EUCLIDEAN AND TRANSFORMATIONAL GEOMETRY

A Deductive Inquiry

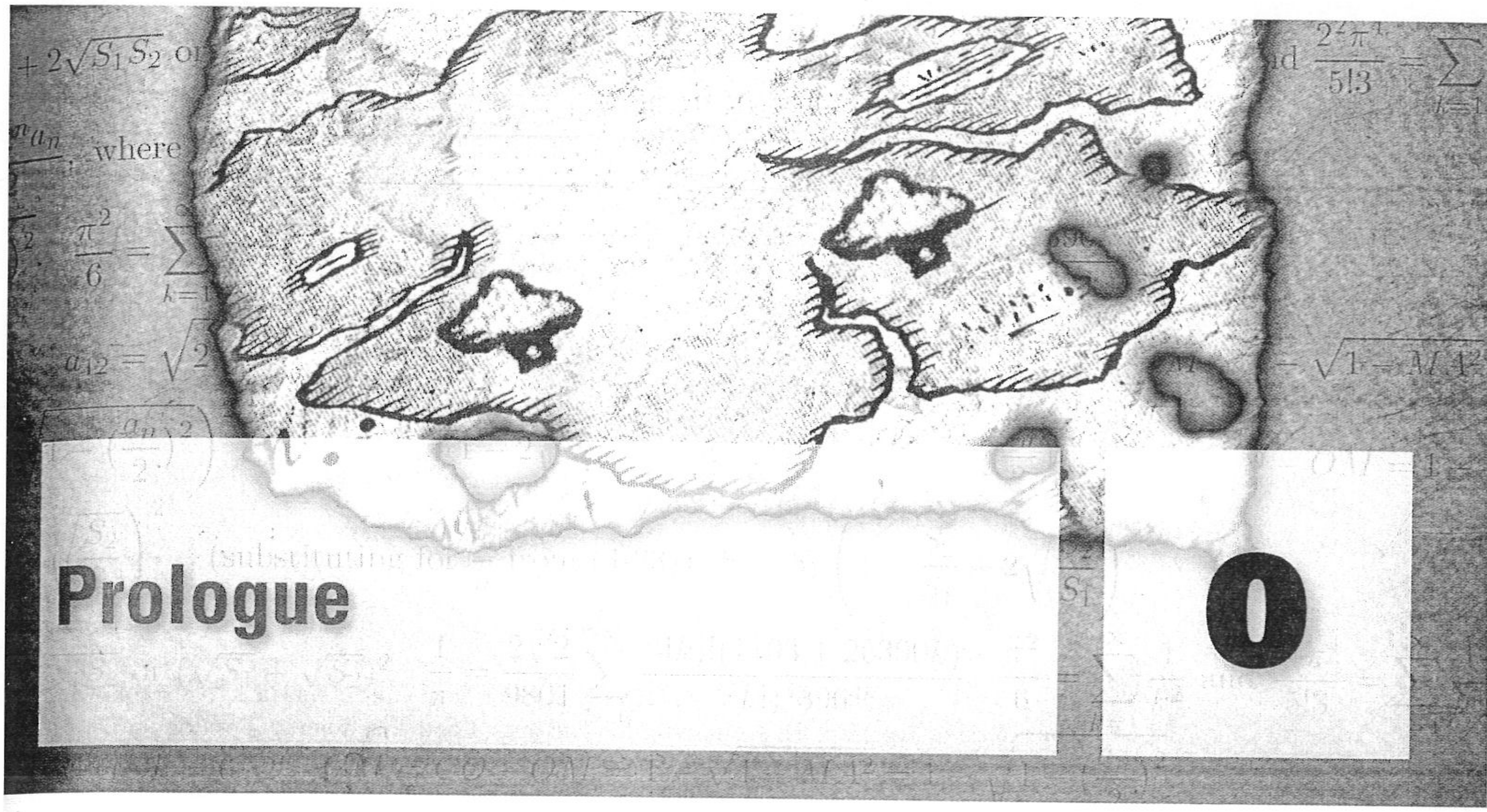
Shlomo Libeskind

University of Oregon

JONES AND BARTLETT PUBLISHERS

Sudbury, Massachusetts

BOSTON TORONTO LONDON SINGAPORE



Prologue 0

It may well be doubted whether, in all the range of science, there is any field so fascinating to the explorer—so rich in hidden treasures—so fruitful in delightful surprises—as Pure Mathematics.

—Lewis Carroll (1832–1898), best known as author of *Alice's Adventures in Wonderland* and *Through The Looking Glass*

Introduction: Surprising Results

Frequently a course in high school geometry starts with a long list of definitions and proofs of theorems that are intuitively obvious. It rarely includes theorems and proofs that would make a lasting impression on the student.

Pause for a moment and recall some geometrical theorems. Are there any on your list that contain an unexpected or surprising result? Which are especially beautiful? Are any of them especially important and, if so, why?

This short section introduces some problems and theorems from Euclidean geometry whose solutions or statements you will very likely find surprising. In several cases, you will be asked to conjecture a solution or a theorem through a sequence of suggested experiments. Later in this text, you will learn how to prove some of these conjectures and statements. Several of the proofs are particularly beautiful owing to their unexpected simplicity and applicability to new problems.

For the work you will be asked to do in this introductory chapter (and in other chapters), you will need a compass, a ruler, and some blank sheets of paper. A geometry utility software such as GSP (Geometer's Sketchpad) is especially helpful in investigating geometrical properties and making conjectures. Throughout the text we will, when appropriate, suggest optional activities using the software. It should be noted, however, that the text is independent of GSP and can be read and studied without the software.

0.1 The Treasure Island Problem

Among his great-grandfather's papers, Marco found a parchment describing the location of a pirate treasure buried on a deserted island. The island contained a coconut tree, a banana tree, and a gallows (Γ) where traitors were hung. A reproduction of the map appears in Figure 0.1. It was accompanied by the following directions:

> *Walk from the gallows to the coconut tree, counting the number of steps. At the coconut tree, turn 90° to the right. Walk the same distance and put a spike in the ground. Return to the gallows and walk to the banana tree, counting your steps. At the banana tree, turn 90° to the left, walk the same number of steps, and put another spike in the ground. The treasure is halfway between the spikes.*

Marco found the island and the two trees but could find no trace of the gallows or the spikes, as both had probably rotted. In desperation, he began to dig at random but soon gave up because the island was too large. Your quest is to devise a plan to find the exact location of the treasure.

If you have spent enough time pondering a solution but could not find one, try the following: On a piece of paper, fix the positions of the two trees. Choose an arbitrary position for the gallows and mark it Γ_1. Follow the directions to find the corresponding spikes and the midpoint between them, T_1. Next choose another position for the gallows Γ_2 and follow the directions to find the corresponding location of the treasure, T_2. Repeat the procedure for at least two more positions of the gallows (GSP is especially convenient here). What do you notice about T_1, T_2, T_3, and T_4? Now try to conjecture how to locate the treasure.

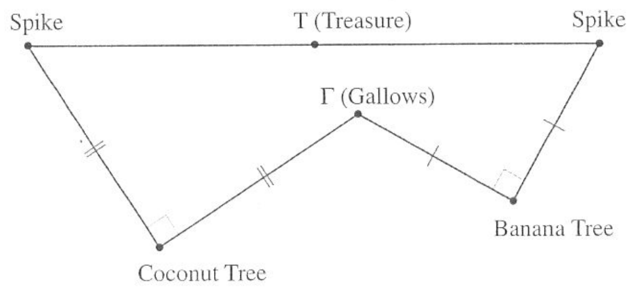


Figure 0.1

0.2 The Nine-Point Circle

The nineteenth century experienced a renewed interest in classical Euclidean geometry. Probably the most spectacular discovery was the **Nine-Point Circle**, which was investigated simultaneously by the French mathematicians Charles Jules Brianchion (1785–1864) and Jean-Victor Poncelet (1788–1867), who published their work jointly in 1821. The theorem is, however, commonly attributed to the German mathematician and high school teacher Karl Wilhem Feurbach (1800–1834), who independently discovered the theorem and published it with some related results in 1822.

With any triangle ABC, nine particular points can be associated with it as shown in Figure 0.2. The first three points—M_1, M_2, and M_3—are the midpoints of the three sides of the triangle. The next three points—N_1, N_2, and N_3—are the midpoints of the segments joining the vertices A, B, and C with the point H; H is the point of intersection of the three altitudes of the triangle (we will prove in Chapter 1 that the three altitudes of any triangle intersect in a single point). The final three points—F_1, F_2, and F_3—are the points of intersection of each altitude with each corresponding side (these points are known as the "feet" of the altitudes).

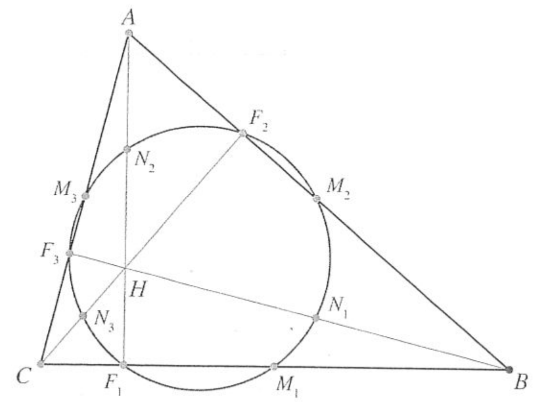


Figure 0.2

The theorem states that all nine points lie on one circle called the Nine-Point Circle. The Nine-Point Circle Theorem and some related results will be proved in Chapter 7.

0.3 Morley's Theorem

In 1899, Frank Morley, professor of mathematics at Haverford College and later at Johns Hopkins University, discovered an unusual property of the trisectors of the three angles of any triangle: If the angle trisectors are drawn for each angle of any triangle, then the adjacent trisectors of the angles meet at vertices of an equilateral triangle. In Figure 0.3, the adjacent trisectors of angles *A* and *B* meet at *D*, the adjacent trisectors of angles *A* and *C* meet at *E*, and the adjacent trisectors of angles *C* and *B* meet at *F*. Morley's Theorem states that triangle *FDE* is equilateral (i.e., its three sides are all the same length). Since 1899, many different proofs of this theorem have been published, including several since 2000. A proof of Morley's Theorem will be presented in Chapter 4, and additional proofs in Chapter 7.

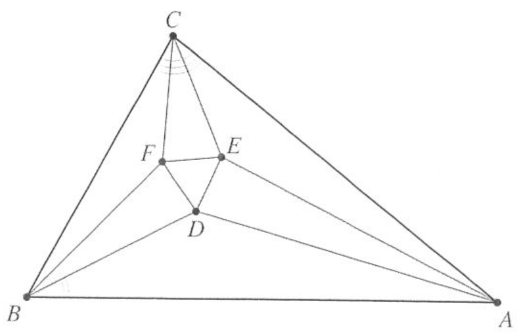


Figure 0.3

0.4 The Hiker's Path

A hiker H in Figure 0.4 needs to get first to the river r and then to her tent T. Find the point X on the bank of the river so that the hiker's total trip $HX + XT$ is as short as possible. (This problem will be investigated in Chapters 1 and 5.)

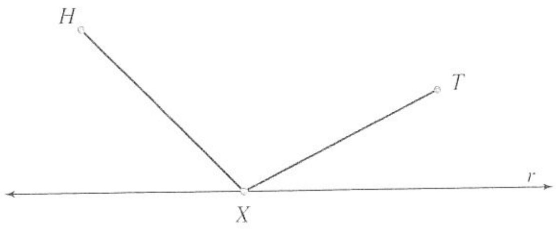


Figure 0.4

0.5 The Shortest Highway

A highway connecting two cities A and B as in Figure 0.5 needs to be built so that part of the highway is on a bridge perpendicular to the parallel banks b_1 and b_2 of a river. Where should the bridge be built so that the path $AXYB$ is as short as possible? (This problem will be investigated in Chapter 5.)

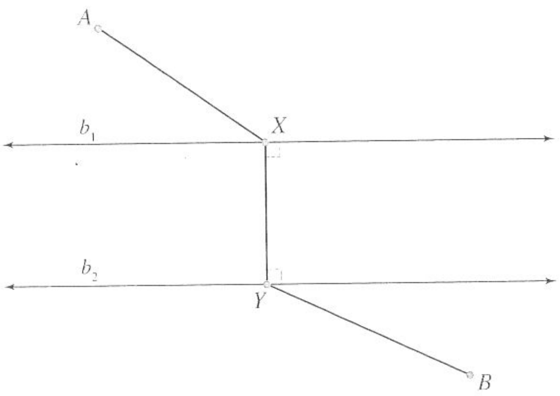


Figure 0.5

0.6 Steiner's Minimum Distance Problem

One of the greatest geometers of all time, and certainly of the nineteenth century, was Jacob Steiner (1796–1863). Born in Switzerland but educated in Germany, Steiner discovered and proved new theorems and introduced new geometrical concepts. In particular, he was interested in the solutions of maximum and minimum problems using purely geometric methods—that is, without using calculus or algebra. Among others, he proved the theorem illustrated in Figure 0.6.

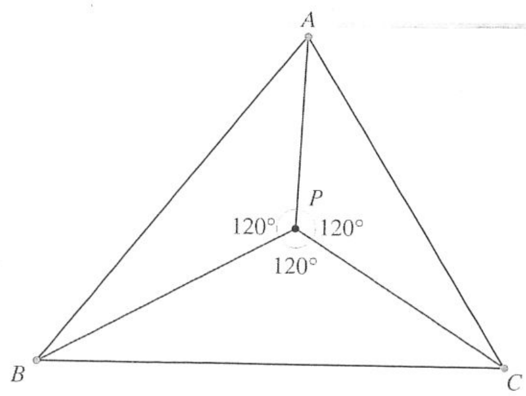


Figure 0.6

If A, B, and C are three points forming a triangle such that each of the angles of $\triangle ABC$ is less than 120°, then the point for which the sum of the distances from P to the vertices of the triangle (that is, $PA + PB + PC$) is at its minimum has the property that each of the angles at P is 120°. Steiner also dealt with the case when one of the angles is 120° or greater than 120°. In addition, he generalized the problem for n points.

Many maximum and minimum problems can be solved more efficiently using purely geometrical methods rather than calculus. Such problems will be investigated using transformational geometry in Chapter 5.

0.7 The Pythagorean Theorem

One of the best-known and most useful theorems in geometry and perhaps all of mathematics is the Pythagorean Theorem, which was discovered in the sixth century B.C.E. (Before the Common Era). There are numerous known proofs of the theorem (*The Pythagorean Proposition* by E. Loomis contains hundreds of them). Do you recall any proof of the Pythagorean Theorem? Can you prove it on your own?

The following is one way to state the theorem: If squares are constructed on the sides of any right triangle (a triangle with a 90° angle), then the area of the largest square equals the sum of the area of the other squares. In Figure 0.7, if the areas of the squares are A, B, and C, then $A + B = C$ or, equivalently, $a^2 + b^2 = c^2$, where a and b are the lengths of the legs of the triangle and c is the length of the hypotenuse (the side opposite the right angle).

Figure 0.7 and Figure 0.8 can be used to justify the Pythagorean Theorem. Can you see how?

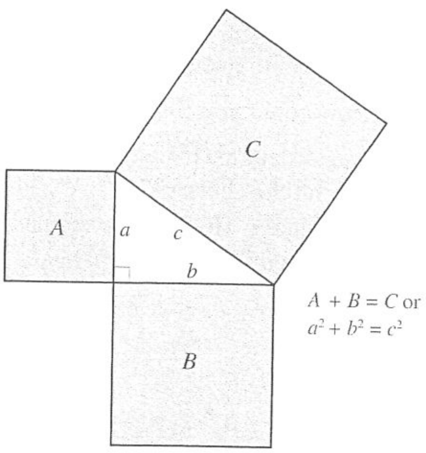


Figure 0.7

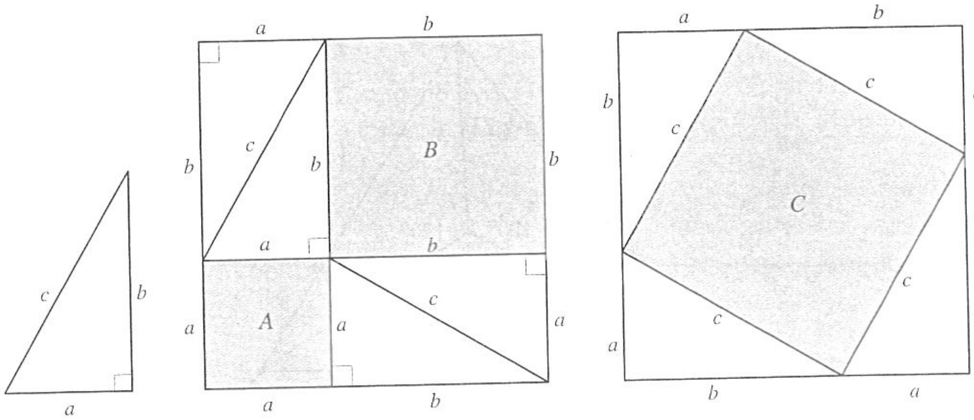


Figure 0.8

In Chapters 3 and 4, we will give several proofs of the Pythagorean Theorem. In Chapter 4 we will also explore what other figures can be constructed on the sides of a right triangle so that the area of the figure constructed on the hypotenuse is equal to the sum of the areas of the figures constructed on the other two sides. We will also generalize the theorem for triangles that are not necessarily right triangles.

Problem Set 0

In each of the following problems, you may use any tools to perform the experiments (a geometry drawing utility such as GSP, the Geometer's Sketchpad, is especially convenient but not necessary).

• **1.** • **a.** Conjecture the solution to the Treasure Island Problem by choosing at least four different positions for the gallows as described in the text.

• **b.** Does your conjecture hold for some Γ positioned below the line connecting the coconut and banana tree? (Make an appropriate construction.)

• **c.** Place Γ at one of the trees and find the corresponding treasure.

• **d.** Place Γ on the line connecting the trees halfway between them and find the corresponding treasure.

• **e.** Based on your experiments in (a) through (d), what seems to be the simplest way to find the treasure?

• **2.** • **a.** Draw a circle on transparent (or see-through) paper. How would you find the center of the circle by folding the circle onto itself?

• **b.** Draw an arc of a circle. How could you find the center now?

• **3.** Let $ABCD$ be any convex quadrilateral. On each side of the quadrilateral, construct a square as shown in Figure 0.9. Find the centers C_1, C_2, C_3, and C_4 of the squares, where C_1 and C_3 are centers of opposite squares. How are the segments C_1C_3 and C_2C_4 related? Repeat the experiment starting with a different quadrilateral.

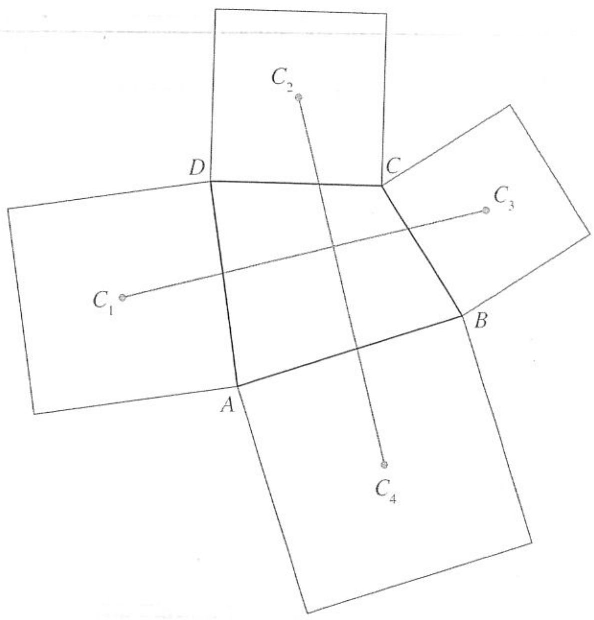


Figure 0.9

4. Choose an arbitrary triangle ABC and construct the corresponding Nine-Point Circle. (You may need the result from Problem 2.)
5. Check Morley's Theorem experimentally for some triangle ABC.
6. Use Figure 0.8 to prove the Pythagorean Theorem.