is a basis of X, then $\{Tx_1, Tx_2, \ldots, Tx_n\}$ is a basis of Y. In all cases (even if T is not invertible), the vectors $\{Tx_1, Tx_2, \ldots, Tx_n\}$ span $R(T)$, the range of T, so that $\dim(R(T)) \leq n$.
  - Given a basis $\beta = \{x_1, x_2, \ldots, x_n\}$ of the n-dimensional vector space X, and a basis $\gamma = \{y_1, y_2, \ldots, y_m\}$ of the m-dimensional vector space X, and $T : X \to Y$ a linear transformation, must know how to construct the $m \times n$ *matrix* $[T]_\beta^\gamma$ *of the linear transformation associated to the given ordered bases.* Here is the recipe:
    1. Find the images under T of the elements of the basis β, namely Tx_j for $j = 1, \ldots, n$.
    2. Find the coefficients of each $Tx_j \in Y$ in the basis γ, namely the column vectors $[Tx_j]_\gamma$ for $j = 1, \ldots, n$.
    3. The column vector $[Tx_j]_\gamma$ is the jth-column of the matrix $[T]_\beta^\gamma$.
  - Canonical example: an $m \times n$ matrix A defines a linear transformation $L_A : \mathbb{R}^n$ into $\mathbb{R}^m$ by multiplying the vector $x \in \mathbb{R}^n$ on the left: $L_A x = Ax$.

    Let α be the standard basis in $\mathbb{R}^n$ and σ the standard basis in $\mathbb{R}^m$, then:
    * A is the matrix representation of $L_A : \mathbb{R}^n \to \mathbb{R}^m$ in the standard bases, $A = [L_A]_\alpha^\sigma$.
    * If B is a $k \times m$ matrix then $L_B : \mathbb{R}^m \to \mathbb{R}^k$, and the composition of $L_B L_A$ corresponds to matrix multiplication BA, that is $L_B L_A = L_{BA}$, or in other words the matrix representation of $L_B L_A : \mathbb{R}^n \to \mathbb{R}^k$ in the standard bases of $\mathbb{R}^n$ and $\mathbb{R}^k$ is BA.
    * The $n \times n$ matrix A is invertible iff the corresponding linear transformation L_A is invertible, moreover $(L_A)^{-1} = L_{A^{-1}}$.
    * We learn that A, B are invertible $n \times n$ matrices iff AB is an invertible $n \times n$ matrix, moreover. $(AB)^{-1} = B^{-1} A^{-1}$.
  - Must understand how all information about $T : X \to Y$ is encoded in the matrix $[T]_\beta^\gamma$. The matrix links in a very precise way the coefficients of $x \in X$ in the basis β to the coefficients of its image $Tx \in Y$ in the basis γ, more precisely,

$$[Tx]_\gamma = [T]_\beta^\gamma \, [x]_\beta.$$

In particular must know the following algebraic rules, paralleling the ones for the canonical example, connecting compositions and inverses:

* If $U : Y \to Z$ is another linear transformation, and $\alpha = \{z_1, \ldots, z_k\}$ is a basis of Z then their composition $U \circ T : X \to Z$, and its matrix representation from basis β to basis α is the product of the corresponding matrix representation of U and T, more precisely

$$[U \circ T]_\beta^\alpha = [U]_\gamma^\alpha \, [T]_\beta^\gamma.$$

* Assume $m = n$ then T is invertible iff the matrix representation is an invertible matrix, moreover,

$$[T^{-1}]_\gamma^\beta = ([T]_\beta^\gamma)^{-1}.$$

- **Change of bases**
  - Given an n-dimensional vector space X and two bases $\beta = \{x_1, x_2, \ldots, x_n\}$ and $\gamma = \{y_1, y_2, \ldots, y_n\}$ in X, the goal is to find a matrix that links the coefficients of a vector x in the basis β to the coefficients of the same x in the basis γ, the so-called *change of bases matrix* A_β^γ,

$$[x]_\gamma = A_\beta^\gamma [x]_\beta.$$

* The matrix is nothing more than the matrix representation of the identity map $I : X \to X$ ($Ix = x$) from the basis β to the basis γ,

$$A_\beta^\gamma = [I]_\beta^\gamma.$$

* Since the identity map is an invertible linear transformation, and $I^{-1} = I$, then the matrix A_β^γ is invertible, and its inverse is the change of bases matrix A_γ^β from the basis γ to the basis β,

$$[A_\beta^\gamma]^{-1} = [I]_\gamma^\beta = A_\gamma^\beta.$$

  - Must know how to compute change of basis matrices, here is the recipe:
    1. Find the coefficients of each element x_j of the basis β in the basis γ, namely the column vectors $[x_j]_\gamma$ for $j = 1, \ldots, n$.
    2. The column vector $[x_j]_\gamma$ is the jth-column of the change of basis matrix A_β^γ.
  - Change of basis matrices allow you to connect different matrix representations of the same linear transformation $T : X \to Y$. More precisely, if β and α are bases of X, and γ and σ are bases of Y, then

$$[T]_\alpha^\gamma = A_\sigma^\gamma \, [T]_\beta^\sigma \, A_\alpha^\beta.$$

  - Note that is $T : X \to X$ and α and β are two bases in n-dimensional vector space X, $[T]_\beta = [T]_\beta^\beta$, that is the matrix representation from basis β to basis β then, there exists an invertible $n \times n$ matrix Q such that,

$$[T]_\alpha = Q^{-1} \, [T]_\beta \, Q.$$

    Who is Q? Nothing other than the change of bases matrix from α to β.
  - In particular if A is an $n \times n$ matrix, and β the standard basis in $\mathbb{R}^n$, and α another basis on $\mathbb{R}^n$, then

$$[L_A]_\alpha = Q^{-1} \, A \, Q, \qquad \text{where} \quad Q = A_\alpha^\beta.$$

    (in such case we say the matrices A and B ($= [L_A]_\alpha$) are *similar*).

Sample problems

Vector Spaces

1. Define what is a vector space over $\mathbb{R}$.

(a) Show that the following subset of $\mathbb{R}^3$ is a subspace (we are considering $\mathbb{R}^3$ with the usual addition and scalar multiplication),
$$V = \{(x, y, z) \in \mathbb{R}^3 : x = 3y, y = x + z\}.$$

(b) Explain why the following subset of $\mathbb{R}^3$ is not a subspace (we are considering $\mathbb{R}^3$ with the usual addition and scalar multiplication),
$$V = \{(x, y, z) \in \mathbb{R}^3 : x = y + z - 1\}.$$

2. Consider the following subspace of $\mathcal{P}_3(\mathbb{R})$ (polynomials of degree less than or equal to 3),
$$X = \{a + bx + cx^2 + dx^3 : a + b = c\}.$$
Find the dimension of X and exhibit a basis of X.

3. Decide whether the following statements are true or false, give a brief explanation supporting your answer. In the following, X is a vector space and $\dim(X) = n \geq 1$.

(a) You can find a collection of $n + 2$ linearly independent vectors in X.

(b) You can find a collection of $n + 1$ vectors that span X.

(c) If V is a subspace of X, then $\dim(V) \leq n$.

(d) If V is a non-trivial subspace of X, and $\beta = \{x_1, x_2, \ldots, x_n\}$ is a basis of X, then you can select vectors in β to obtain a basis for V.

4. Show that two vectors $u, v \in X$ are linearly independent if and only if the vectors $u + v$ and $v - u$ are linearly independent.

Linear Transformations

5. Show that the mapping $T : \mathcal{P}_2(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ is a linear transformation, where T is defined for each polynomial $p(x) = a + bx + cx^2$, to be
$$T(p(x)) = \begin{pmatrix} 2p''(0) & p'(0) + p(0) \\ 0 & p(1) \end{pmatrix}.$$
Is T one-to-one? Is T onto? Find the nullspace and the range of T.

6. Let $T : \mathbb{R}^4 \to \mathbb{R}^4$ be a linear transformation defined by
$$T(x_1, x_2, x_3, x_4) = (x_4, x_1, x_2, x_3).$$

(a) Show that T is one-to-one and onto, hence invertible. Find the inverse transformation.

(b) Let α be the standard basis in $\mathbb{R}^4$, and let β be the following basis in $\mathbb{R}^4$,
$$\beta = \left\{ \begin{pmatrix} 1 \\ 2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

Find the matrix representations $[T]_\alpha^\alpha$, $[T^{-1}]_\alpha^\alpha$, and $[T]_\beta^\alpha$.

7. Find all linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$ such that $T(1,2,0) = (3,-1)$ and $T(4,-1,1) = (2,2)$. Is such T one-to one? Is it onto?

8. Let $T : X \to X$ be a linear transformation, $\dim(X) = 3$, assume $\alpha = \{x_1, x_2, x_3\}$ and $\beta = \{y_1, y_2, y_3\}$ are two bases on X. Let the matrix representation

$$[T]_\beta^\beta = \begin{pmatrix} 1 & -3 & 2 \\ 0 & 1 & -1 \\ 2 & -1 & 0 \end{pmatrix}.$$

(a) If $x = 2x_1 - x_2 + x_3$ what is Tx?

(b) Suppose that the change of basis matrix from α to β is the following matrix,

$$A_\alpha^\beta = \begin{pmatrix} 1 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$

verify that the change of basis matrix from β to α is

$$A_\beta^\alpha = \begin{pmatrix} 1 & -1 & -1 \\ 0 & -1 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$

(c) Find the matrix representation T in the basis α, that is $[T]_\alpha^\alpha$.

9. (Exercise 7 in Section 2.5) In $\mathbb{R}^2$ let ℓ be the line through the origin $y = mx$, with $m \neq 0$. Find an expression for $T(x,y)$ where,

(a) T is the reflection of $\mathbb{R}^2$ about the line ℓ.

(b) T is the projection on ℓ along the line perpendicular to ℓ (orthogonal projection).

0.1 Applications: reading assignments for Spring Break

- Lagrange Interpolation Formula, and interpolating polynomials (p. 51-53).
- Incidence Matrices (p.94-96).
- Homogeneous linear differential equations (Section 2.7, p. 127-140).