What you need to know for your Second Midterm and Review Problems
Math 313: Complex Analysis - Fall 2013

On Tuesday Nov 5th, we will have a review session and we will work on the review problems that are at the end of this document. There is no homework this week to give time to study and prepare for the exam.

Second Midterm will be on Thursday Nov 7, 2013. For this exam you need to know basic complex integration and tricks (meaning use if possible theorems and formulas to compute contour integrals), and basics of power and Taylor series expansions. No books or notes will be allowed. I will write on the blackboard (and on the first page of the test) for you: The Cauchy Integral Formulas for analytic f and the Cauchy product formula.

Complex Integration - Chapter 4: Section 4.1-4.6

1. Must know the definition of a contour integral of a complex valued function f over the contour Γ given by a parametrization $z : [a, b] \to \Gamma$ (with initial point $z_0 = z(a)$ and endpoint $z_1 = z(b)$

$$\int_\Gamma f(z)dz := \int_a^b f(z(t))z'(t)dt.$$

2. Must be able to calculate a contour integral by definition. This may entail finding an admisible parametrization $z(t)$ for the contour, identifying the interval of parametrization $[a, b]$, finding $z'(t)$, putting all the ingredients together and being able to effectively compute $\int_a^b f(z(t))z'(t)dt$. The curves we are considering are mostly segments or circular arcs, perhaps glued together, in which case the contour integral is the sum of the contour integrals over the piecewise contours. If orientation is reversed then the sign of the contour integral changes.

3. You can often bypass using the definition to compute a contour integral. You should be able to identify when you are in a setting where you can appeal to one or more of the following theorems or formulas:

   (i) Fundamental Theorem of Calculus (FTC): if the integrand f has an *antiderivative* F defined on a *domain* (you don't need here a simply connected domain, in particular punctured domains are OK), then the contour integral is the difference between the values of the antiderivative F at the end point and initial point, or in other words, the contour integral is independent of the path[1]:

$$\int_\Gamma f(z)dz = \int_{z_0}^{z_1} f(z)dz = F(z_1) - F(z_0).$$

   (ii) Cauchy Integral Theorem (CIT): If the integrand is *analytic* in a *simply connected domain* then integrals along *closed* contours are ZERO.

   (iii) Cauchy Integral Formula (CIF): if f is *analytic* on a *simply connected domain* containing a *simple closed positively oriented contour* Γ, and z_0 is a point *interior* to the contour, then

$$f(z_0) = \frac{1}{2\pi i}\oint \frac{f(z)}{z - z_0}dz, \qquad \text{and} \qquad f^{(n)}(z_0) = \frac{n!}{2\pi i}\oint \frac{f(z)}{(z - z_0)^{n+1}}dz.$$

   (iv) Deformation Theorem: if f is *analytic* in a *domain* D and one contour Γ_1 can be *continuously deformed inside the domain* D into another contour Γ_2 then their contour integrals are equal.

4. Practical tips:

   (i) If you are asked to compute a contour integral and the contour is NOT closed, then either you have an antiderivative (either is given or you are able to find it) and you can use the Fundamental Theorem of Calculus (FTC), or you have to use the definition in which case you either have a parametrization or you have to produce one.

   (ii) If the contour is *closed* AND the function is *analytic* in a *simply connected domain* containing the contour, then the integral is zero by the Cauchy Integral Theorem.

[1] The following partial reciprocal is also true, if the contour integrals of a *continuous* complex function f on a domain are independent of the path or equivalently the integrals over closed contours are zero, then the function has an antiderivative F.

(iii) If the contour is *closed*, the function f is *analytic* inside the contour except at a few points but it has an *antiderivative* then the integral is zero by the FTC. For example $f(z) = (z - z_0)^{-n}$ for z_0 inside the countour, and $n \geq 2$ has an antiderivative $F(z) = \frac{(z-z_0)^{n+1}}{n+1}$. Alternatively in this example, you could use the Cauchy integral Formula applied to the function $g(z) = 1$ to get $2\pi i g^{(n)}(z_0) = 0$ because $g'(z) = g''(z) = \cdots = g^{(n)}(z) = 0$. (See next item (iv).) Main non-example is the case $n = 1$ where $\oint_{|z-z_0|=r} \frac{1}{z-z_0} dz = 2\pi i$ (counterclockwise), this we calculated by definition and is the basis for the Cauchy Integral Formula.

(iv) If the contour is a simple closed curve positively oriented and the function is analytic except at a few points inside the contour, you may be able to use the Cauchy Integral Formula (CIF). First identify all the bad points for the integrand f and decide how many are inside the contour, the ones outside disregard. If more than one bad point is inside the contour, you need to deform the contour so that only one bad point is inside each component of the deformed curve. The contour integral will then be the sum of the smaller contour integrals each containing only one bad point in their interior. Suppose then that the curve is one of those components, say Γ_0 (simple closed positively oriented) and z_0 is the only bad-point inside Γ_0, then the integrals you will be asked to compute are such that you can rewrite the integrand in a way where the CIF applies to a different function $g(z) = (z - z_0)^{k+1} f(z)$ that is now analytic in a simply connected domain containing Γ_0 and its interior to get:

$$\oint_{\Gamma_0} f(z)dz = \oint_{\Gamma_0} \frac{g(z)}{(z - z_0)^{k+1}} dz = 2\pi i g^{(k)}(z_0).$$

(if orientation is reversed, you pick up a negative sign).

5. Should be able to estimate contour integrals using the "triangle inequality":

$$\left| \int_\Gamma f(z)dz \right| \leq \int_\Gamma |f(z)||dz| \leq M\ell(\Gamma),$$

where f is bounded on Γ, say $|f(z)| \leq M$, and $\ell(\Gamma) = \int_\Gamma |dz| := \int_a^b |z'(t)|dt$ is the length of the contour.

6. Should know Liouville's Theorem: a function that is analytic and bounded on the whole complex plane must be a constant function. Should also know the maximum modulus principle: if f is analytic in a domain and its modulus $|f(z)|$ reaches a maximum at a point z_0 in the domain, then f is constant in the domain.

7. An important byproduct of all the integration theory is that derivatives of analytic functions f are themselves analytic and hence derivatives of all orders exist and f, f', f'',..., $f^{(n)}$,... are all analytic.

8. The fact that derivatives of analytic functions are analytic and the footnote in page 1 imply Morera's Theorem: if f is continuous on a domain and the integrals over all closed contours are zero then f is analytic in the domain.

Power and Taylor series expansions - Sections 5.1, 5.2 and 5.3

1. You should know that a series of complex numbers $\sum_{n=0}^{\infty} w_n$ is said to converge to w if and only if the series $\sum_{n=0}^{\infty} x_n$ and $\sum_{n=0}^{\infty} y_n$ of the real and imaginary parts of w_n ($w_n = x_n + iy_n$) converge to x and y, the real and imaginary parts of w ($w = x + iy$). This is equivalent to having the sequence of partial sums $S_N = \sum_{n=0}^{N} w_n$ converges to w, that is $\lim_{N\to\infty} \sum_{n=0}^{N} w_n = w$. Usual convergence tests for real series do work here: divergence test (if terms don't go to zero then series diverges), root and ratio test, absolute convergence test (if the series of the absolute values $\sum_{n=0}^{\infty} |w_n|$ converges so does the original series).

2. Given a power series centered at z_0, that is

$$\sum_{n=0}^{\infty} a_n(z - z_0)^n = a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + a_3(z - z_0)^3 + \ldots,$$

you should know that there is *radius convergence* $R \geq 0$ such that the series converges for all complex numbers z such that $|z - z_0| < R$, the series diverges for all complex numbers z such that $|z - z_0| > R$, and for those complex numbers on the circle $|z - z_0| = R$ anything can happen. The series converges *uniformly* for all $|z - z_o| \leq r$ provided $r < R$.

3. You should be able to calculate the radius of convergence R for a given power series. If $\lim_{n\to\infty} |a_n|^{1/n} = L$ then the radius of convergence is its reciprocal, $R = 1/L$. Likewise if $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = L$ then the radius of convergence is its reciprocal, $R = 1/L$. It is useful to remember that $\lim_{n\to\infty} n^{1/n} = 1$.

4. Canonical example: geometric series $\sum_{n=0}^{\infty} z^n$ has radius of convergence $R = 1$ and the series converges to the function $1/(1-z)$ if $|z| < 1$, and diverges otherwise (if $|z| \geq 1$).

5. A power series with positive radius of convergence $R > 0$ defines an analytic function f,

$$\sum_{n=0}^{\infty} a_n(z-z_0)^n =: f(z), \quad \text{on the domain} \quad |z-z_0| < R.$$

   We can differentiate term-by-term the series to get the derivative $f'(z) = \sum_{n=0}^{\infty} na_n(z-z_0)^{n-1}$ (this term-by-term differentiation is permitted because both the power series and the differentiated power series have the same radius of convergence and converge uniformly on smaller discs). Furthermore we can keep differentiating term-by-term to get higher order derivatives, and evaluating at z_0 we conclude that the coefficients are the *Taylor coefficients* of f: $a_n = \frac{f^{(n)}(z_0)}{n!}$. Likewise we can integrate term by term power series as long as the contour of integration remains inside the disc of convergence.

6. Two power series centered at z_0 that are equal on $|z-z_0| < r$ for some $r > 0$ must have exactly the same coefficients (they will be the Taylor coefficients of the same analytic function). That is,

$$\text{if } \sum_{n=0}^{\infty} a_n(z-z_0)^n = \sum_{n=0}^{\infty} b_n(z-z_0)^n \text{ for all } |z-z_0| < r \text{ then } a_n = b_n \text{ for all } n \geq 0.$$

7. An analytic function f on a disc $|z-z_0| < R$ has a power series expansion centered at z_0, its *Taylor series*, $\sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n$, convergent at least on the given disc (if R were the radius of the largest disc where the function is analytic then it will coincide with the radius of convergence of its Taylor series).

8. The Taylor polynomial of degree N associated to an analytic function f on a domain containing z_0 and centered at z_0 is $P_N(z) = \sum_{n=0}^{N} \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n$. Item 7 means that $\lim_{N\to\infty} |P_N(z) - f(z)| = 0$. It is of practical interest to understand the rate at which the absolute value of the remainder $R_N(z) = P_N(z) - f(z)$ goes to zero. Formulas for the Taylor remainder are extremely useful, one of them is given by a contour integral similar to the Cauchy integral formula and ameanable to estimates (positively oriented):

$$R_N(z) = (z-z_0)^{N+1} \frac{1}{2\pi i} \oint_{|z-z_0|=r} \frac{f(w)}{(w-z_0)^{N+1}(w-z)} dw.$$

9. You should be able to calculate the Taylor series of an analytic function (or at least the first three non-zero terms) by definition: differentiating, evaluating at z_0, and dividing by the appropriate factorial to find the Taylor coefficients.

10. You should know the MacLaurin ($z_0 = 0$) series for e^z, $\sin z$, $\cos z$, $\frac{1}{1-z}$, and polynomials.

11. You should be able to use these known Taylor series to calculate Taylor series of functions that use the known ones as building blocks. For example finding the series of e^{z^2}, $z^3 \cos z$, or $\frac{1}{(1-z)^2}$ (the last one recognizing it as the derivative of $\frac{1}{1-z}$) should be painless. Or to get Taylor series centered at different points, if $f(z) = \sin(z-2i)$ then just plugging in $z-2i$ into the MacLaurin series of $\sin z$ will give you the Taylor series of f centered at $2i$. Also noticing that $\mathrm{Log}(1-z)$ is analytic in the disc $|z| < 1$ and its derivative is $\frac{-1}{1-z}$ then its Taylor series can be found integrating term by term from $w = 0$ to $w = z$ the Maclaurin series of $\frac{-1}{1-z}$ and using the Fundamental Theorem of calculus for contour integrations of functions with an antiderivative. Same integration term by term technique allows to find Taylor expansion centered at z_0 of an antiderivative F of the analytic function f at z_0, since $F(z) := \int_{z_0}^{z} f(w)dw$ for $|z-z_0| < R$. You should be able to give a short explanation why the above are valid calculations.

12. You should know how to use the Cauchy Product Formula to calculate the product of two power series centered at the same point. If $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ and $\sum_{n=0}^{\infty} b_n(z-z_0)^n$ converge for $|z-z_0| < r$, then their product is another power series $\sum_{n=0}^{\infty} c_n(z-z_0)^n$ where the coefficients are $c_n = \sum_{k=0}^{n} a_k b_{n-k}$.

Review Problems

1. True or false. If true give short explanation, if false give a counterexample and a corrected statement. All closed contours are positively oriented.

   (a) For every function analytic at each point of a closed contour Γ we have $\oint_\Gamma f(z)dz = 0$.

   (b) The arclength of a contour Γ with admissible parametrization $z = z(t)$, for $a \leq t \leq b$, the length of Γ can be computed by the formula $\int_a^b z'(t)dt$.

   (c) The function $f(z) = 1/z$ has an antiderivative in the punctured plane $\mathbb{C} \setminus \{0\}$.

   (d) $\displaystyle\oint_{|z|=1} \frac{1}{z} dz = \oint_{|z|=1} \bar{z} dz.$ (e) $\displaystyle\oint_{|z|=1} \frac{1}{z} dz = \oint_{|z|=1} \frac{1}{|z|} dz$ (f) $\displaystyle\oint_{|z|=1} \frac{1}{z} dz = \oint_{|z+i|=2} \frac{1}{z} dz.$

2. Evaluate $\int_\Gamma z\cos(z^2)dz$, where Γ is the contour that goes on a straight line from $z = 0$ to $z = i + 1$ and then from $z = i + 1$ to $z = 1 + 2i$.

3. Evaluate $\int_\Gamma (2\bar{z} - z)dz$, where Γ is parametrized by $x = -t$, $y = t^2 + 2$ and $0 \leq t \leq 2$.

4. Calculate $\int_\Gamma \left(z^{1/2} + 3\frac{1}{z^{1/2}}\right)dz$ where Γ is the arc of the circle $z = 4e^{it}$, $-\pi/2 \leq t \leq \pi/2$. You choose an appropriate branch for the square root function

5. Evaluate $\displaystyle\oint_{|z|=2} \left(z + \frac{1}{z} + \frac{3}{z^3} + \frac{2}{(z+3i)}\right)dz$ and $\oint_{|z|=2} \frac{5z^3+2z-1}{(z-i)^3} dz$ (counterclockwise).

6. Explain why $\oint_\Gamma \frac{\sin z}{e^{z^2}} dz = 0$, where Γ is a closed contour? For what contours are we sure $\int_\Gamma \mathrm{Log} z dz = 0$?

7. Explain why an antiderivative for $\sin z^2$ exists. Find an antiderivative for $\sin z^2$.

8. Represent the function $\frac{2z+1}{z-1}$ by its Maclaurin series, and find the region of convergence.

9. Find the region of convergence of the complex power series:

$$\sum_{n=0}^{\infty} n!(z-1)^n, \qquad \sum_{n=0}^{\infty} n(z+i)^n, \qquad \sum_{n=1}^{\infty} \frac{z^n}{n^2}.$$

10. Find the Maclaurin series for $f(z) = ze^{2z^2}$ and $g(z) = \cos^2 z$.

11. Find the Taylor series centered at $z_0 = 1$ for $f(z) = (z-1)e^{-3z}$ and for $g(z) = \frac{1}{z}$.

12. Find the Taylor series centered at $z_0 = \pi/4$ for $f(z) = \cos z$.

13. Find the first three terms of the Maclaurin series for $f(z) = \tan z$ and determine its radius of convergence.

14. Consider the function $f(z) = \mathrm{Log}(1 + z)$. What is the largest circle centered at the origin within which f is analytic? Find the Maclaurin series of f and its radius of convergence. Find the Maclaurin series of $\mathrm{Log}(1 - z)$. Find a Maclaurin series for $\mathrm{Log}\left(\frac{1+z}{1-z}\right)$.

15. Find an analytic function on $|z| < 1$ whose Maclaurin series is $\sum_{n=0}^{\infty} n^2 z^n$. Hint: consider the function $1/(1-z)$, differentiate, multiply by z, differentiate again and multiply once more by z.

16. Show that if $|f(z)| \leq 10$ for all $|z| = 3$ then $\left|\oint_{|z|=3} \frac{f(z)}{z^2+z^3} dz\right| \leq 10\pi/3$.

17. Find the Taylor remainder $R_N(z)$ for the function $f(z) = e^z$ centered at $z_0 = 0$. Find $N > 0$ such that you can ensure $|R_N(100)| < 10^{-5}$.

18. If f is entire and its third derivative is bounded on the complex plane, what type of function must f be?

19. If an entire function f is such that $|f(z)| = 2$ on the circle $|z - (2-3i)| = 4$. Is it possible that $f(1) = \sqrt{3} - \sqrt{2}i$? Is it possible that $f(1) = \sqrt{3} + i$? If yes, identify a function f with the prescribed property.