WHAT YOU NEED TO KNOW FOR YOUR FIRST MIDTERM!
Math 313: Complex Analysis - Fall 2013

Tomorrow Tuesday Sep 24th, we will have a review session. A page with review problems has been posted on the webpage since Thursday, and there is no homework this week. I expect you have working diligently on these review problems. Here is the link to the review page:

http://www.math.unm.edu/~crisp/courses/math313/review1.pdf

First Midterm will be on Thursday Sep 24, 2013. No books or notes will be allowed. I will write on the blackboard for you:

- definitions of: $\text{Log}z$, $\log z$, $\sin z$, $\cos z$, $\sinh z$, $\cosh z$, and z^b for b a complex number (although you should know them!!!),

- the result from vector analysis about partial derivatives of a real valued function u(x,y) being zero on a "domain" imply the function u(x,y) is a constant,

- what it means for a limit to be infinity, for a limit as z goes to infinity exist, and what it means for a limit as z goes to infinity to be infinity,

- the Cauchy-Riemann equations in polar coordinates.

You need to know:

1. Definitions of and how to calculate: e^z, $|z|$ (absolute value or modulus of z), $\bar{z}$ (conjugate of z), $\text{Arg}z$ (principal value argument of z), $\arg z$, $\text{Arg}_\alpha z$ (when argument is in between α and $\alpha + 2\pi$), n-th roots of z: $z^{1/n}$, $\sin z$, $\cos z$, $\sinh z$, $\cosh z$, $\text{Log}z$, $\log z$, and z^b for b a complex number.. You should know the functions $\arg z$, $\log z$, z^b, are multivalued functions (for b not a natural number), and one can make them single-valued by choosing a branch of $\arg z$ (say $\text{Arg}_\alpha z$).

2. How to write a complex number z in cartesian form: $z = x + iy$ with $x, y \in \mathbb{R}$; and when z is not zero, in polar form $z = re^{i\theta}$. You must know Euler's formula.

3. How to find the real and imaginary parts, u and v, of a function $f(z) = u(x,y) + iv(x,y)$. If z is not zero you should also know how to write u and v in terms of the polar coordinates of z: $u(r,\theta)$ and $v(r,\theta)$.

4. You should be able to calculate limits as z goes to z_0 of $f(z)$ including the possibility of the limit being infinity, and/or z_0 being infinity.

5. You should be able to identify continuous functions $f(z)$ (suffices to check the real and imaginary part are continuous functions of x and y). This is most useful tool to calculate limits, if $f(z)$ is continuous at z_0 then limit as z goes to z_0 of $f(z)$ is equal to $f(z_0)$.

6. You should be able to decide when a function $f(z) = u(x,y) + iv(x,y)$ is differentiable or not at a point. Either by definition or by checking Cauchy-Riemann equations. Knowledge of derivatives of common functions (such as e^z, $\cos z$, $\sin z$, etc) can be used to calculate limits of the form $0/0$, for example $\lim_{z \to 0} \frac{\sin z}{z}$ is equal to $\frac{d}{dz} \sin z \Big|_{z=0} = \cos 0 = 1$. **Warning**: it is not enough to have u, v being polynomials of x and y or other functions for which you can compute partial derivatives to conclude a function is differentiable in the complex sense. Think for example the function $f(z) = x - iy = \bar{z}$... is nowhere differentiable.

7. You should know the Cauchy-Riemann (CR) equations in Cartesian coordinates, and how to use them in either Cartesian or polar form to decide if a function is differentiable at a point or not. If CR equations fail at a point then the function is NOT differentiable at that point. If CR equations hold at a point AND partial derivatives of u and v are continuous at the point THEN the function is differentiable at the point.

8. You should be able to decide when a function is analytic at a point. Need the function differentiable at the point but that is NOT ENOUGH, you need to verify the function is differentiable in a disc centered at the given point. You should know that: $\bar{z}$ (the conjugate of z) is a nowhere differentiable function of z, hence nowhere analytic. Likewise, $|z|^2$ is only differentiable at zero (can check by definition at zero, everywhere else CR fail), hence is nowhere analytic. In general if a function is differentiable only at a point or a curve such as a line, a circle, a parabola, etc. then it is nowhere analytic because you cannot fit a disc inside the curve.

9. You should know what is a entire function and how to identify it (analytic in $\mathbb{C}$). You should know that polynomials in z, in particular monomials: z, z^2, ..., the exponential function e^z and hence trigonometric $\sin z$, $\cos z$, and hyperbolic functions $\sinh z$ and $\cosh z$, are all entire functions. On the other hand functions such as Arg$z$, and hence Log$z$, $z^{1/n}$ and z^b where $b \in \mathbb{C}$ is not a natural number are not defined at $z = 0$, moreover they are discontinuous on the ray $\theta = pi$, so they are analytic on $r > 0$, $-\pi < \theta < \pi$ (a slit domain).

10. You should be able to identify the domain of definition of a given function $f(z)$, and recognize whether it is a "domain" (open and connected). You should understand the importance of this hypothesis (being connected) in results such as: if f is analytic in a domain and $f'(z)$ is zero the the function is a constant function, or if an analytic function on a domain has a zero real part or a zero imaginary part, then f must be a constant function.

11. You should know that the familiar differentiation formulas hold for complex differentiation: addition/subtraction, product, quotient and chain rule, and be able to use these formulas to calculate derivatives of more elaborated functions. Careful with quotient rule, no zeros in denominator are allowed. Continuity, differentiability, analyticity are preserved under addition/subtraction, products, composition (careful with composition: $f \circ g(z)$ you need g continuous/differentiable/analytic at z and f continuous/differentiable/analytic at $g(z)$ to conclude the same for $f \circ g(z) = f(g(z))$). As for quotients, as long as z is not a zero in the denominator, continuity/differentiability/analyticity are preserved at z for a quotient of functions wich are respectively continuous/differentiable/analytic at z.

12. You should know how to decide if a real-valued function $w(x, y)$ is harmonic. Either by definition: satisfies Laplace's equation, or because you can identify an analytic function such that $w(x, y)$ is either its real or imaginary part (see next item).

13. You should know that the real and imaginary parts of an analytic function are harmonic functions. If the analytic function $f(z) = u(x, y) + iv(x, y)$ then v is the harmonic conjugate of u. Be careful u is NOT the harmonic conjugate of v, who is it? The real and imaginary part of elementary analytic functions: z, e^z, Log$z$, z^2, z^3,... are harmonic functions in their domains and you can use them.

14. Given a harmonic function $u(x, y)$ on a disc or on the complex plane (later we will learn that more generally in a simply connected domain) we can always construct a harmonic conjugate by calculating partial derivatives of u, use Cauchy-Riemann equations to get partial derivatives of v, then integrating $\partial v / \partial x$ with respect to x, and $\partial v / \partial y$ with respect to y. If you are able to identify u as the real part of an analytic function then its imaginary part will be a harmonic conjugate and you are done.

15. You should be able to use your newly acquired knowledge of harmonic functions (real and imaginary parts of analytic functions) to solve Laplace's equation given very simple boundary values. The geometries we can access with this simple minded approach are: horizontal and vertical strips, washers, wedges, and shifts of those (or rotations). Horizontal and vertical lines are level curves for the imaginary and real parts of $f(z) = z$, circles and rays are level curves for the real and imaginary parts of Log$_\alpha z$ (here when dealing with wedges, must be careful to choose a branch so that the slit is NOT in the wedge). The book has some more complicated boundary conditions various values on segments along a line, I will not test you on those.