1. (a) $\lim_{z\to 2+i} |z^2-4| = |(2+i)^2-4| = |4+2i-1-4| = \sqrt{(-1)^2+2^2} = \sqrt{5}$

(b) $\lim_{z\to\infty} \frac{3z^2+iz}{z^2+2z+3i} = \lim_{z\to\infty} \frac{z^2(3+\frac{i}{z})}{z^2(1+\frac{2}{z}+\frac{3i}{z^2})} = 3$

since $\lim_{z\to\infty}\frac{1}{z} = 0 \iff \lim_{z\to 0} z = 0$

(c) $\lim_{z\to\infty} \sin z$ D.N.E. since if $z = iy$ (along vertical path $x=0$)

Then $z\to\infty \iff y\to\infty$

and $\lim_{y\to\infty}\sin(iy) = \lim_{y\to\infty}\frac{e^{i(iy)}-e^{-i(iy)}}{2i}$

$= \lim_{y\to\infty}\frac{e^{-y}-e^{y}}{2i} = \infty$

However along a horizontal path $y=0$, $z=x\to\infty$

$\lim_{x\to\infty}\sin x$ DNE (oscillates between 1 and -1).

(d) $\lim_{z\to 3i}\frac{e^z}{z^2+9} = \infty$ because $\lim_{z\to 3i}\frac{z^2+9}{e^z} = \frac{(3i)^2+9}{e^{3i}}$

$= \frac{-9+9}{e^{3i}} = 0$

$\left(\lim_{z\to z_0} f(z) = \infty \iff \lim_{z\to z_0}\frac{1}{f(z)} = 0\right)$

2. (i) $f(z) = \frac{(z+2)^3}{(z^2+2iz-1)^4} = \frac{(z+2)^3}{(z+i)^2}$

(a) f is a rational function continuous except at zeros of denominator $z=-i$. $\mathbb{C}\setminus\{-i\}$

(b) f is differentiable on $\mathbb{C}\setminus\{-i\}$

(c) f is analytic on $\mathbb{C}\setminus\{-i\}$

(ii) $f(z) = z|z| = (x+iy)\sqrt{x^2+y^2} = x\sqrt{x^2+y^2} + iy\sqrt{x^2+y^2}$

(a) f is the product of two continuous functions, hence is continuous everywhere.

[continuation 2 (ii)]

(b) CR in cartesian coordinates $u(x,y) = x\sqrt{x^2+y^2}$, $v(x,y) = y\sqrt{x^2+y^2}$

$$\frac{\partial u}{\partial x} = \sqrt{x^2+y^2} + x\cdot\frac{2x}{2\sqrt{x^2+y^2}} = \frac{y^2+2x^2}{\sqrt{y^2+x^2}}$$

$$\frac{\partial u}{\partial y} = \frac{x\cdot 2y}{2\sqrt{x^2+y^2}} = \frac{xy}{\sqrt{x^2+y^2}}$$

$$\frac{\partial v}{\partial x} = \frac{xy}{\sqrt{x^2+y^2}}$$

$$\frac{\partial v}{\partial y} = \sqrt{x^2+y^2} + \frac{y\cdot 2y}{2\sqrt{x^2+y^2}} = \frac{x^2+2y^2}{\sqrt{y^2+x^2}}$$

Partial derivatives continuous on $\mathbb{C}\setminus\{0\}$

(1) $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \Rightarrow y^2+2x^2 = x^2+2y^2$
$x^2 = y^2$
$x = \pm y$

(2) $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \Rightarrow xy = 0 \Rightarrow$ $x=0$ or $y=0$; $x=0 \overset{(1)}{\Downarrow} y=0$; $y=0 \overset{(2)}{\Downarrow} x=0$ — CR holds

Cauchy-Riemann equations could only hold when $z=0$, nowhere else. Let us check by definition differentiability at $z=0$

$$\lim_{z\to 0}\frac{z|z|-0}{z-0} = \lim_{z\to 0}\frac{z|z|}{z} = \lim_{z\to 0}|z| = 0$$

f is differentiable at $z=0$ only.

(c) f is nowhere analytic, because is only differentiable at a point.

Notice that polar coordinates for $z\neq 0$ $z=re^{i\theta}$ are nice for this problem:

$u(r,\theta) = r^2\cos\theta$ $\quad \frac{\partial u}{\partial r} = 2r\cos\theta \quad \frac{\partial u}{\partial \theta} = -r^2\sin\theta$

$v(r,\theta) = r^2\sin\theta$ $\quad \frac{\partial v}{\partial r} = 2r\sin\theta \quad \frac{\partial v}{\partial \theta} = r^2\cos\theta$

CR in polar coordinates: $r>0$

$\frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r} \Rightarrow -r^2\sin\theta = -2r^2\sin\theta \Rightarrow \theta = 0$

$\frac{\partial v}{\partial \theta} = +r\frac{\partial u}{\partial r} \Rightarrow r^2\cos\theta = 2r^2\cos\theta \Rightarrow \theta = \frac{\pm\pi}{2}$

They hold nowhere is $z\neq 0$.

(iii) $f(z) = \operatorname{Im} z - i\operatorname{Re} z = y - ix = -i(x+iy) = -iz$

(c) $f(z)$ is a multiple of z hence is everywhere analytic, that is it is an entire function

(b) f entire $\Rightarrow$ f differentiable everywhere

(c) f differentiable everywhere $\Rightarrow$ f continuous everywhere

[continuation 2]

(iv) $f(z) = 3\bar{z} = 3(x - iy) = 3x - i(3y)$

(a) since real and imaginary parts of $f(z) = u + iv$ are polynomials: $u(x,y) = 3x$, $v(x,y) = -3y$. They are continuous ⇒ f is continuous

(b) f is nowhere differentiable (despite the fact that u & v are polynomials!). We can check this by definition or verifying CR eqns. do not hold. $\frac{\partial u}{\partial x} = 3 \neq \frac{\partial v}{\partial y} = -3$.

(c) Since f is nowhere differentiable it is nowhere analytic.

Summary:

(a) (ii), (iii), (iv)

(b) (iv)

(c) (i) analytic on $\mathbb{C} \setminus \{3 - i3\}$

(ii) & (iv) are nowhere analytic

(iii) analytic everywhere.

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3. (i) $f(z) = \frac{z^2 - |z|^2}{2} = \frac{x^2 + 2xyi - y^2 - (x^2 + y^2)}{2} = -y^2 + xyi$

(a) f is Not entire, if it were then $|z|^2 = z^2 - 2f(z)$ will also be entire and is not.

Also note that CR eqns only hold when $z = 0$

$u = -y^2$ $\quad \frac{\partial u}{\partial x} = 0 \quad \frac{\partial u}{\partial y} = -2y \quad \frac{\partial u}{\partial x} = 0 = x = \frac{\partial v}{\partial y} \Rightarrow x = 0$

$v = xy$ $\quad \frac{\partial v}{\partial x} = y \quad \frac{\partial v}{\partial y} = x \quad \frac{\partial u}{\partial y} = -2y = -y = -\frac{\partial v}{\partial x} \Rightarrow y = 0$

[continuation 3(i)]

(b) $\operatorname{Im} f(z) = v(x,y) = xy$, $\frac{\partial^2 v}{\partial x^2} = 0$ $\frac{\partial^2 v}{\partial y^2} = 0$

$$\Rightarrow \Delta v = \nabla^2 v = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0 + 0 = 0$$

So $\operatorname{Im} f(z)$ is harmonic on $\mathbb{C}$.

(ii) $g(z) = (\bar{z})^2 + 4z\operatorname{Re} z - 4(\operatorname{Re} z)^2 =$

$= (x - iy)^2 + 4(x+iy)\cdot x - 4x^2$

$= x^2 - 2xyi - y^2 + 4x^2 + 4yxi - 4x^2$

$= (x^2 - y^2) + 2xyi = (x+iy)^2 = z^2$

(a) $g(z) = z^2$ hence is entire

(b) Since g is entire both its real and imaginary parts are harmonic.

Could have also verified CR eqns:

$u(x,y) = x^2 - y^2$ $\quad \frac{\partial u}{\partial x} = 2x = \frac{\partial v}{\partial y}$ ✓ $\quad \frac{\partial^2 v}{\partial x^2} = 0$

$v(x,y) = 2xy$ $\quad \frac{\partial u}{\partial y} = -2y = -\frac{\partial v}{\partial x}$ $\quad \frac{\partial^2 v}{\partial y^2} = 0$

and Laplace's eqn $\Delta v = \nabla^2 v = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0 + 0 = 0$

Summary

(a) g is entire, f is Not

(b) both $\operatorname{Im} f$ and $\operatorname{Im} g$ are harmonic fcns

in fact $\operatorname{Im} g = 2xy = 2 \operatorname{Im} f$.

4. (a) $\text{Log}((1+i)^3) = \text{Log}\left(\left(\sqrt{2}\cdot e^{i\frac{\pi}{4}}\right)^3\right) = \text{Log}\left(2^{3/2}\cdot e^{i\frac{3\pi}{4}}\right)$

$= \ln(2^{3/2}) + i\,\text{Arg}\left(2^{3/2} e^{i\frac{3\pi}{4}}\right)$

$= \frac{3}{2}\ln 2 + i\,\frac{3\pi}{4}$

(They are equal)

$3\,\text{Log}(1+i) = 3\,\text{Log}\left(\sqrt{2}\, e^{i\frac{\pi}{4}}\right) = 3\left[\ln\sqrt{2} + i\,\text{Arg}\left(\sqrt{2}\, e^{i\frac{\pi}{4}}\right)\right]$

$= \frac{3}{2}\ln 2 + i\,\frac{3\pi}{4} = \text{Log}((1+i)^3)$

(b) $\text{Log}((-1+i))^2 = \text{Log}\left(\left(\sqrt{2}\, e^{\frac{3\pi}{4}i}\right)^2\right) = \text{Log}\left(2\cdot e^{\frac{3\pi}{2}i}\right)$

$-1+i = \sqrt{2}\, e^{\frac{3\pi}{4}i}$

$= \ln 2 + i\,\text{Arg}\left(2\, e^{\frac{3\pi}{2}i}\right)$

$= \ln 2 + i\left(-\frac{\pi}{2}\right)$

$2\,\text{Log}((-1+i)) = 2\,\text{Log}\left(\sqrt{2}\, e^{\frac{3\pi}{4}i}\right) = 2\ln\sqrt{2} + 2i\,\text{Arg}\left(\sqrt{2}\, e^{\frac{3\pi}{4}i}\right)$

$= 2\ln\sqrt{2} + 2i\,\text{Arg}\left(\sqrt{2}\right)$

$= \frac{2}{2}\ln 2 + 2i\,\frac{3\pi}{4}$

(They are different.)

$= \ln 2 + i\left(\frac{3\pi}{2}\right) \neq \ln 2 + i\left(\frac{-\pi}{2}\right)$

5. (a) $\left(\frac{i}{2+2i}\right)^{1/5} = \left(\frac{e^{i\pi/2}}{2\sqrt{2}\, e^{i\pi/4}}\right)^{1/5} = \frac{1}{(2\sqrt{2})^{1/5}}\cdot\left(e^{i\pi/4}\right)^{1/5}$

$$\boxed{= 2^{-3/10}\cdot e^{i\left(\frac{\pi}{20} + \frac{2k\pi}{5}\right)} \quad k = 0,1,2,3,4}$$

(b) $|\cos z| = \left|\frac{e^{iz} + e^{-iz}}{2}\right| = \left|\frac{e^{ix-y} + e^{-ix+y}}{2}\right| =$

$= \left|\frac{e^{-y}\cdot e^{ix} + e^{y} e^{-ix}}{2}\right| = \left|\frac{e^{-y}(\cos x + i\sin x) + e^{y}(\cos x - i\sin x)}{2}\right|$

[continuation 5(b)]

5(b) $|\cos z| = \left| \cos x \left(\frac{e^{-y} + e^{y}}{2} \right) - i \sin x \left[\frac{e^{+y} - e^{-y}}{2} \right] \right|$

$= |\cos x . \cosh y - i \sin x \sinh y|$

$= \sqrt{\cos^2 x \cosh^2 y + \sin^2 x \sinh^2 y}$

$z = 3 - i \quad , \quad x = 3 \quad y = -1$

$$\boxed{|\cos(3 - i)| = \sqrt{\cos^2 3 . \cosh^2(-1) + \sin^2 3 \sinh^2(-1)}}$$

(c) $z^i = e^{i \log z} = e^{i(\ln r + i \arg z)} = e^{-\arg z} \cdot e^{i \ln r}$

$= e^{-(\operatorname{Arg} z + 2k\pi)} e^{i \ln r} \quad k \in \mathbb{Z}$

$z = 2 - 2i = 2\sqrt{2}\, e^{-\frac{\pi}{2} i} \quad \operatorname{Arg} z = -\frac{\pi}{2} \quad r = 2\sqrt{2}$

$$\boxed{(2-2i)^i = e^{+\frac{\pi}{2} - 2k\pi} \cdot (\cos(\ln 2\sqrt{2}) + i \sin(\ln 2\sqrt{2}))}$$

6. $\{z \in \mathbb{C} : |z| < 1\}$ is a simply connected domain

$u(x,y) = 3xy^2 - x^3$ must find conjugate harmonic using CR eqns

$\frac{\partial u}{\partial x} = 3y^2 - 3x^2 = \frac{\partial v}{\partial y} \rightarrow v = \int (3y^2 - 3x^2)\, dy = y^3 - 3x^2 y + \varphi(x)$

$\frac{\partial u}{\partial y} = 6xy = -\frac{\partial v}{\partial x} \rightarrow v = \int -6xy\, dx = -3x^2 y + \psi(y)$

[continuation 6]

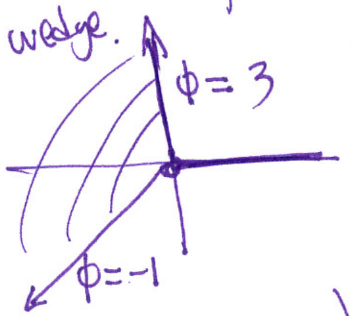
6. hence $v(x,y) = y^3 - 3x^2y + C$ works.

Let $f(z) = 3xy^2 - x^3 + i\,(y^3 - 3x^2y)$

$$= -\left(x^3 - 3xy^2 + i\,(3x^2y - y^3)\right)$$

$$= -(x+iy)^3 = -z^3 \quad \text{(entire)}$$

The function $g(z) = u(z) = 3xy^2 - x^3$ is nowhere analytic because is real valued. Alternatively doesn't satisfy CR eqns.

7. 

Consider $\phi(z) = a\,\mathrm{Arg}_0 z + b$

$z = re^{i\,\mathrm{Arg}_0 z}$

$\phi(r, \frac{\pi}{2}) = a\frac{\pi}{2} + b = 3$

$\phi(r, \frac{5\pi}{4}) = a\frac{5\pi}{4} + b = -1$

solve linear system

$a\left(\frac{\pi}{2} - \frac{5\pi}{4}\right) = 4$

$a\pi\left(-\frac{3}{4}\right) = 4 \Rightarrow \boxed{a = -\frac{16}{3\pi}}$

$b = 3 - \frac{a\pi}{2} = -3 - \frac{16}{3\pi}\cdot\frac{\pi}{2} = -3 - \frac{8}{3} = \boxed{-\frac{17}{3} = b}$

$$\therefore \boxed{\phi(r,\theta) = -\frac{16}{3\pi}\,\theta - \frac{17}{3} \qquad 0 < \theta < 2\pi}$$

θ is harmonic being imaginary part of $\mathrm{Log}_0 z$.

8. $e^{iz} = e^{i(x+iy)} = e^{ix-y} = e^{-y}e^{ix}$

$= e^{-y}(\cos x + i\sin x) = e^{-y}\cos x + i e^{-y}\sin x$

$\sin z := \frac{e^{iz} - e^{-iz}}{2i} = \frac{e^{-y}\cos x + ie^{-y}\sin x - (e^{y}\cos(-x) + ie^{y}\sin(-x))}{2i}$

$= i\cos x\left(\frac{e^{-y} - e^{+y}}{2}\right) + \sin x\left(\frac{e^{-y} + e^{y}}{2}\right)$

$\text{Im}(e^{iz}) = e^{-y}\sin x = \sin z \iff y = 0$

ANSWER: no only when $y=0$, $z=x$.

9. $f(z) = \sinh(z^2) - 2z\cos(3z)$

This is an analytic fcn, in fact entire because is a differene of composition and products of analytic functions

$\sinh z = \frac{e^{z} - e^{-z}}{2}$ $\quad \frac{d}{dz}\sinh z = \frac{e^{z} + e^{-z}}{2} = \cosh z$

Using familiar rules of differentiation which hold in ~~the context of~~ complex differentiation:

$f'(z) = 2z\cosh(z^2) - (2\cos(3z) - 6z\sin(3z))$

$$\boxed{f'(z) = 2z\cosh(z^2) - 2\cos(3z) + 6z\sin(3z)}$$