Math 313 (Complex Variables for Engineering): **course goals**

Proofs of main theorems should be given in full details or sketched, time permitting. Main ideas and methods of proofs can also be illustrated on enlightening examples. "ε-δ" proofs can be demonstrated on simple examples, including $\lim_{z\to z_0} |z| = |z_0|$, $\lim_{z\to\infty} \frac{1}{z} = 0$, $\lim_{z\to 0} \frac{1}{z} = \infty$.

Students should acquire knowledge and skills listed below. The topics and examples can be split between lectures and homework.

Mathematical maturity:

◇ write solutions in complete and mathematically correct sentences

◇ logically justify main steps of solutions by referring to appropriate results

◇ go beyond solving template problems; understand and apply definitions and theorems to problems that have not been solved in class or textbook

◇ comprehend abstract solutions not involving numbers or specific functions

Complex numbers:

◇ apply connections between complex numbers, points, and vectors in the plane

◇ find the modulus of a complex number

◇ perform operations on complex numbers in Cartesian and polar coordinates

◇ know Euler and De Moivre formulas and how to use them to derive trigonometric formulas

◇ know how to find the n-th roots of a complex number and how to construct them geometrically using the n-th roots of unity

◇ apply triangle and reverse triangle inequalities

◇ identify properties of subsets of $\mathbb{C}$: open, closed, bounded, (simply) connected, domain; interior, exterior, boundary of a set

◇ represent sets of complex numbers graphically; describe them (1) verbally by referring to geometric properties of the sets and (2) using set notations $\{z \in \mathbb{C} : z \text{ satisfies property } A\}$

◇ know the definition of the extended complex plane (to be applied in problems on limits and Möbius transformations)

Functions of a complex variable:

◇ find the domain and range of a function

◇ determine whether a function is one-to-one and onto

- ◇ represent a function of (x, y) variables as a function of $(z, \bar{z})$ and of (r, θ) variables
- ◇ find the real and imaginary parts of a function of $z = x + iy$; in particular, of functions z^2, $1/z$, e^z, $\sin(z)$, $\cos(z)$
- ◇ find a limit and verify continuity by applying properties of limits
- ◇ know the definition and properties of complex differentiable functions
- ◇ derive Cauchy-Riemann equations in Cartesian coordinates (lecture), in polar coordinates (homework), and also in $(z, \bar{z})$ coordinates (homework)
- ◇ verify complex (non)differentiability by (1) definition, (2) reducing the problem to functions whose (non)differentiability is well known (and arguing by contradiction), and (3) via Cauchy-Riemann equations
- ◇ know classical examples of nowhere differentiable functions $\bar{z}$, $|z|$, $\mathrm{Re}(z)$, $\mathrm{Im}(z)$
- ◇ compare real differentiability and complex differentiability (the level of the examples is to be adjusted to the background of the students)
- ◇ geometric interpretation of the derivative (homework)
- ◇ know the concepts and classical examples of a multiple valued function (argument, logarithm, and complex/irrational power) and of a continuous/analytic branch of such function

Analytic and harmonic functions:

- ◇ understand definition of analyticity and difference between complex differentiability and analyticity
- ◇ verify (non)analyticity and (non)harmonicity of a function
- ◇ apply (in particular, in boundary value problems) the facts that the real and imaginary parts of an analytic function are harmonic functions
- ◇ find analytic functions with prescribed real or imaginary part; write the respective functions in terms of z treated as a single unit
- ◇ prove facts like "if the real part of an analytic function is a constant in a domain, then the function is a constant itself" by applying the Cauchy-Riemann equations and also by the open mapping theorem
- ◇ know definitions and properties of elementary functions in the complex plane: polynomial, rational, exponential, logarithmic, complex power, trigonometric, and inverse trigonometric functions
- ◇ know that (anti)derivatives of analytic functions in (simply connected) domains are analytic and apply these properties in integration
- ◇ apply Cauchy's estimates, Liouville's theorem, maximum/minimum modulus principle in simple proofs

Series representations for analytic functions:

- ◇ apply tests for convergence of numeric series
- ◇ recognize a power series; find its radius and interval of convergence; get familiar with uniform convergence of a power series; know termwise differentiation and integration of a power series
- ◇ know the definitions of Taylor and Laurent series and Abel's theorem regarding their regions of convergence
- ◇ find the region of convergence and analyticity of a Taylor series and of a Laurent series
- ◇ find expansion of an analytic function into Laurent series by reducing the problem to elementary functions (geometric series and exponential function) via algebraic manipulations, differentiation, and/or integration
- ◇ understand that the formula for the series term in the expansion of an analytic function depends on the center and region of convergence of the series
- ◇ know uniqueness theorem for analytic functions and its corollary ("isolated zeros")
- ◇ apply Rouché's theorem and argument principle
- ◇ classify isolated singularities and determine behavior of a function near a singularity; in particular, know Casorati-Weierstrass and Picard's theorem
- ◇ find residues of a function in a given domain

Conformal mappings:

- ◇ apply open mapping property of an analytic function
- ◇ apply preservation of connectivity by a continuous function
- ◇ know Riemann mapping theorem
- ◇ recognize a Möbius transformation and its composition with other transformations
- ◇ apply properties of a Möbius transformation: conformality on the extended complex plane (analyticity, bijection, preservation of angles), preservation of the class of lines and circles, preservation of orientation, preservation of symmetry, preservation of cross ratio
- ◇ find a Möbius transformation that maps three given points to other three given points
- ◇ find a conformal mapping of a domain whose boundary consists of (arcs of) circles and of (segments of) lines to another domain of such type

Boundary value problems:

- ◇ find a function that is harmonic in a washer-, wedge-, slab-, or wall-shaped region and whose boundary values are given

- ◇ solve a boundary value problem in a complicated region by finding a conformal mapping that maps to a simple region named above

Complex integration:

- ◇ get familiar with the definition of a contour integral (limit of Riemann sums) and use it to derive estimates for integrals
- ◇ evaluate an integral by using properties of integration (like in calculus); in particular, by decomposing a rational function component of the integrand into partial fractions
- ◇ apply the Fundamental Theorem of Calculus to evaluate a complex integral; parametrize a contour, if needed
- ◇ evaluate an integral over a complicated loop by continuously deforming it to a simpler loop
- ◇ evaluate an integral over a complicated contour by building up contours that simplify calculations
- ◇ evaluate an integral by applying Cauchy's integral theorem and/or Cauchy's integral formula for (a derivative of) an analytic function
- ◇ evaluate an integral by applying Cauchy's residue theorem
- ◇ evaluate a complicated integral by applying an appropriate synthesis of methods
- ◇ estimate an integral by applying triangle and/or reverse triangle inequality
- ◇ apply Jordan's lemma to justify calculation of integrals
- ◇ apply an analog of the residue theorem for the limit of integrals over arcs of circles with radii approaching zero to justify calculation of principle value integrals
- ◇ know how to derive the aforementioned result for a specific function
- ◇ calculate real integrals by complex methods (trigonometric integrals over a segment, Fourier type integrals, integrals involving roots and logarithms over a half axis)